NIKHEF

NATIONAAL INSTITUUT VOOR KERNFYSICA EN HOGE-ENERGIEFYSICA

NIKHEF-H/91-03
December 1990

Ashtekar Formalism with Real Variables

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Abstract

A new approach to canonical gravity is presented, which is based on the Ashtekar formalism. But in contrast to Ashtekar's variables, this formulation does not need complex quantities nor does it lead to second class constraints. This is achieved using $SO(3,1)$ as a gauge group instead of complexified $SO(3)$. Because of the larger gauge group additional first class constraints are needed which turn out to be cubic and quartic in the momenta.

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Abstract

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1 Introduction

The oldest approach to quantum gravity is the canonical quantization of Einstein's theory of general relativity [1, 2]. However, taking Einstein's theory in the metric formulation, the constraints on the phase space are non-polynomial, which leads to as yet unsolved difficulties in the quantization program.

In 1986 an important break through was made by Ashtekar [3], who showed, that the phase space of general relativity can be embedded in the phase space of a Yang-Mills theory. This work was based on an observation made by Sen [4], that the constraints of gravity may be encoded in complexified $SU(2)$ spin connections on a 3-dimensional manifold. The main point in Ashtekar's formulation of general relativity is, that all constraints are polynomial in the new variables, which he introduced. In this context, the metric tensor is a derived quantity and, due to the polynomial nature of the constraints, it may be degenerate. In 2+1 dimensional gravity it turned out, that this possible degeneracy is an essential aspect of the vacuum structure of the corresponding quantum theory [5]. A detailed discussion of the relation between Witten's work in 2+1 dimensions [5] and gravity in 3+1 dimension in terms of Ashtekar's variables was given by Bengtsson [6].

Using Ashtekar's formulation of gravity, a large class of exact solutions to all quantum contraints has been obtained for the first time [7, 8, 9]. Although this is substantial progress, there are still several open problems, which have to be solved before one has a theory of quantum gravity. One key problem is that a hermitian inner product on the physical state space has not been defined yet. This difficulty is related to the fact, that Ashtekar's variables are complex and describe a complexified theory of gravity. In the classical theory one has to impose reality constraints on the variables which restrict the theory to its real part. On the quantum level those reality constraints have to be absorbed by an appropriate hermitian scalar product on the physical state space. This problem would be simplified, if one could find a formulation of Ashtekar's formalism with real variables. Some attemps were made to derive a real formulation [10, 11], but they led to constraint systems with second class constraints.

In the following we present a purely real action, which leads to Ashtekar's constraints. The symmetry group of this action is $SO(3,1)$ and therefore some additional constraints are needed. They turn out to be fairly simple, being cubic and quartic in the momenta. Moreover, the whole constraint system is still first class.

2 The Einstein action

The Einstein-Hilbert action for pure gravity in 3+1 space-time dimensions is

$$S = \int d^4x \; e\, e^{\mu}_{a} e^{\nu}_{b} \eta^{bc} R_{\mu\nu}{}^{a}_{c} \,, \tag{2.1}$$

where e^{μ}_{a} is the vierbein, e is the determinant of the inverse of e^{μ}_{a}, η^{ab} =diag$(-,+,+,+)$ and $R_{\mu\nu}{}^{a}{}_{b} = \partial_{[\mu}\omega_{\nu]}{}^{a}_{b} + \omega_{[\mu}{}^{c}_{b}\omega_{\nu]}{}^{a}_{c}$ is the field strength for the $SO(3,1)$-connection $\omega_{\mu}{}^{a}_{b}$. We use the first order formalism, in which the vierbein and the connection are treated as independent variables.

The action (2.1) is invariant under local coordinate and local Lorentz-transformations. Therefore from the Hamiltonian point of view it is a constrained system and it requires great care to obtain a correct phase-space description of the theory. Moreover, the definition of the momentum variable conjugate to the space-like connection $\omega_{i}{}^{a}_{b}$:

$$\pi^{i}{}^{a}_{b} = e e^{[0}_{b} e^{i]}_{c} \eta^{ca} \tag{2.2}$$

shows, that the constraints are non-polynomial in the basic variables e^{μ}_{a}. This complicates the the Hamiltonian analysis further, to the point where solution of the constraints and quantization of the system becomes an intractable problem.

