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THE BEST-FIT UNIVERSE

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Inflation provides very strong motivation for a flat Universe, Harrison-Zel'dovich (constant-curvature) perturbations, and cold dark matter. However, there are a number of cosmological observations that conflict with the predictions of the simplest such model—one with zero cosmological constant. They include the age of the Universe, dynamical determinations of Ω, galaxy-number counts, and the apparent abundance of large-scale structure in the Universe. While the discrepancies are not yet serious enough to rule out the simplest and "most well motivated" model, the current data point to a "best-fit model" with the following parameters: $\Omega_B \simeq 0.03$, $\Omega_{\rm CDM} \simeq 0.17$, $\Omega_\Lambda \simeq 0.8$, and $H_0 \simeq 70\,{\rm km\,sec^{-1}\,Mpc^{-1}}$, which improves significantly the concordance with observations. While there is no good reason to expect such a value for the cosmological constant, there is no physical principle that would rule such out.

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Introduction

Over the past decade the infusion of ideas from particle physics with implications for the earliest history of the Universe and a growing body of cosmological data that can test these implications have led to a renaissance in cosmology. Several key cosmological parameters that seemed to be beyond the realm of explanation or prediction, can now be "predicted" by very well motivated theories of the early Universe. Among them are the baryon asymmetry of the Universe, the curvature radius of the Universe, the spectrum of primeval density perturbations, and the quantity and composition of matter in the Universe. Knowledge of these parameters is crucial to understanding how the Universe evolved to its present state—especially how structure formed.

Foremost of the attractive scenarios of the early Universe is inflation.[1] It provides a comprehensive scenario for the earliest history of the Universe and makes a number of robust predictions: (i) spatially-flat Universe;[a] (ii) Harrison–Zel'dovich spectrum of scale-invariant curvature perturbations;[2] and (iii) a spectrum of relic gravitational waves.[3]

[a] The spatially-flat Einstein–de Sitter model is favored for other reasons as well: (i) Temporal Copernican Principle—if $\Omega \neq 1$ the deviation of Ω from unity grows as a power

(Some might dispute the "robustness" of these predictions. For example, it is not impossible for the density perturbations to be nonscale invariant,[4] and isocurvature perturbations can also arise.[5] It is possible to have just enough inflation so that Ω today is less than unity, although such a model begs the question of why the curvature radius is just today becoming comparable to the Hubble radius and would likely be in conflict with the observed large-scale isotropy and homogeneity as our inflationary region would be comparable in size to the present Hubble radius (see Silk and Turner[4]). The three inflationary predictions mentioned above are about as robust as theoretical predictions come! For further discussion of the "inflationary paradigm" see Ref. 6.)

The first two of these predictions have very important implications for structure formation. They provide the initial data for the structure formation problem: the nature of the density perturbations, and the quantity and composition of matter in the Universe. Taking the simplest flat Universe model—one with zero cosmological constant—flatness ($\Omega_{\rm TOT} = 1$) together with the primordial nucleosynthesis constraint to the baryon density[7]—$0.011h^{-2} \lesssim \Omega_B \lesssim 0.019h^{-2}$—imply that most of the matter in the Universe must be nonbaryonic.[b] (The present Hubble parameter $H_0 = 100h\,{\rm km\,sec^{-1}\,Mpc^{-1}}$ and $\Omega_i = \rho_i/\rho_{\rm CRIT}$ is the fraction of critical density contributed by species i.) There are a number of early Universe relics that are promising candidates for the nonbaryonic component of the mass density: an axion of mass $10^{-6}\,$eV to $10^{-4}\,$eV; a neutralino of mass from about 10 GeV to about 3 TeV; and a light neutrino of mass $90h^2\,$eV. (For a discussion of particle dark-matter candidates see Refs. 11.)

The neutrino is referred to as hot dark matter: Relic neutrinos have velocities close to the speed of light around the time the Universe became matter dominated and perturbations on scales less than about $40\,{\rm Mpc}/(m_\nu/30\,{\rm eV})$ are damped by neutrino free streaming. Structure forms from the "top down:" large objects (superclusters) form and then fragment into galaxies. Inflation-produced density perturbations and hot dark matter seem to be ruled out because galaxies form too late.[13]

The axion and neutralino behave as cold dark matter: Around the time the Universe becomes matter-dominated they have very small velocities, free streaming is unimportant, and perturbations on small scales survive unscathed.[c] Inflation-produced density perturbations and cold dark matter is a far more promising scenario. Indeed, some (including this author) have called it the most well motivated, most detailed, and most successful model for structure formation yet proposed![15]

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of the scale factor, begging one to ask why Ω is just now beginning to differ from unity; and (ii) structure formation—in spatially-open models there is less time for the growth of density perturbations and larger initial perturbations are required; in fact, low Ω models with adiabatic density perturbations are inconsistent with the isotropy of the CMBR.

