USSR STATE COMMITTEE FOR UTILIZATION OF ATOMIC ENERGY
INSTITUTE FOR HIGH ENERGY PHYSICS

И Ф В Э 89-193
ОТФ

M.V.Saveliev
Institute for High Energy Physics, Serpukhov

A.M.Vershik
Leningrad State University, Leningrad

NEW EXAMPLES OF CONTINUUM GRADED LIE ALGEBRAS

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Abstract

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We give several new examples of continuum graded Lie algebras which provide an additional elucidation of these algebras. Here, in particular, the Kac-Moody algebras, the algebra S_0Diff T^2 of infinitesimal area-preserving diffeomorphisms of the torus T^2, the Fairlie, Fletcher and Zachos sine-algebras, etc., are described as special cases of the cross product Lie algebras.

Аннотация

Вершик А.М., Савельев М.В. Новые примеры континуальных градуированных алгебр Ли: Препринт 89-193. - Серпухов, 1989. - 16 с., библиогр. 8:

Мы приводим несколько новых примеров континуальных градуированных алгебр Ли, которые проясняют понимание этих алгебр. В числе них находятся алгебры Каца-Муди, алгебра S_0 Diff T^2 инфинитезимальных диффеоморфизмов тора T^2, сохраняющих объем, синус-алгебры Фаэри, Флетчера и Закоса, и т.д., которые описываются как специальные случаи алгебр Ли скрещенных произведений.

1. INTRODUCTION

After the publication of our first papers [1,2] where we introduced a new type of infinite-dimensional Lie algebras (called continuum Lie algebras there) with, generally speaking, an infinite - dimensional Cartan subalgebra and a contiguous set of roots, we have had many interesting discussions and stimulating conversations with physicists and mathematicians. Here it is noteworthy that there is a recent keen interest of theoreticians in applications of infinite-dimensional Lie algebras of various types (for example, in strings and membranes approaches, in the SU(∞) gauge theories, the theory of nonlinear evolution and wave dynamical systems, etc.). Recall that special examples of the continuum algebras considered as Z-graded Lie algebras are Kac-Moody algebras, algebras of Poisson brackets, algebras of vector fields on a manifold, current algebras, algebras of diffeomorphisms of a manifold, algebras with the Cartan operator (a generalization of the Cartan matrix) being the Hilbert operator, etc. In April, 1989 when paper [2] was completed and appeared as a preprint, Ch. Devchand and D.B. Fairlie acquainted one of us (M.S.) with the series of papers [3,5-7] where the authors had introduced and made identification of some generalizations of the Witt- and Virasoro-type algebras. Then, as soon as the end of April, D.B. Fairlie illuminated a connection between Example 2.3 of our paper [2] and the algebra from Ref. [5].

In our present paper we consider several new examples of the continuum Z-graded Lie algebras. Moreover, we describe the Kac-Moody algebras, the algebra S_0 Diff T^2 of infinitesimal area-preserving diffeomorphisms of the torus T^2, the Fairlie - Fletcher - Zachos sine-algebra, etc., as special cases of the cross product Lie algebras.

2. GENERAL DEFINITIONS

First of all, let us review in a few words the definition of infinite-dimensional continuum Lie algebras introduced in paper [2], however in a more general formulation.

Let E be an arbitrary associative (noncommutative in general) algebra over the field Φ (R or C); $K_{\pm,0}$ are three bilinear mappings $E\times E\to E$; $\hat{g}\equiv g_{-1}\oplus g_0\oplus g_{+1}$ is a local Lie algebra whose subspaces g_i, $i=0,\pm1$ (as vector spaces) are isomorphic to E and their elements $X_i(\varphi)$, $\varphi\in E$, satisfy the defining relations

$$[X_0(\varphi),X_0(\psi)]=X_0([\varphi,\psi]),\quad [X_0(\varphi),X_{\pm1}(\psi)]=X_{\pm1}(K_\pm(\varphi,\psi)),$$
$$[X_{+1}(\varphi),X_{-1}(\psi)]=X_0(K_0(\varphi,\psi)), \tag{1}$$

for all $\varphi,\psi\in E$; $[\varphi,\psi]\equiv\varphi\psi-\psi\varphi$. Here the Jacobi identity for $\hat{g}$ is equivalent to the conditions