In the case of 2+1 dimensional gravity a way out of these difficulties was shown by Witten [5]. Here the action can be rewritten as a polynomial in the fields:

$$S = \int d^3x \; \epsilon^{\rho\mu\nu} e^{a}_{\rho} R_{\mu\nu a} \,. \tag{2.3}$$

The canonical momenta are now given by:

$$\pi^{i}_{a} = \epsilon^{i}_{a} \,. \tag{2.4}$$

Witten has shown that this action leads to a finite dimensional and therefore exactly soluble quantum system [5]. The keypoint is here that the dreibein e^{μ}_{a} has not only the interpretation of a linear map from the tangent-space to the fibre, which is isomorphic to the tangent-space, but it can also be viewed as a linear map from the space of 2-forms on tangent-space to the algebra of the gauge-group $SO(2,1)$. This naturally leads to the interpretation of the e^{μ}_{a} as a connection 1-form of the translation-group.

In this paper we show how Witten's construction can be generalised to 3+1 dimensional space-time. In the next section we present our set of variables and the action. Then we turn to the analysis of the constraints and the Hamiltonian formulation.

3 3+1 dimensional gravity as a polynomial gauge theory

The key to the generalization of the action (2.3) to the case of 3+1 dimensional gravity is to introduce a new fundamental object $T^{\mu\nu a}{}_{b}$, which maps 2-forms on tangent-space to the algebra of the local Lorentz group $SO(3,1)$. In the action (2.1) this is taken to be a composite quantity

$$T^{\mu\nu b}{}_{a} = \frac{1}{2} e e_{a}^{[\mu} e_{c}^{\nu]} \eta^{cb} \tag{3.1}$$

allowing us to write the Einstein-Hilbert action in the form

$$S = \int d^4x \, T^{\mu\nu}_{\alpha} R^{\alpha}_{\mu\nu} \,, \tag{3.2}$$

where α is a $SO(3,1)$ index, representing an antisymmetrized index pair $[ab]$ and taking values (1,...,6). We now give $T^{\mu\nu}_{\alpha}$ the status of an independent field and develop a first order formulation of the action (3.2), such that eq.(3.1) is a solution of the field equations for $T^{\mu\nu}_{\alpha}$.

In order to achieve this, we first note that in general $T^{\mu\nu}_{\alpha}$ has 36 components which is more than the right-hand side of eq.(3.1) has, even when it is formulated with a general non-diagonal fibre metric. The additional degrees of freedom have to be removed by an appropriate constraint equation:

$$T^{\mu\nu}_{\alpha} T^{\rho\sigma\,\alpha} = \frac{1}{2} \left(q^{\mu\sigma} q^{\nu\rho} - q^{\mu\rho} q^{\nu\sigma} \right) \,, \tag{3.3}$$

where $q^{\mu\nu}$ is an arbitrary symmetric tensor-density satisfying[1]

$$\det q^{\mu\nu} \leq 0 \,. \tag{3.4}$$

That eq.(3.3) is the the correct constraint can be seen by first noting that a solution is given by:

$$q^{\mu\nu} = \tilde{e}_{a}^{\mu} \tilde{e}_{b}^{\nu} \eta^{ab} \tag{3.5}$$

$$T^{\mu\nu a}{}_{b} = \frac{1}{2} \tilde{e}_{b}^{[\mu} \tilde{e}_{c}^{\nu]} \eta^{ca} \tag{3.6}$$

where $\tilde{e}_{a}^{\mu}$ is the densitized vierbein to the fibre isomorphic to $\mathbb{R}^4$. Eq.(3.3) and (3.5) are invariant under $GL(4)$- transformations on the fibre, but eq.(3.6) is only invariant under the scale-transformations of $GL(4)$ (the precise definition of the scalar product on the $SO(3,1)$-algebra and the corresponding symmetries are given in the appendix).