[b] Several means for avoiding the nucleosynthesis constraint have been suggested;[8,9] the one that has attracted the most interest involves inhomogeneities in the baryon-to-photon ratio, produced by a strongly first-order quark-hadron transition.[9] While this scenario initially looked promising, it is now clear that the light element abundances predicted severely conflict with the observed abundances.[10]

[c] There is an intermediate possibility—referred to as warm dark matter—where the damping scale is about 1 Mpc (scale of galaxies); this case arises for a relic of mass about 1 keV with abundance about one-tenth that of a neutrino species.[14]

Successes of Cold Dark Matter

Specifically, the cold dark matter scenario is: a flat Universe whose composition is $\Omega_B \sim 0.1 \ll \Omega_{\rm CDM} \sim 0.9$, with $h \sim 0.5$ (to have a sufficiently old Universe) and inflation-produced Harrison–Zel'dovich curvature perturbations whose spectrum after the epoch of matter–radiation equality is[15]

$$|\delta_k|^2 = \frac{A\,k}{(1+\beta k+\omega k^{1.5}+\gamma k^2)^2}. \tag{1}$$

Here δ_k is the amplitude of the Fourier component of comoving wavenumber k ($\equiv 2\pi/\lambda$), A is an overall normalization constant, $\beta = 1.7(\Omega h^2)^{-1}\,{\rm Mpc}$, $\omega = 9.0(\Omega h^2)^{-1.5}\,{\rm Mpc}^{1.5}$, and $\gamma = 1.0(\Omega h^2)^{-2}\,{\rm Mpc}^2$. The epoch of matter–radiation equality, $t_{\rm EQ} = 4.4\times 10^{10}\,(\Omega h^2)^{-2}\,{\rm sec}$ and $T_{\rm EQ} = 5.5\,(\Omega h^2)\,{\rm eV}$, is when subhorizon-sized perturbations can begin to grow. The *rms* mass fluctuation on scale λ, $(\delta M/M)_\lambda$, is related to δ_k by: $\delta M/M \simeq k^{3/2}|\delta_k|/\sqrt{2}\pi$. The spectrum given by Eq. (1) is characterized by $\delta M/M \rightarrow \lambda^{-2}$ for $\lambda \gg \lambda_{\rm EQ} = 13(\Omega h^2)^{-1}\,{\rm Mpc}$ and $\delta M/M \rightarrow$ const for $\lambda \ll \lambda_{\rm EQ}$ (more precisely, $\rightarrow \ln\lambda$). While the horizon-crossing amplitude of the inflation-produced curvature perturbations is scale-invariant, the spectrum of perturbations at matter–radiation equality has a scale: $\lambda_{\rm EQ} = 13(\Omega h^2)^{-2}\,{\rm Mpc}$. That scale arises because subhorizon-sized density perturbations remain roughly constant in amplitude until the Universe becomes matter-dominated, and $\lambda_{\rm EQ}$ is the scale that crosses the horizon at matter–radiation equality. Note that the spectrum is a function of $\lambda/\lambda_{\rm EQ}$ only, and thus "shifts" right or left as $(\Omega h^2)^{-1}$. (For reasons that will soon become clear I have retained the Ω dependence throughout.)

Since the spectrum of perturbations decreases with increasing scale, small structures form first and larger structures form later[d] ("bottom up" or hierarchical structure formation). Typical galaxies form relatively recently, red shifts $z \sim 1$ to 2, although "rare" objects such as QSOs and large radio galaxies can form earlier.[16] The formation of a galaxy begins with the dark matter halo; baryons within the extended halo then dissipate energy and collapse to form a disk. Rich clusters too should have formed relatively recently. The prediction of relatively recent galaxy and cluster formation is consistent with "deep" CCD exposures that reveal few high red-shift objects.[17] The successes of cold dark matter are many; they include:[18]

- Provides a detailed and comprehensive scenario
- Correctly accounts for many properties of galaxies
  - Number densities of galaxies of different types
  - Internal properties of halos (flat rotation curves, rotation velocities, and mass densities)
- Accounts for observed galaxy clustering
- Predicts correct number density of clusters
- Accounts for clustering of clusters
- Predicts anisotropies of the cosmic microwave background radiation (CMBR) that are consistent with current limits and accessible in near-term experiments

In short, cold dark matter (CDM) is the most detailed and successful scenario of structure formation yet developed. The CDM Paradigm has served to focus and sharpen

[d] Mergers also probably play an important role in the formation of larger objects.

the questions that we ask about the formation of structure. At the very least CDM has served—and continues to serve—as a foil for observations.