$$K_\pm([\varphi,\psi],\chi)=K_\pm(\varphi,K_\pm(\psi,\chi))-K_\pm(\psi,K_\pm(\varphi,\chi)),$$
$$[\psi,K_0(\varphi,\chi)]=K_0(K_+(\psi,\varphi),\chi)+K_0(\varphi,K_-(\psi,\chi)) \tag{2}$$

which are assumed to be satisfied. Let $g'(E;K)$ be a Lie algebra freely generated by $\hat{g}$, and J be the largest homogeneous ideal which has a trivial intersection with g_0. Then $g(E;K)=g'(E;K)/J$ is called a continuum contragredient Lie algebra. Clearly, it is a Z-graded Lie algebra, $g=\oplus_{n\in Z}g_n$. (Note, that this gradation does not coincide, generally speaking, with a root-type decomposition).

All the examples considered in [2] belong to a special case of this formulation when E is an associative commutative algebra (possibly, without unity) over Φ, and the mappings

$$K_+ = -K_- = K \text{ and } K_0 \equiv S \text{ have a linear form, namely}$$
$$K_+(\varphi,\psi) = -K_-(\varphi,\psi) = \psi \cdot K\varphi, \quad K_0(\varphi,\psi) = S(\varphi.\psi).$$

Here the Jacobi identity is satisfied automatically. Moreover, the substitution $X_0(S\varphi) \to X_0(\varphi)$ reduces the defining relations for this case to the form with $K \to KS \equiv \tilde{K}$, $S \to I$, and the operator $\tilde{K}$ is called the Cartan operator.

Later, while referring to defining relations (1), we put the number of these formulas as (1_b), (1_l) and (1_{ls}) if the corresponding operators K and S are bilinear, linear and linear with S=I, respectively, and E is Abelian.

Let us emphasize once more that our starting point was just the classical Cartan theory of finite-dimensional simple Lie algebras. We show now how the latter agree with our axioms. For this case $E = C^n$ and is supplied with a coordinate multiplication; $g_0 \simeq E$ and it has the basis h_i, $1 \leq i \leq \text{rank} g < \infty$. $g_{\pm 1} \simeq E$ are the subspaces with the basis $X_i^{\pm}$ of the root vectors corresponding to the positive and negative simple roots of g. The isomorphism between g_0 and $g_{\pm 1}$ is given by the correspondence between their basis elements h_i and $X_i^{\pm}$ provided by the relations $[X_i^+, X_i^-] = h_i$. So, here we have $S = I$, and the operator K coincides, as is easy to verify, with the Cartan matrix of the corresponding finite-dimensional Lie algebra g^f, $k = (k_{ij})$, $[h_i, X_j^{\pm}] = \pm k_{ji} X_j^{\pm}$. In fact, most of the examples considered in paper [2] are just the continuum analogues of this case.

3. CROSS PRODUCT LIE ALGEBRAS

In this Section we construct the main examples of Z-graded continuum Lie algebras and somewhat more general examples of

Z-graded Lie algebras with arbitrary Cartan subalgebras, namely the cross product Lie algebras.

The class of Lie algebras coming after the finite-dimensional case (in the sense of their complexity) are infinite-dimensional Lie algebras with a finite-dimensional Cartan subalgebra, in particular, the Kac-Moody algebras. Here we demonstrate that the Kac-Moody algebras are a special case of the algebras associated with the cross product Lie algebras (see Example 2.6 in [2]). We think such understanding of the Kac-Moody algebras to be important. That is why we consider this problem in greater detail despite the fact that we have already shown in Ref. [2] that the Kac-Moody algebras are included in our general scheme (see Example 2.1 in [2]). For simplicity, we shall consider the example of the series A_n. An analogous discussion for the other classical series (B_n, C_n and D_n) will be presented in our next publication.