[1] Actually one also has to require that $q^{00} < 0$, because this parameter plays the role of the lapse-function in the Hamiltonian formalism. For a detailed discussion, see the next section.

Because of the equivalence of the $SL(4)$- and $SO(3,3)$- algebras the complete set of solutions can be obtained from eq.(3.6) together with $SL(4)$-transformations so that they are of the form:

$$q^{\mu\nu} = \bar{e}^{\mu}_{a}\bar{e}^{\nu}_{b}\chi^{ab} \quad \text{and} \quad T^{\mu\nu a}{}_{b} = \frac{1}{2}\bar{e}^{[\mu}_{b}\bar{e}^{\nu]}_{c}\chi^{ca} , \tag{3.7}$$

where χ^{ab} is a general fibre metric. Although this fibre-structure does not appear explicitly, it will be useful in order to prove some algebraic properties of the $T^{\mu\nu}_{\alpha}$ on the constraint surface. This construction is only possible in 4 dimension, because it depends on the equivalence of the $SO(3,3)$ and the $SL(4)$ algebras.

Note that eq.(3.3) is the place where the metric in a densitized form comes in but it only plays the role of an auxiliary field.

The first order action generalizing (3.2) and implementing the constraints is now

$$S = \int d^4x \, T^{\mu\nu}_{\alpha} R^{\alpha}_{\mu\nu} + M_{\mu\nu\rho\sigma}\left(T^{\mu\nu}_{\alpha}T^{\rho\sigma\,\alpha} - \frac{1}{2}\left(q^{\mu\sigma}q^{\nu\rho} - q^{\mu\rho}q^{\nu\sigma}\right)\right) , \tag{3.8}$$

where $M_{\mu\nu\rho\sigma}$ is a Lagrange multiplier field introduced to impose the constraint.

We now analyse the field equations obtained from this action and show that for $\det q^{\mu\nu} < 0$ it is classically equivalent to the Einstein-Hilbert action. Thereby the geometrical structure of the theory will become more clear.

$$\begin{aligned} R^{\alpha}_{\mu\nu} + 2M_{\mu\nu\rho\sigma}T^{\rho\sigma\,\alpha} &= 0 && (3.9)\\ q^{\rho\sigma}M_{\mu\rho\sigma\nu} &= 0 && (3.10)\\ D_{\mu}T^{\mu\nu}_{\alpha} &= 0 && (3.11)\\ T^{\mu\nu}_{\alpha}T^{\rho\sigma\,\alpha} - \frac{1}{2}\left(q^{\mu\sigma}q^{\nu\rho} - q^{\mu\rho}q^{\nu\sigma}\right) &= 0 . && (3.12) \end{aligned}$$

D_{μ} denotes the $SO(3,1)$-covariant derivative. If $\det q^{\mu\nu} \neq 0$, then $T^{\mu\nu}_{\alpha}$ is also non-degenerate and $M_{\mu\nu\rho\sigma}$ is by eq.(3.9)the Riemann-tensor where all indices are Lorentz-covariant. Eq.(3.10) then states that all empty space solutions are Ricci-flat.

In order to see the geometrical meaning of eq.(3.11) on the constraint surface, the fibre-structure given in (3.7) can be used. However, it is more convenient to use the form $T^{\mu\nu a}{}_{b} = \frac{1}{2}e\sqrt{-\chi}e^{[\mu}_{b}e^{\nu]}_{c}\chi^{ca}$, where the determinants $\chi = \det(\chi_{ab})$ and $e = \det(e^{a}_{\mu})$ appear explicitly. A multiplication with the inverse of $T^{\mu\nu a}{}_{b}$, i.e., with $T_{\rho\sigma}{}^{a}{}_{b} = \frac{1}{2}(e\sqrt{-\chi})^{-1}e^{a}_{[\rho}e^{c}_{\sigma]}\chi_{cb}$. brings eq.(3.11) into a $GL(4)$-invariant form (see the appendix):