Shortcomings of Cold Dark Matter

Cold dark matter is not without its shortcomings—perhaps serious enough to lead to its demise. For the most part its successful predictions involve the Universe on small scales—say less than about $20h^{-1}$ Mpc—where the observational data are relatively well established; its shortcomings involve observations on larger scales—where the data and their interpretations are less certain. The shortcomings of cold dark matter include:

- Predicts cluster–cluster correlation function amplitude that is about a factor of three too small
- Seems to predict less clustering on scales $\gtrsim 20h^{-1}$ Mpc than is indicated by recent determinations of the angular correlation function for the APM catalogue[19]
- May not be able to account for the large voids and the distribution of galaxies on thin surfaces surrounding voids, as seen in the CfA slices and in other surveys[20]
- May not be able to account for coherent structures as large and as thin as the so-called Great Wall[21]
- May not be able to account for the large bulk motion (about $700\,\mathrm{km\,s^{-1}}$) of the local $50h^{-1}$ Mpc neighborhood[22]
- May not be able to account for the "regularity" in red shifts seen in the recent pencil-beam survey of Broadhurst et al.[23]

These problems involve measurements that are on less firm ground and/or whose interpretations are less quantitative. For example, several authors have emphasized that the amplitude of the cluster–cluster correlation function[24] may have been overestimated due to selection effects in the Abell catalogue.[25] At present, there is no quantitative measure of the large-scale structure seen in the surveys mentioned, and to some eyes, numerical simulations of cold dark matter produce voids, Great Walls, and even red shift periodicity.[26]

The peculiar velocity field is a very powerful probe of the density field: Inhomogeneities in the matter distribution lead to peculiar velocities, and in linear theory $(\delta v/c)_\lambda \sim \Omega^{0.6}(\lambda/H_0^{-1})(\delta\rho/\rho)_\lambda$. The peculiar velocity field is almost unique in its ability to probe the density field; most other observations, e.g., red shift surveys, only determine the distribution of bright galaxies. However, peculiar velocities are difficult to measure because an accurate, independent measure of the distance to a galaxy is required. Moreover, the interpretation of the data is subtle. *If,* as the bulk motion data seem to indicate, a Great Attractor of mass $10^{16}\,M_\odot$ at a distance of about $40h^{-1}$ Mpc exists, this poses a real difficulty for CDM.

All of the above observations suggest that the cold dark matter scenario is deficient in large-scale power. There are other worrisome cosmological data:

(1) Age problems. The present age of a matter-dominated Einstein–de Sitter model $t_0 = 2/3H_0 \simeq 6.5h^{-1}$ Gyr. If the Hubble constant is greater than $65\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, then the age of the Universe is less than 10 Gyr, an age that is at best marginally consistent with other independent determinations. Conventional CDM all but requires $h = 0.5$. Likewise, the age of the Universe at a given epoch, $t(z) = 2H_0^{-1}/3(1+z)^{3/2}$, scales as H_0^{-1}; for a larger value of H_0 there is less time for an object at a given red shift to have evolved to its observed state. This may already be a problem for some high red-shift objects that appear highly evolved.[17]

(2) The Ω problem. Measurements of the mass density clearly indicate that the luminous component of the mass density is very small: $\Omega_{LUM} \lesssim 0.01$. Determinations of the mass density "associated with bright galaxies" indicate that $\Omega_{ABG} \simeq 0.1 - 0.3$, far greater than Ω_{LUM}, but significantly less than the predicted value of unity. There are ways of accommodating this disappointing fact. Since no rotation curve of a spiral galaxy has been seen to "turn over," the mass associated with spiral galaxies could be considerably greater than present estimates, even enough to provide $\Omega = 1$. Likewise, it is possible that the core radii of clusters are much larger than the distribution of galaxies indicate (e.g., if dynamical friction has caused galaxies to sink deep into the cluster potential). There is also the possibility that there is considerable mass density in unseen, low-luminosity galaxies that are more smoothly distributed—so-called biased galaxy formation.