First, define the cross product Lie algebras and prove that the ross product is a Z-graded Lie algebra in the above sense. Let E be a commutative associative algebra, while G is a group of its automorphisms with the generators T. (In what follows, $G=Z$ or $G=Z_p\equiv Z/pZ$). We shall call a cross product $g(E;T)$ such a Lie algebra which consists of finite sums of the form $\sum_n \varphi_n \otimes W^n$ with the bracket

$$[\varphi\otimes W^m, \psi\otimes W^n] = (\varphi\cdot T^n\psi - \psi\cdot T^m\varphi)\otimes W^{m+n}, \qquad (3)$$

where φ, $\psi\in E$. Note that the algebra is infinite-dimensional even if the group G is finite (this is the case with T being a periodic operator). It is easy to convince oneself that $g(E;T)$ is a Lie algebra. If one assumes that $W^N=I$, N being the period of T, then the algebra in question becomes a finite-dimensional Lie algebra (see below). More exactly, if $T^N=I$ then $(\sum_n \varphi_n \otimes W^{nN})$ forms an ideal, and the quotient is a finite-dimensional Lie algebra.

Let us show now that the algebra $g(E;T)$ belongs to the class of Z-graded algebras which we are interested in, and determine the corresponding Cartan operator. For this aim define $g_0 \simeq E$, $g_{+1} \simeq E \otimes W$, while g_{-1} is identified with E as $\varphi \to T^{-1}\varphi \otimes W^{-1}$. Similarly, for g_{-n} we have $\varphi \to T^{-n}\varphi \otimes W^{-n}$. Then the operators K and S take the form

$$K\varphi = (I-T)\varphi, \quad S\varphi = (I-T^{-1})\varphi. \tag{4}$$

Now pass to the algebra $\tilde{g}$ via the substitution $\varphi \to (I-T^{-1})\varphi$ (cf. the corresponding procedure of transition from (1) to (1_s)). For this mapping the invariants (under the action of T), in particular constants, turn into zeros, i.e., $g_0 \to \tilde{g}_0 = g_0/g_0^T$ where $g_0^T = (a \in g_0,\ Ta=a)$, and correspondingly, $g \to \tilde{g}$. Then the operator $\tilde{K} \equiv KS$ has the form $\tilde{K} = 2I-T-T^{-1}$.

Consider now the following concrete examples of this quite general construction.

3a. Kac-Moody Algebras as Cross Products.

Let $E = C^n$ with a coordinate multiplication, T be any cyclic permutation of the coordinates. Then the algebra $\tilde{g}(E;T)$ is exactly the centreless Kac-Moody algebra for the series $\tilde{A}_n$.

Now let us have a closer look at it. Consider $x=(x_1,\dots,x_n) \in E$, $(Tx)_i = x_{(i+1) \bmod n}$. Then

$$g_0 \simeq (x_i, \sum_i x_i = 0) = C^{n-1}; \quad g_{sn} \simeq C^{n-1}, s \in Z; \quad g_i \simeq C^n, i \neq sn;$$

and, hence, $\tilde{g}(E;T) = \sum_{k \in Z} \oplus C^{r(k)} \otimes W^k$ where

$$r(k) = \begin{cases} n-1, & \text{if } k = sn,\ s \in Z \\ n & \text{otherwise.} \end{cases}$$

Here the bracket is defined by the formula

$$[x \otimes W^k, y \otimes W^l] = (x_i y_{(i+k) \bmod n} - x_{(i+l) \bmod n}\, y_i)_{i=1}^{n} \otimes W^{k+l}.$$

The above mentioned gradation of $\tilde{g}$ is known and clearly does not coincide with the loop gradation. (The reader can compare the usual description of this gradation with ours.) Our aim was to show that the natural description of this algebra obtained in terms of cross products is simpler. The case of the series B, C and D will be considered elsewhere.

3b. Lie Algebras Associated with Circle Rotation.

This example is very interesting. It has been considered in our paper [2] and earlier from a somewhat different point of view, also in Ref. [3]. Let E be a space of the trigonometric polynomials, $T \equiv T_{2\lambda}$ is the operator of rotations, i.e. $T\, e^{2\pi i n t} = e^{2\pi i n(t+2\lambda)}$ where λ is irrational. For this case the algebra $g(E;T)$ is a continuum contragradient Lie algebra with the operators $\dot{K}$ and S entering relations (1_I) of the form

$$K_\lambda = -i\mathfrak{I}\, \exp(\lambda\, \partial/\partial t)\cdot \mathrm{sh}(\lambda\, \partial/\partial t), \quad S_\lambda = K_{-\lambda},$$

or, after factorization over the constants,

$$K_\lambda \longrightarrow \tilde{K}_\lambda \equiv K_\lambda S_\lambda = \mathfrak{I}^2\, \mathrm{sh}^2(\lambda\, \partial/\partial t), \quad S_\lambda \longrightarrow I.$$