$$T^{\mu\nu}_{\alpha}D_{\mu}T^{\alpha}_{\rho\sigma} = \frac{e\sqrt{-\chi}}{4}e^{[\mu}_{a}e^{\nu]}_{b}\chi^{bc}D_{\mu}\left(\left(e\sqrt{-\chi}\right)^{-1}e^{a}_{[\rho}e^{d}_{\sigma]}\chi_{dc}\right) . \tag{3.13}$$

If a gauge is chosen where the fibre metric is diagonal, such that $\chi_{ab} = \eta_{ab}$, then eq.(3.11) or eq.(3.13) is just the condition for a vanishing torsion:

$$D_{[\mu}e^a_{\nu]} = 0 \ . \tag{3.14}$$

Therefore one can conclude, that the action (3.8) describes general relativity (in the case $\det q^{\mu\nu} \neq 0$) formulated on a general frame-bundle with a $SO(3,1)$-connection. Although the frame-bundle structure is not used as starting point in the formulation of the action, it comes in with the constraints on the new variables $T^{\mu\nu}_\alpha$. In that sense the constraints restrict the abstract $SO(3,1)$-connection and the vector-space isomorphism $T^{\mu\nu}_\alpha$ between the 2-form on tangent-space and the $SO(3,1)$-algebra to those parts of the configuration space, where this frame-bundle interpretation is possible. The usual tetrad-formulation can be obtained by using the $GL(4)$ invariance of the action to restrict the frames to orthogonal frames.

We end this section with a remark on the $GL(4)$ invariance of the action (3.8). Suppose the constraints on $T^{\mu\nu}_\alpha$ are solved and the action (3.8) is of the same form as (2.1), but with a general non-diagonal fibre metric. This action may be viewed as an $GL(4)$ gauge invariant action, i.e., the connection is not restricted to the Lorentz-algebra a priori. But then the metric postulate in the vierbein-gauge ($\chi_{ab} = \eta_{ab}$) reduces the connection to a SO(3,1)-field. A detailed discussion of this subject in a somewhat different context is given in [12, 13].

4 The hamiltonian and constraints

In the Hamiltonian formalism of mechanics and field theory, time plays a priviliged role. It is used to determine the momenta conjugate to the coordinates, and the Hamiltonian itself is by definition the generator of time-translation in the sense of Poisson-brackets. However, the time-variable thus singled out is the parameter determining the evolution in phase space. In a reparametrization invariant theory, where the coordinates determine the geometry of some n dimensional manifold Σ, the evolution of the system leads to a n+1 dimensional manifold with a $\mathbb{R} \times \Sigma$-topology, but the evolution parameter is not necessarily the same as the time variable in the geometrical sense.

In gravity with the metric or the tetrad as basic variables one can use those and single out a unique timelike direction, which is orthogonal to the spacelike 3-manifold. This leads in the Hamiltonian framework to a geometry with a $\mathbb{R} \times \Sigma$-topology, where $\mathbb{R}$ is the timelike direction and Σ is an arbitrary topology of the space-like part[2]. However,

[2]If Σ is not closed, one has to introduce appropriate boundary conditions on the fields. We assume in the following that Σ is a closed manifold.

in our approach the metric is not an independent dynamical variable but a derived quantity. As a consequence, one does not know a priori which is the time-direction. Then the 'time'-evolution of the solution to the constraints still has the topology $\mathbb{R}\times\Sigma$, but the $\mathbb{R}$ -part need not represent geometrical time. In other words, if one choose e.g. the $\frac{\partial}{\partial x^0}$-derivative as the time-derivative in the *Hamiltonian* sense, one cannot decide by the constraints or 'time'-evolution if it is a timelike vector in the *geometrical* sense. This ambiguity can be avoided for solutions with a non-degenerate metric (i.e. solutions with a geometric interpretation) by a suitable choice of initial conditions.