(It should be mentioned that some determinations of Ω do give values close to unity; e.g., The reconstruction of the local peculiar velocity field using the distribution of matter as determined by the IRAS catalogue of infrared-selected galaxies provides a preliminary determination: $\Omega^{0.6}/b = 1.0 \pm 0.3$, where $1 \lesssim b \lesssim 3$ is the biasing factor.[27] Loh and Spillar[28] have used the galaxy count–red shift test with a sample of about 1000 field galaxies—red shifts out to 0.75—to infer $\Omega = 0.9^{+0.7}_{-0.5}$.[e])

(4) Galaxy counts. The number of galaxies observed in a given solid angle $d\omega$ and given red shift interval dz depends upon the number density of galaxies $n_{GAL}(z)$ and the cosmological model:

$$\frac{dN_{GAL}}{d\omega dz} = \frac{n_{GAL}(z)[zq_0 + (q_0 - 1)(\sqrt{2q_0 z + 1} - 1)]^2}{H_0^3(1+z)^3 q_0^4 [1 - 2q_0 + 2q_0(1+z)]^{1/2}},$$

$$\simeq z^2 n_{GAL}(z)[1 - 2(q_0 + 1)z + \cdots]/H_0^3, \tag{2}$$

where $q_0 = \Omega/2$ (for $\Lambda = 0$), and the second expression is valid for small z. For fixed number density of galaxies, the galaxy count increases with decreasing Ω (or q_0) because of the increase in spatial volume. The test has great cosmological leverage. Recent deep galaxies counts indicate an excess of galaxies at higher red shifts—indicative of a low value of Ω.[29] (If galaxy mergers are very important—as they may well be in CDM—the number density of galaxies at higher red shifts would be expected to be larger.)

To summarize, the shortcomings of cold dark matter are deficient large-scale structure, deficient galaxy counts, the age problems, and the Ω problem.[f] No one of these problems is sufficiently troublesome to falsify the cold dark matter paradigm—yet—but taken together they are worrisome. As we shall, the addition of a cosmological constant simultaneously addresses all of these problems.

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[e] Their result has drawn much criticism; in part because their red shifts are not spectroscopically determined (they are determined by multi-band photometry) and because their results are sensitive to the assumptions that they make about galactic evolution.[28]

[f] It is interesting to note that a neutrino-dominated Universe could also help with the Ω problem and the deficiency of large-scale structure. Because of their high velocities neutrinos tend to remain more smoothly distributed, and because structure forms from the "top down" there is more power on large scales.

A Relic Cosmological Constant[g]

The basic idea is simple; retain the flat Universe model, but add a cosmological constant. The model I propose here is: (i) Hubble constant of around $70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ ($h = 0.7$)—a nice compromise value; (ii) $\Omega_B \sim 0.03$—near the central value implied by nucleosynthesis; (iii) $\Omega_{\rm CDM} \sim 0.17$—sufficiently greater than the baryonic component so that the mass density is dominated by that of the cold dark matter; (iv) Ω_Λ—cosmological constant corresponding to an energy density $\rho_\Lambda \equiv \Omega_\Lambda \rho_{\rm CRIT} \simeq 3.2\times 10^{-47}\,\mathrm{GeV}^4 = (2.4\times 10^{-3}\,\mathrm{eV})^4$. I am not wed to these particular values and I simply use this set for definiteness. I will leave the question of motivation for the end.

For this model the total matter contribution is $\Omega_{\rm NR} = 0.2$, and today the vacuum energy density dominates the matter energy density by a factor of four. In general the ratio $\rho_{\rm NR}/\rho_\Lambda = 0.25(1+z)^3$. At red shifts greater than about $z_\Lambda \simeq 0.59$ the matter energy density dominates, and the model behaves just a flat, CDM model.[h] To determine when this model becomes matter dominated one simply sets $\Omega h^2 = \Omega_{\rm NR} h^2 \simeq 0.098$: $T_{\rm EQ} = 0.54\,\mathrm{eV}$; $t_{\rm EQ} \simeq 4.5\times 10^{12}\,\mathrm{sec}$; and $z_{\rm EQ} \simeq 2300$. Once the radiation energy density is negligible ($z \ll z_{\rm EQ}$), the scale factor evolves as

$$R(t) = \left(\frac{\Omega_{\rm NR}}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(3\sqrt{\Omega_\Lambda}H_0 t/2\right), \tag{3}$$

where the value of the scale factor today is taken to be one.

The Ω Problem

A cosmological constant behaves just like a uniform mass density (with equation of state $p = -\rho$). As such, it would not affect determinations of Ω based upon dynamics (galactic halos and cluster virial masses). These measurements of the masses of tightly bound systems are insensitive to the contribution of a uniform background energy density because the average density in these objects is much greater than the average density of the Universe. Likewise, determinations of Ω based upon the peculiar velocities induced by the clumpy matter distribution would only reveal the clumpy, matter component. Thus, all current dynamical determinations that indicate $\Omega \simeq 0.1 - 0.3$, would be consistent with a flat Universe ($\Omega = 1$) with $\Omega_{\rm NR} = 0.2$.