Then, if we choose the basis of g_n, $g = \oplus_{n\in Z}\, g_n$, as

$$Y_{\bar{n}} = \mathfrak{I}/2i\, \exp in_2(t+n_1\lambda) \otimes T^{n_1},$$

the commutation relations for the algebra in question take the form

$$[Y_{\bar{m}}, Y_{\bar{n}}] = \mathfrak{I}\, \sin \lambda(\bar{m}\times\bar{n})\cdot Y_{\bar{m}+\bar{n}}. \qquad (5)$$

Here $\bar{m}=(m_1,m_2)$, and $\bar{n}=(n_1,n_2)$ are 2-dimensional integer vectors, $\bar{m}\times\bar{n} \equiv m_1 n_2 - m_2 n_1$; $\mathfrak{I}$ is some constant; $\dim Y_{\bar{n}} = 1$. This basis coincides, up to the inessential re-notations, with the basis from Ref. [3] where a central extension of this algebra was also considered. Note that the elements X_ϵ, $\epsilon = 0, \pm 1$, entering (1_I)

are expressed via $Y_{\bar{n}}$ as $X_{\epsilon}(\exp(int)) = (2/i\hbar) Y_{n,\epsilon} \exp(-in|\epsilon|\lambda)$; the roots are identified with $\hbar n \, \mathrm{sh}(\lambda \, \partial/\partial t)\delta(t-t')$; the dynamic equation (of the continuous Toda - type lattice) associated with this algebra by means of the construction given in Ref. [2] has the form

$$\partial^2 \rho/\partial z_+ \partial z_- = 2 \exp \rho(t) - \exp \rho(t+\lambda) - \exp \rho(t-\lambda), \quad \rho \equiv \rho(z_+, z_-; t)$$

It is interesting to consider the degeneration of this algebra for the rational values of $\lambda/2\pi$, i.e. $\lambda = 2\pi(M/N)$ with the mutually simple $M, N \in Z$. Then the algebra is isomorphic to the tensor product g_N of the Kac-Moody series $\tilde{A}_N$ and $C[\tau^{-1},\tau]$, i.e., $\tilde{g}_N = \tilde{A}_N \otimes_{T_{2\lambda}} C[\tau^{-1},\tau]$. The consideration of the quotient algebra g_0/J^{λ} (J is the ideal generated by the elements $(\exp(2\pi i N p t) - 1)$ for all $p \in Z$) leads to $\tilde{A}_N$. The interpretation given in Ref. [3] corresponds to an additional factorization over the subspaces with the gradation indices multiple of N and leads to the series A_N.

Among the algebras considered within the general scheme of Example 2.6 in Ref. [2], the case in question is distinguished by the fact that the operator T has a complete (discrete) system of eigenfunctions. Thus, the whole algebra which we have initially introduced as a Z-graded algebra, can be represented as a Z^2-graded algebra with one-dimensional components (This is what was done, in fact, in Ref. [3]). Any other diffeomorphism (see Example 2.6 in [2]) with a discrete spectrum leads to the same class of algebras, for example, shifts on a d-dimensional torus,

$$[Y_{m_0,\bar{m}}, Y_{n_0,\bar{n}}] = \hbar \sin \bar{\lambda} \, (m_0 \bar{n} - n_0 \bar{m}) \, Y_{m_0+n_0,\bar{m}+\bar{n}},$$

where $\bar{m}$ and $\bar{n}$ are d-dimensional integer vectors.
Note the fact that Example 2.6 is a limiting case for the series A_N (in accordance with Theorem 6 of [2]), while its other limi-

ting case is given by Example 2.3 (in [2]) of the Poisson bracket algebra which we will say a few words about.

3c. Limiting Case Associated With $S_0\mathrm{Diff}T^2$.