We now turn to the Hamiltonian corresponding to the action (3.8). The only non-vanishing momenta are those conjugate to ω_i^a:

$$\pi^i_\alpha = 2T^{0i}_\alpha \, . \tag{4.1}$$

The Hamiltonian of the system is therefore:

$$\begin{aligned} H = -\int d^4x \; T^{ij}_\alpha R^\alpha_{ij} + \omega^\alpha_0 D_i\pi^i_\alpha \;&+\; M_{0i0j}\left(\pi^i_\alpha\pi^{j\alpha} - 2\left(q^{0i}q^{0j} - q^{00}q^{ij}\right)\right) \\ &+\; 2M_{0ijk}\left(\pi^i_\alpha T^{jk\alpha} - q^{0k}q^{ij} + q^{0j}q^{ik}\right) \\ &+\; M_{ijkl}\left(T^{ij}_\alpha T^{kl\alpha} - \frac{1}{2}\left(q^{il}q^{jk} - q^{ik}q^{jl}\right)\right) \, . \end{aligned} \tag{4.2}$$

The canonical momenta of ω^α_0, $q^{\mu\nu}$, T^{ij}_α and $M_{\mu\nu\rho\sigma}$ are zero by primary constraints, which lead to the following system of secondary constraints:

$$\frac{\delta H}{\delta\omega^\alpha_0} = D_i\pi^i_\alpha = 0 \tag{4.3}$$

$$\frac{\delta H}{\delta q^{\mu\nu}} = 2q^{\rho\sigma}M_{\mu\rho\sigma\nu} = 0 \tag{4.4}$$

$$\frac{\delta H}{\delta T^{ij}_\alpha} = R^\alpha_{ij} + 2M_{0kij}\pi^{k\alpha} + 2M_{ijkl}T^{kl\alpha} = 0 \tag{4.5}$$

$$\frac{\delta H}{\delta M_{0i0j}} = \pi^i_\alpha\pi^{j\alpha} - 2\left(q^{0i}q^{0j} - q^{00}q^{ij}\right) = 0 \tag{4.6}$$

$$\frac{\delta H}{\delta M_{0ijk}} = \pi^i_\alpha T^{jk\alpha} - \left(q^{0k}q^{ij} - q^{0j}q^{ik}\right) = 0 \tag{4.7}$$

$$\frac{\delta H}{\delta M_{ijkl}} = T^{ij}_\alpha T^{kl\alpha} - \frac{1}{2}\left(q^{il}q^{jk} - q^{ik}q^{jl}\right) = 0 \, . \tag{4.8}$$

The last three constraints (4.6)-(4.8) are algebraic constraints on the momenta. In order to analyse them further it is convenient to adopt the ADM-notation for the densitized metric tensor $q^{\mu\nu}$:

$$q^{00} = -N^{-1} \quad \text{with} \quad N > 0 \tag{4.9}$$

$$q^{0i} = N^{-1}N^i \tag{4.10}$$

$$q^{ij} = h^{ij} - N^{-1}N^iN^j \quad \text{with} \quad \det h^{ij} \geq 0 \, . \tag{4.11}$$

Note that $\det h^{ij} = 0$ is also allowed and that h^{ij} may have signature $(+,-,-)$, but together with $N > 0$ $g^{\mu\nu}$ still has the correct signature of a Minkowski-metric. The condition $N \neq 0$ is necessary, because N and N^{-1} will be used in the following. The physical meaning of this condition will become clear in the next section. Eqs.(4.6)-(4.8) now become:

$$\pi^i_\alpha \pi^{j\alpha} - 2N^{-1}h^{ij} = 0 \tag{4.12}$$

$$\pi^i_\alpha T^{jk\alpha} - N^{-1}\left(h^{ij}N^k - h^{ik}N^j\right) = 0 \tag{4.13}$$

$$T^{ij}_\alpha T^{kl\alpha} - \frac{1}{2}\left(h^{il}h^{jk} - h^{ik}h^{jl} - N^{-1}\left(N^jN^kh^{il} + N^iN^lh^{jk} - N^jN^lh^{ik} - N^iN^kh^{jl}\right)\right) = 0\,. \tag{4.14}$$