The Age Problems

As is well appreciated the addition of a cosmological constant *increases* the age of a flat Universe. The age of a Λ model is

$$t(z) = \frac{2H_0^{-1}}{3\sqrt{\Omega_\Lambda}} \sinh^{-1}\left[\sqrt{\Omega_\Lambda/\Omega_{\rm NR}}/(1+z)^{3/2}\right]; \tag{4a}$$

$$t_0 \equiv t(z=0) = \frac{2H_0^{-1}}{3\sqrt{\Omega_\Lambda}} \sinh^{-1}\left[\sqrt{\Omega_\Lambda/\Omega_{\rm NR}}\right] = \frac{2H_0^{-1}}{3\sqrt{\Omega_\Lambda}} \ln\left[\frac{1+\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_{\rm NR}}}\right]. \tag{4b}$$

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[g] Cosmologists dating back to Einstein have "resorted" to a cosmological constant to solve their problems—Einstein to obtain static solutions, Hoyle, and Bondi and Gold to resolve the age crisis when the Hubble time was only 2 Gyr, and more recently Turner, Steigman, and Krauss[30] and Peebles[31] to solve the Ω problem.

[h] The very recent transition to vacuum energy domination occurs because the ratio of matter energy density to vacuum energy density varies rapidly, as R^{-3}.

The present age of a Λ-model is always greater than $2H_0^{-1}/3$ and for $\Omega_\Lambda = 0.8$, $t_0 = 1.1H_0^{-1} \simeq 15.5\,\text{Gyr}$, an age which is comfortably consistent with the age as determined from the radioactive elements, from the oldest globular clusters, and from white dwarf cooling (e.g., see Ch. 1 of Ref. 6 and references therein). Moreover, a Λ model is older than its matter-dominated counterpart at any given epoch, so that objects at a given red shift have had more time to evolve. For $z \gg z_\Lambda$, $t(z) \to 2H_0^{-1}/3\sqrt{\Omega_{\rm NR}}(1+z)^{3/2}$, which is a factor of $\Omega_{\rm NR}^{-1/2}$ older than a flat, matter-dominated model; at these early epochs the "best-fit model" is a factor of 1.6 older than the conventional CDM model.

Large-scale Structure

The spectrum of density perturbations at matter-radiation equality, $(\delta M/M) \propto k^{3/2}|\delta_k|$, decreases monotonically with λ and its wavelength scale is determined by the value of Ωh^2. The spectrum "shifts" to larger length scales as Ωh^2 is decreased. Supposing that the spectrum is normalized on the scale $\lambda = 8h^{-1}\,\text{Mpc}$ (a common normalization is: $\delta M/M \simeq 1$ for $\lambda \simeq 8h^{-1}\,\text{Mpc}$), decreasing Ωh^2 increases the power on all scales greater than the normalization scale. Put another way, the ratio of the characteristic scale in the spectrum, $\lambda_{\rm EQ} = 13(\Omega h^2)^{-1}\,\text{Mpc}$, to the scale of nonlinearity in the Universe, $\lambda_{\rm NL} \simeq 8h^{-1}\,\text{Mpc}$, is $\lambda_{\rm EQ}/\lambda_{\rm NL} \simeq 1.6/\Omega h$; in the "best-fit model" this ratio is a factor of 3.5 greater than in a model with $\Omega = 1$ and $h = 0.5$ (conventional cold dark matter, or the "most well motivated model"), implying more power on large scales. Needless to say, this can only help with the problem of deficient large-scale structure.

To be specific, if the spectrum of perturbations is normalized by $(\delta M/M)_{\lambda=8h^{-1}\,{\rm Mpc}} = 1$,[i] I find that: $A = 4.4\times 10^6$ for $\Omega = 1$ and $h = 0.5$ (conventional CDM) and $A = 2.5\times 10^7$ for $\Omega_{\rm NR} = 0.2$ and $h = 0.7$ ("best-fit model"). On large scales ($\lambda \gg \lambda_{\rm EQ}$) $\delta M/M \propto \sqrt{A}/\lambda^2$; it follows that $\delta M/M$ for the "best-fit model" is a factor of 4.7 bigger on large scales.