Taking in (5) $\hbar^{-1}=\lambda$, in the limit $\lambda\to 0$ one comes [3] to the algebra $S_0\mathrm{Diff}T^2$ (see Examples 2.3 and 2.4 in [2]),

$$[Y^0_{\bar{m}},\ Y^0_{\bar{n}}] = (\bar{m}\times\bar{n})Y^0_{\bar{m}+\bar{n}}, \tag{6}$$

i.e., the centreless algebra of the infinitesimal area-preserving diffeomorphisms of the torus T^2. (This form of writing the commutation relations for the elements of the algebra was first published in [4]). The identification of this type has been done and used in Refs. [3, 5-7] and others.

3d. Vector Fields of a Manifold.

The vector fields on a manifold (see Example 2.7 in [2]) represent also the limiting case of the cross product Lie algebras.

4. CROSS PRODUCT LIE ALGEBRAS WITH THE CARTAN OPERATOR DEFINED ON A NONCOMMUTATIVE ASSOCIATIVE ALGEBRA.

Now we will give an example of a more general situation when the subalgebra g_0 is not Abelian. Let T be a diffeomorphism, for example rotations of a circle, or any other diffeomorphism. Then the corresponding defining relations (1) generate rather a general class of the continuum Lie algebras $g(E;T)$ which involves the above - mentioned Examples from Section 3.

We begin with a coordinate basis and introduce the following commutation relations (in the sense of (9)) for the elements $Y_{\bar{n}}$ ($\dim Y_{\bar{n}}=1$) of the Lie algebra $g(E;T)$:

$$[Y_{\vec m}, Y_{\vec n}] = A_{\vec m, \vec n} Y_{\vec m + \vec n}, \qquad (7)$$

with

$$A_{\vec m, \vec n} = i\Im\cos\lambda(\vec m \times \vec n) \begin{cases} \mp 1, \text{if } m_1 \equiv 0 (\text{mod } 3),\ n_1 \equiv \pm 1 (\text{mod } 3); \\ -\frac{1}{2}, \text{if } m_1 \equiv 1 (\text{mod } 3), n_1 \equiv -1 (\text{mod } 3); \\ 0 \text{ for all other values of the indices } m_1 \text{ and } n_1 \end{cases}$$

Realize the elements $Y_{\vec n}$ in the form:

$$Y_{3n_1, n_2} = \frac{\Im}{2i} \begin{pmatrix} e^{in_2(t+n_1\lambda)} \otimes T^{n_1} & 0 \\ 0 & -e^{in_2(t+n_1\lambda)} \otimes T^{n_1} \end{pmatrix},$$

$$Y_{3n_1+1, n_2} = \frac{\Im}{2i} \begin{pmatrix} 0 & e^{in_2(t+n_1\lambda)} \otimes T^{n_1} \\ 0 & 0 \end{pmatrix},$$

$$Y_{3n_1-1, n_2} = \frac{\Im}{2i} \begin{pmatrix} 0 & 0 \\ e^{in_2(t+n_1\lambda)} \otimes T^{n_1} & 0 \end{pmatrix}, \qquad (8)$$

where $T \equiv T_{2\lambda}$. So, the algebra under consideration is $C(Z) \otimes_\lambda Sl(2, C[\tau^{-1}, \tau])$, i.e. the cross product of current on the circle $S_{(1)}$ (with the values in the Laurent polynomials on some other circle $S_{(2)}$) and the shift on the circle $S_{(2)}$. Here instead of $Sl(2,C) \otimes_C C[\tau^{-1}, \tau]$ one can take the algebra $g^f \otimes C[\tau^{-1}, \tau]$ with an arbitrary semisimple finite-dimensional Lie algebra g^f. Note that, as follows from relations (7), (8), the transition $\lambda \to 0$ is not correct due to the presence of the cosine-type structural constants.