This system of constraints still contains the auxiliary fields T^{ij}_α, h^{ij}, N^i and N, which makes it hard to see what are really the constraints on the dynamical variables π^i_α. However, as was shown in the previous section, on the constraint-surface there is a fibre-structure with a vierbein and a fibre-metric available and with eq.(3.7) one can set:

$$\pi^{ia}_b = \bar{e}^{[0}_b\bar{e}^{i]}_c\chi^{ca} \quad \text{and} \quad T^{ija}_b = \frac{1}{2}\bar{e}^{[i}_b\bar{e}^{j]}_c\chi^{ca}\,. \tag{4.15}$$

It is straightforward to show the following relation:

$$T^{ij}_\alpha = \frac{1}{2}Nf_{\alpha\beta\gamma}\pi^{i\beta}\pi^{j\gamma} - \frac{1}{2}N^{[i}\pi^{j]}_\alpha\,, \tag{4.16}$$

where $f_{\alpha\beta\gamma}$ are $SO(3,1)$ structure constants. This equation can now be used to eliminate the T^{ij}_α from eqs.(4.13) and (4.14) and with the help of eq.(4.12) one derives:

$$f_{\alpha\beta\gamma}\pi^{i\alpha}\pi^{j\beta}\pi^{k\gamma} = 0 \tag{4.17}$$

$$f_{\alpha\beta\gamma}f^\alpha_{\zeta\eta}\pi^{i\beta}\pi^{j\gamma}\pi^{k\zeta}\pi^{l\eta} + \frac{1}{2}\left(\pi^i_\alpha\pi^{k\alpha}\pi^j_\beta\pi^{l\beta} - \pi^i_\alpha\pi^{l\alpha}\pi^j_\beta\pi^{k\beta}\right) = 0\,. \tag{4.18}$$

These are the new constraints, which restrict the π^i_α to a subspace of the $SO(3,1)$-algebra. Note that this subspace is *not* a subalgebra of $SO(3,1)$ as a consequence of eq.(4.16). Only the quantities T^{ij}_α given by (4.16) form a subalgebra, which is the $SO(3)$ or the $SO(2,1)$ in case h^{ij} is non-degenerate, or a subalgebra of those, if h^{ij} is degenerate. Actually, one still has to check if

$$\det \pi^i_\alpha\pi^{j\alpha} = \det(2N^{-1}h^{ij}) \geq 0$$

for the solutions of eq.(4.17) and (4.18). The proof, that this is the case, is given in the appendix. One therefore concludes that the constraints (4.17) and (4.18) together with (4.16) are completely equivalent to (4.6)-(4.8).

It is at this point that the ambiguity in the 'time'-direction discussed in the beginning of this section appears. In the case that det $\pi^i_\alpha \pi^{j\alpha} > 0$, the constraints are not sufficient to restrict $\pi^i_\alpha \pi^{j\alpha}$ to a positive definite matrix, i.e., to a matrix with signature $(+,+,+)$. The $\pi^i_\alpha \pi^{j\alpha}$ also may have signature $(+,-,-)$. These two possibilities correspond to the two possible choices for the 3 dimensional algebra of the T^{ij}_α.