Growth of Density Perturbations

Subhorizon-sized, linear density perturbations grow as the scale factor during the matter-dominated regime ($z \lesssim z_{\rm EQ} \simeq 23000\Omega h^2$), and remain roughly constant in amplitude when the Universe is radiation dominated, curvature dominated ($z \lesssim z_{\rm CURV} \simeq \Omega^{-1} - 2$; $z_{\rm CURV} \simeq 3$ for $\Omega = 0.2$), or vacuum-energy dominated ($z_\Lambda \simeq [\Omega_\Lambda^{-1} - 1]^{1/3} - 1 \simeq 0.59$). For a nonflat, $\Omega = 0.2$ model the reduction in the growth of perturbations relative to a flat model is very significant: about a factor of 20. By contrast, in flat-Λ models perturbations grow almost unhindered until the present (see Refs. 31 and 32). In the "best-fit model" the growth factor is only a factor of 0.8 less than $z_{\rm EQ}$, or about 1800. For comparison, in the conventional CDM model the growth factor $z_{\rm EQ} \simeq 5800$, only about a factor of three more growth.

Microwave Anisotropies

For conventional CDM the predicted CMBR temperature anisotropies are about a factor of three or so below the current level of observed isotropy (depending upon the angular scale and biasing factor b).[33] One might worry that because the "best-fit model" has more power on large scales and the growth factor for perturbations is smaller the predicted CMBR anisotropies might violate current bounds. That is not the case. The

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[i] I have used the "top hat" window function [$W(r) = 1$ for $r \le r_0$ and $= 0$ for $r \ge r_0$] to define M, so that $(\delta M/M)^2 = (9/2\pi^2)\int_0^\infty k^2|\delta_k|^2[\sin(kr_0)/k^3r_0^3 - \cos(kr_0)/k^2r_0^2]^2\,dk$, where $r_0 = 8h^{-1}\,\text{Mpc}$.

reason involves the angular size on the sky θ of a given scale λ at epoch z:

$$\theta(\lambda, z) = \lambda / r(z); \tag{5a}$$

$$r(z) = \int_{t(z)}^{t_0} \frac{dt}{R(t)} = \frac{2H_0^{-1}}{3\Omega_\Lambda^{1/6}\Omega_{\rm NR}^{1/3}} \int_{\sinh^{-1}\left[\sqrt{\Omega_\Lambda/(1+z)^3\Omega_{\rm NR}}\right]}^{\sinh^{-1}\left[\sqrt{\Omega_\Lambda/\Omega_{\rm NR}}\right]} \frac{du}{\sinh^{2/3} u}, \tag{5b}$$

where $r(z)$ is the coordinate distance to an object at red shift z. In a flat, matter-dominated model $r(z) = 2H_0^{-1}\left[1 - 1/\sqrt{1+z}\right] \to 2H_0^{-1}$ for $z \gg 1$, and $\theta(\lambda, z \gg 1) \simeq 34.4''\,(\lambda/h^{-1}\,{\rm Mpc})$. For the "best-fit model" $r(z \gg 1) \simeq 3.9H_0^{-1}$ and $\theta(\lambda, z \gg 1) \simeq 17.7''\,(\lambda/h^{-1}\,{\rm Mpc})$.

In a flat Λ-model the horizon is further away and a given length scale has a smaller angular size. Since the temperature fluctuations on a given angular scale are related to the density perturbations on the length scale that subtends that angle at decoupling, in the "best-fit model" temperature fluctuations on a given angular scale are related to density perturbations on a *larger* scale λ. While the "best-fit model" has more power on a *fixed* (large) length scale, a fixed angle θ corresponds to a *larger* length scale, where the amplitude of perturbations is smaller because $\delta M/M$ decreases with λ.

Consider the temperature fluctuations on large-angular scales ($\theta \gg 1^\circ$); they arise due to the Sachs–Wolfe effect and $(\delta T/T)_\theta \simeq (\delta\rho/\rho)_{\rm HOR}/2$ on the scale $\lambda(\theta)$ when that scale crossed inside the horizon. For the Harrison–Zel'dovich spectrum the horizon-crossing amplitude is constant, so that $\delta T/T$ is independent of angular scale (for $\theta \gg 1^\circ$). The CMBR quadrupole anisotropy is related to the amplitude of the perturbation that is just now crossing inside the horizon: $\lambda_{\rm HOR} \sim 2H_0^{-1} \sim 12000\,{\rm Mpc}$ (conventional CDM) and $\lambda_{\rm HOR} \sim 3.9H_0^{-1} \sim 16700\,{\rm Mpc}$ ("best-fit model"). Evaluating the normalized spectra on these scales it follows that the large-angle temperature fluctuations in the "best-fit model" are only a factor of 1.2 larger than for conventional CDM, in spite of the fact that the "best-fit model" has significantly more power on large scales.