This algebra is a Z-graded algebra, it admits various gradations. First of all, we can represent g as $g = \oplus_{n_1 \in Z} g_{3n_1+\epsilon}$, $\epsilon = 0, \pm 1$, with the subspaces

$$g_{3n} = (Eh \otimes T^n),\ g_{3n\pm 1} = (EX_\pm \otimes T^n).$$

Here E is any ring, for example, a ring of polynomials or a ring of functions on a torus, circle, etc ; h and $X_{\pm}$ are the basis elements of sl(2,C). The bracket (see above) can be written in the form

$$[U_1 \otimes T^m,\ U_2 \otimes T^n] = (U_1 \cdot T^m(U_2) - U_2 \cdot T^n(U_1))_o \otimes T^{m+n}. \quad (9)$$

Here the local part $g_{-1} \oplus g_0 \oplus g_{+1}$ coincides with sl(2, E); the multiplication $U_1 \cdot U_2$ is an ordinary matrix product of the corresponding elements; the subscript "o" in $(\ )_o$ denotes the traceless part of the round bracket, which means the quotient of the algebra by the center. For this gradation the local part is a subalgebra and it does not generate the algebra g as a whole.

Then, it is possible to use the loop-like gradation

$$g = \bigoplus_{n_1 \in Z} g_{n_1},\quad g_{n_1} \simeq sl(2, C[\tau^{-1}, \tau]) \otimes T^{n_1},$$

or, in a more general case, $g_{n_1} \simeq \tilde{g}^f \otimes T^{n_1}$, and define the diffeomorphism by the formula $T\varphi(t) = \varphi(Tt) \in g^f$, so that the bracket is

$$[U_1(t)T^m,\ U_2(t)T^n] = (U_1(t) \cdot U_2(T^m t) - U_2(t) \cdot U_1(T^n t))_0 \cdot T^{m+n}. \quad (10)$$

For this gradation the subspace g_0 is not Abelian.

The "discrete" version of this algebra (in the sense of Ref. [2]) is obtained, for example, if one takes instead of a circle the finite group Z_p. Then, in particular, for p=2 we come to a Z-graded algebra of constant growth with dim g_n=2.

Note that if one considers the cross product of gl(2) with bracket(3) where "." is the symbol of multiplication in the associative algebra GL, then the subalgebra which is the cross product of the diagonal matrices, is isomorphic to (5)

The examples considered above associated with the cross product, contain the algebra of the functions with a zero integral as the Cartan subalgebra. This fact becomes clear from the formulas of Sections 3 and 4 if one excludes the operator S from the defining relations. As a centre we can adjoint constants obtaining a one-dimensional centre as a direct term. However, sometimes, as in the case of the Kac-Moody algebras, there is a nontrivial central extension, i.e. a nontrivial cocycle. A general investigation of the problem of the central extensions for the algebras discussed in this paper will be given elsewhere. Here we consider only one example which is contained (in a basis form) in Ref. [3].

Let us write down the bracket with a cocycle for the algebra, generated by the rotation of the circle, in the following form

$$[\varphi \otimes T_\lambda^n, \psi \otimes T_\lambda^m] = (\varphi T^n \psi - \psi T^m \varphi) \otimes T_\lambda^{m+n} + \delta_{m+n,0}\, n \int_0^{2\pi} dt\ \varphi(t)\psi(t+n\lambda).$$

Then, by a direct check one gets convinced that this is a nontrivial 2-cocycle which defines the central extension of the algebra $g(E;T)$. The (Jacobi) identity for the cocycle uses a concrete form of the transformation. It is not clear, however, whether a similar 2-cocycle exists for every transformation T, or not.

The limiting case of the given example is, as has already been mentioned above, the Poisson bracket algebra on the torus T^2. Here the cocycle has the form

$$[\varphi, \psi] = (\partial\varphi/\partial s\ \partial\psi/\partial t - \partial\varphi/\partial t\ \partial\psi/\partial s) + \iint ds\, dt (\mathfrak{I}_1 \partial\varphi/\partial s + \mathfrak{I}_2 \partial\varphi/\partial t)\psi,$$

where $\mathfrak{I}_{1,2}$ are some parameters. There are also other cocycles; the problem of their complete description seems quite interesting.

Reduction of the subspace g_0.

It is possible to attach to the example considered above the structure of the Z-graded algebra with a (commutative) Cartan subalgebra g_0. This transition is analogous to the transition from the loop gradation to the principal gradation of the Kac-Moody algebras. For this aim let us put $\bar{g}_0 = C[\tau^{-1},\tau]$, i.e., in basis (7) the subspace $\bar{g}_0$ is the linear hull of the elements $Y_{3n_1,0}$. Further, we take $\bar{g}_{+1} = C[\tau^{-1},\tau] \oplus C[\tau^{-1},\tau]$. Namely, in basis (7) the subspace $\bar{g}_{+1}$ is the linear hull of the elements $Y_{3n_1+1,0}$ and $Y_{3n_1-1,1}$, while the corresponding elements of $\bar{g}_{-1}$ are $Y_{3n_1-1,0}$ and $Y_{3n_1+1,1}$.