The amplitude of the temperature fluctuations on small angular scales ($\theta \ll 1^\circ$) is proportional to the amplitude of the density perturbations at the time of decoupling ($z_{\rm DEC} \sim 1000$), on the scale $\lambda(\theta)$. In the "best-fit model" perturbations have grow by a factor of about $0.8z_{\rm DEC}$ since decoupling, while those in the "most well motivated model" have grown by a factor of $z_{\rm EQ}$. On the other hand the length scale corresponding to the angular scale θ is larger for the "best-fit model." The net result is that the temperature fluctuations on an angular scale of 1° are also only about a factor of 1.2 larger.

Galaxy Counts—and Other Kinematic Tests

Because the coordinate distance to an object of given red shift is greater in a flat Λ model, there is greater volume per red shift interval per solid angle, which increases the number of galaxies in $dzd\omega$. (The galaxy count test is discussed in more detail in Refs. 31 and 32.) To see roughly how this goes, consider the deceleration parameter of Sandage[34]

$$q_0 \equiv -\frac{\ddot{R}}{R_0 H_0^2} = \Omega(1 + 3p/\rho)/2 = (1 - 3\Omega_\Lambda)/2 \simeq -1.2, \tag{6}$$

where Ω is the total energy density ρ divided by the critical energy density and p is the total pressure. From Eq. (2) one can see that the galaxy-number count is increased:

$dN_{\rm GAL}/dz = z^2 n_{\rm GAL}[1-3z+\cdots]$ compared to $z^2 n_{\rm GAL}[1+0.4z+\cdots]$. The addition of a cosmological constant can significantly increase the galaxy count.

There are other "kinematic tests" that could prove useful; they include the red shift–luminosity test (Hubble diagram), angle–red shift test, and look-back time–red shift test (for further discussion see Ref. 32). For small red shift different models can be parameterized in terms of Sandage's q_0. In this regard, the "best-fit model" is characterized by $q_0 = -1.2$ (for comparison, a conventional low-Ω, negatively curved model has $q_0 = \Omega/2$). To date, none of these tests have proved definitive, though some put great stock in the potential of the infrared ($2.2\mu m$ or K band) version of the Hubble diagram.[35]

Large-scale Motions

The *rms* peculiar velocity of a volume defined by the "window function" $W(r)$, averaged over all such volumes in the Universe, is

$$\langle v^2\rangle = \frac{1}{2\pi^2}\int_0^\infty k^2|\mathbf{v}_k|^2|W(k)|^2dk, \tag{7}$$

where in the linear perturbation regime the Fourier component of the peculiar velocity field $\mathbf{v}_k(t) = -i\mathbf{k}R(t)\dot{\delta}_k(t)/|\mathbf{k}|^2$. For a flat, matter-dominated Universe, $|\mathbf{v}_k| = H_0|\delta_k|/k$; while for a flat model with a cosmological constant $|\mathbf{v}_k| = \Omega_{\rm NR}^{0.57}H_0|\delta_k|/k$ (see Refs. 31 and 36). Using a gaussian window function $[W_{r_0}(r) = \exp(-r^2/2r_0^2)]$ and normalizing the spectrum as above, the *rms* peculiar velocity expected on the scale $r_0 = 50h^{-1}$ Mpc is[36]

$$v_{50} \simeq 83h^{-0.9}\,{\rm km\,s^{-1}} \simeq 160\,{\rm km\,s^{-1}} \qquad (\Omega=1,\ h=0.5);$$

$$v_{50} \simeq 83\Omega_{\rm NR}^{-0.33}h^{-0.9}\,{\rm km\,s^{-1}} \simeq 200\,{\rm km\,s^{-1}} \qquad (\Omega_{\rm NR},\ h=0.7).$$

While the *rms* peculiar velocity on the scale of 50 Mpc is still far short of $700\,{\rm km\,s^{-1}}$, it is larger, owing to fact that there is more power on large scales.[j]

Concluding Remarks

Introducing a cosmological constant helps to resolve all the shortcomings of conventional cold dark matter ($\Omega_{\rm CDM}=0.9$, $\Omega_B=0.1$, and $h=0.5$). In particular it eases the age problems, resolves the Ω problem, increases the number of galaxies expected in a given red shift interval, and leads to more power on large scales. At the same time, the predicted CMBR temperature anisotropies are only a factor of 1.2 larger than for conventional CDM.

The model can be tested in a number of ways, although the usual dynamical means of inferring Ω are not sensitive to Ω_Λ. Given its large deceleration parameter, $q_0 = 0.5(1-3\Omega_\Lambda) \simeq -1.2$, several of the classic kinematic tests—angle–red shift, galaxy count-red shift, lookback-time–red shift, and red shift-luminosity—may prove useful. It may well be that new tests which key on the the "hallmarks" of the Λ model—larger volume, older Universe, and more distance to the horizon—can be found.[k]

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[j] The comparison of theoretical expectations to the peculiar velocity data of the Seven Samurai[22] is far more complicated than just computing $\langle v^2\rangle$ for a gaussian window function.[37] The point I wish to make here is that adding a cosmological constant increases peculiar velocities.

[k] Along these lines, E.L. Turner has recently used the frequency of multiple image lensing of QSOs to argue against a large value of Λ, perhaps even precluding $\Omega_\Lambda = 0.8$.[38]

The "best-fit" model has implications for dark matter. The abundance of a relic particle species $\Omega_X h^2$ is related its fundamental properties (mass, couplings, etc.). In the "best-fit model" the value of $\Omega_X h^2$ is a factor of almost three smaller than in the conventional CDM model. For a thermal relic such as a neutralino, the relic abundance $\Omega_X h^2$ is proportional to the inverse of the annihilation cross section, implying that the annihilation cross section is about a factor of three larger. This fact has implications for dark matter searches: (i) The rates for indirect detection schemes that rely upon the annihilation products, e.g., high-energy neutrinos from annihilations in the sun, or high-energy positrons from annihilations in the halo, are increased by a factor of about three; (ii) The rates for direct detection, e.g., in bolometric detectors, which depend upon the cross section for elastic scattering with matter, are increased by the same factor because the elastic scattering cross section is related to the annihilation cross section by crossing symmetry.

One might wonder if in a Λ model one could revive hot dark matter, or do away with exotic dark matter all together. The revival of a neutrino-dominated Universe does not seem likely. The required neutrino mass is about 8 eV, implying a neutrino-damping scale of about 150 Mpc, which would further exacerbate the problems of a neutrino-dominated Universe. In a baryon-dominated Λ model perturbations are damped (by photon diffusion) on scales smaller than $\lambda_{\rm SILK} \simeq 1\,(\Omega_B h^2)^{-3/4}\,{\rm Mpc} \simeq 20\,{\rm Mpc}$, which results in the original Zel'dovich pancake scenario with supercluster-sized baryon pancakes. Because perturbations continue to grow until almost the present this scenario is better than a nonflat, low-Ω baryon-dominated model; however, this scenario is likely to have problems with CMBR anisotropies (re-ionizing the Universe might relieve this problem on angular scales smaller than about 7°).[m]

Finally, let me address the motivation for the "best-fit model." As its name suggests, it is a model motivated by observation and not aesthetics: Conventional cold dark matter is clearly better motivated. While the conventional CDM model has one question to answer—why the ratio of the baryon density to that of cold dark matter is of order unity[41]—in the "best-fit model" one must also address "why now?"—why is the cosmological constant just now becoming dynamical important? (This problem is similar to the flatness problem, where the question is, why is the curvature radius just now becoming comparable to the Hubble radius?) Moreover, there is the issue of the cosmological constant itself: At present there is every reason to expect a cosmological constant $\rho_\Lambda = \Lambda/8\pi G \sim m_{\rm Pl}{}^4$ which is some 122 orders of magnitude larger than observations permit[l] (Supersymmetry *might* be able to help in this regard, reducing the estimate to $\rho_\Lambda \sim G_F^{-2}$, which is only 56 orders of magnitude too large!) The strongest statement that one can make in defense of a relic cosmological constant of the desired size is that no good argument exists for *excluding* it!

The additional of a cosmological constant to the cold dark matter paradigm resolves a number of its apparent shortcomings—and is the sole motivation for introducing it. Cold dark matter *sans* cosmological constant is still the most well motivated model. One should

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[m] Peebles[39] has recently discussed this possibility. While re-ionization might be able to erase the primary CMBR fluctuations, the secondary fluctuations may be problematic.[40]

[l] There is one interesting explanation of why the cosmological constant is "probably" zero: Coleman and others[42] have argued that due to wormhole effects the wavefunction of the Universe is very sharply peaked at zero cosmological constant.

keep in mind that the observations or their interpretations could change, and cold dark matter could once again become both the most well motivated model and the best-fit model.

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