Here the subspaces $\bar{g}_{\pm 1}$ generate the algebra $g = \oplus_n \bar{g}_n$ as a whole, and, therefore, the elements of $\bar{g}_{\pm 1}$ can be considered as the Chevalley generators.

This representation is nothing but the cross product of the graded Kac-Moody algebra $\tilde{A}_1$, which acts on $C[\tau^{-1},\tau]$ in accordance with the formula $T\,\varphi(\theta) = \varphi(\theta+\lambda)$.

The subspaces $\bar{g}_{\pm 1}$ in the construction of the present Section, in distinction with the general definition given in Section 2, are only modules over E, but not already isomorphic (as the linear spaces) to the algebra E.

5. SUPPLEMENT TO THE LIST OF CONTINUUM LIE ALGEBRAS OF TEMPERATE GROWTH.

5a. Lie Algebra of Polynomial Differential Operators.

This example appeared as a result of the discussion with B.L. Feigin. Let g be a Lie algebra of polynomial differential operators of one variable (t) with the generators ∂ and τ (a derivative with respect to t and multiplication by t, respectively), $\deg\partial=-1$, $\deg\tau=+1$; $[\partial,\tau] = I$. Then g becomes a graded Lie algebra of the type involved.

Really, let E be a ring of polynomials of one variable h, i.e., $E=C[h]=(\sum_n C_n h^n)$. Identify g_0 with E by means of the mapping $\varphi \to \varphi(\tau\cdot\partial)$, $\varphi\in E$. The identification of $g_{\pm 1}$ with E is performed using the formulas for monomials:

$$X_{+1}(\varphi_n)\simeq(\tau\cdot(\partial\cdot\tau)^n),\quad X_{-1}(\varphi_n)\simeq((\partial\cdot\tau)^n\cdot\partial)$$

with $\varphi_n(h)=h^n$; here $X_0(\varphi_n)\simeq((\partial\cdot\tau)^n)$. Then the algebra g is a particular case of the algebra with defining relations (1_1), for which

$$K\varphi(h)=\varphi(h)-\varphi(h-1),\quad S\varphi(h)=h\varphi(h)-(h+1)\varphi(h+1), \tag{11}$$

$\varphi\in E$, or, equivalently, with defining relations (1_{1S}), for which

$$\tilde{K}\varphi(h)\equiv K\cdot S\varphi(h)=2h\varphi(h)-(h+1)\varphi(h+1)-(h-1)\varphi(h-1). \tag{12}$$

(Cf. the discrete version of Example 2.6 in Ref. [2].)

5b. Versions of $gl(\infty)$.

The simplest Z-graded Lie algebras with a "discrete" Cartan subalgebra which is not finite-dimensional (in distinction from the Kac-Moody algebras, the classical finite-dimensional case including) are various types of the $gl(\infty)$ algebra. Let

$E'=\sum_{i\in N}\oplus C$ be a direct sum (a space of finite sequences);

$E''=(\lambda 1,\lambda\in C)\oplus E'=\{C=(C_i),\ i\in N \mid \exists\lambda(C),\ i_0(C):\ C_i=\lambda \text{ for } i>i_0(C)\}$;

$E'''=\prod_{i\in N} C=\{(C_i): C_i\in C\}$.

Here the spaces E', E'' and E''' are supplied with coordinate multiplication, and for all the cases in question the operators K and S on these spaces have the form

$$K=I-T_1 \text{ with } T_1((C_i)_{i=1}^{\infty}) = (C_2, C_3, \ldots),$$

$$S=I-T_2 \text{ with } T_2((C_i)_{i=1}^{\infty}) = (0, C_1, C_2, \ldots).$$

Then the corresponding Lie algebras $g(E;K,S)=gl(\infty)$ are: $gl(E';K,S)=gl(N)_{fin}$ is a Lie algebra of matrices with a finite support;

$gl(E'';K,S)=gl(N)_{diag}$ is a Lie algebra of infinite ("diagonal") matrices which, up to the finite matrices, have the form $a_{ij}=a(i-j)$, $i,j\in N$;

$gl(E''';K,S)=gl(N)_{a.diag}$ is a Lie algebra of infinite ("almost diagonal") matrices, i.e. $a_{ij}=0$ for $|i-j|>k$; $i,j\in N$ (matrices with a support in the diagonal strip).

All these matrices can be changed for the matrices (a_{ij}) with $i,j\in Z$ if

$$E'\to\tilde{E}'=\sum_{i\in Z}\oplus C,\ E''\to\tilde{E}''=(\lambda I,\lambda\in C)\oplus\tilde{E}',\ E'''\to\tilde{E}'''=\prod_{i\in Z} C.$$

The corresponding Lie algebras are

$gl(Z)_{fin}$, $gl(Z)_{diag}$, and $gl(Z)_{a.diag}$ (see Ref. [8]).

So we have six Lie algebras. It is interesting to note that the root systems are the same for the first three algebras ($gl(\infty)$: o—o—o ..); the difference between them lies just in their Cartan subalgebras, i.e., E',E'' and E'''. A similar situation takes place also for the remaining three algebras.

Note that the operator S is not, generally speaking, invertible. The algebra $gl(\infty)_{diag}$ is adjoint to the associative algebra which is a semigroup algebra of the so-called bicyclic semigroup.

It is obvious that the algebra A_∞ in the standard understanding (i.e., that it is finitary infinite matrices with a zero trace) can be represented in the framework of our scheme as the algebra $g(E;K,S)$. Here E is the space of finitary sequences (or the other one), $S=I$, while the operator $K=(2\delta_{ij}-\delta_{i+1\,j}-\delta_{i-1\,j})_{i,j=1}^{\infty}$ is the same as the operator $\tilde{K}\equiv KS=(I-T_1)(I-T_2)$ considered above. For the last three algebras $\tilde{K}=(2\delta_{ij}-\delta_{i+1\,j}-\delta_{i-1\,j})_{\pm1}$ with $i,j\in Z$. In these cases the operator S is invertible, $T_2=T_1^{-1}$, and the algebras $gl(Z)_{fin}$, $gl(Z)_{diag}$ and $gl(Z)_{a.diag}$ are cross products in the above-mentioned sense with $E=\tilde{E}'$, $\tilde{E}''$, $\tilde{E}'''$, and $T(\equiv T_1)$ is a biliteral shift in Z.

Note that those of the algebras discussed here which are continuum ones, are not embedded into $gl(\infty)$ due to the discreteness of the root system of the latter. Therefore, in particular, their modules, cocycles, etc., are not "extended" through $gl(\infty)$. This fact distinguishes the continuum algebras from, for example, the Kac-Moody algebras.

6. CONTRACTION

The last comment concerns the transition to contracted algebras which we consider by the example of continuum Lie algebras with a commutative associative algebra E and linear operators K and S. Following the standard technique by Inönü-Wigner, supply the elements $X_{\pm 1}(\varphi)$ in defining relations (i_1) with the contraction parameter χ as follows:

$$X_{\pm 1}(\varphi) = \chi\, X^c_{\pm 1}(\varphi),\quad X_0(\varphi) = X^c_0(\varphi),$$

and let χ tend to infinity. Then the resulting (contracted) algebra g^c would coincide with its local part defined by the relations

$$[X^c_0(\varphi), X^c_0(\psi)] = 0,\quad [X^c_0(\varphi), X^c_{\pm 1}(\psi)] = \pm X^c_{\pm 1}(\psi K\varphi), \qquad (13)$$

$$[X^c_{+1}(\varphi),\ X^c_{-1}(\psi)] = 0$$

(cf. (i_1)), i.e. it is a continuum analogue $M_2(E)$ of the algebra of the 2-dimensional plane motion group M_2. The subspaces $g^c_{\pm n}$ of the algebra $g^c = \oplus_{n\in Z} g^c_n$ with $|n|>1$ are absent; this can be verified by a direct check using the Jacobi identity.

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Савельев М.В. и др.

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