So far we only discussed the last three of the constraints given by (4.3)-(4.8). The first one is the Gauß-law and needs no further discussion. What is still missing are the constraints, which generate the spatial diffeomorphism group and time translation. They are given by (4.4) and (4.5), but those equations contain the non-dynamical fields and therefore they are not easy to handle. However, there is a much nicer way to present them: they can be written in a form equivalent to the Ashtekar constraints

$$H^i = \pi^j_\alpha R^\alpha_{ij} \tag{4.19}$$

$$H = f_{\alpha\beta\gamma} \pi^{i\alpha} \pi^{j\beta} R^\gamma_{ij} \,. \tag{4.20}$$

Together with the Gauß-law

$$G_\alpha = D_i \pi^i_\alpha \tag{4.21}$$

and the algebraic constraints on the momenta

$$C = \epsilon_{ijk} f_{\alpha\beta\gamma} \pi^{i\alpha} \pi^{j\beta} \pi^{k\gamma} \tag{4.22}$$

$$C_{mn} = \epsilon_{mij} \epsilon_{nkl} \left(f_{\alpha\beta\gamma} f^\alpha_{\zeta\eta} \pi^{i\beta} \pi^{j\gamma} \pi^{k\zeta} \pi^{l\eta} + \pi^i_\alpha \pi^{k\alpha} \pi^j_\beta \pi^{l\beta} \right) \tag{4.23}$$

the Ashtekar constraints form the complete set of constraints for the system given by the action (3.8), which is equivalent to the constraint system (4.3)-(4.8), but with all auxiliary fields removed.

5 The constraint algebra

Before proving that the system of constraints given above leads to the action we started with, we first turn to the algebra of the constraints. There are two Poisson brackets of the constraints, which need some discussion. The first one is

$$\{\xi_1 H, \xi_2 H\} = -4 f_{\alpha\beta\gamma} f^\alpha_{\zeta\eta} \xi_{[1} \partial_k \xi_{2]} R^\gamma_{ij} \pi^{i\eta} \pi^{j\beta} \pi^{k\zeta} \tag{5.1}$$

In the case of Ashtekar's variables the gauge group is the $SO(3)$ and the structure constants have the following nice property, which is needed to close the constraint algebra:

$$f_{\alpha\beta\gamma} f^\alpha_{\zeta\eta} = \delta_{\beta\zeta} \delta_{\gamma\eta} - \delta_{\beta\eta} \delta_{\gamma\zeta} \,. \tag{5.2}$$

This identity does not hold for the $SO(3,1)$ algebra in general. A similiar problem arises with the other Poisson-bracket:

$$\{\xi^{ij}C_{ij},\xi H\} = 8\xi^{ij}\epsilon_{ikl}\epsilon_{jmn}f^{\alpha\beta}_{\gamma}f_{\alpha\zeta\eta}\pi^{l\gamma}\pi^{m\zeta}\pi^{n\eta}A^k_\beta - 8\xi^{ij}\epsilon_{ikl}\epsilon_{jmn}\pi^l_\alpha\pi^{n\alpha}\pi^{m\beta}A^k_\beta \quad (5.3)$$

with

$$A^k_\alpha = f_{\alpha\beta\gamma}D_i\left(\xi\pi^{k\beta}\pi^{i\gamma}\right) .$$

Here once again the fibre structure on the constraint surface can be used to derive the desired algebraic properties. With (4.15) one can show for a general antisymmetric matrix $A \in SO(3,1)$ that

$$Tr\left([A,\pi^i][\pi^j,\pi^k]\right) = \frac{1}{2}Tr(A\pi^{[k})Tr(\pi^{j}\pi^{i}) . \quad (5.4)$$

With this identity the two Poisson brackets close on the constraint surface and one obtains the following constraint algebra:

$$\begin{aligned}
\{\xi^\alpha_1 G_\alpha,\xi^\beta_2 G_\beta\} &= f^\gamma_{\alpha\beta}\xi^\alpha_1\xi^\beta_2 G_\gamma & (5.5)\\
\{\xi^\alpha G_\alpha,\xi H\} &= \{\xi^\alpha G_\alpha,\xi^i H_i\} = \{\xi^\alpha G_\alpha,\xi C\} = \{\xi^\alpha G_\alpha,\xi^{ij}C_{ij}\} = 0 & (5.6)\\
\{\xi^i_1 H_i,\xi^j_2 H_j\} &= \xi^i_{[1}\partial_i\xi^j_{2]}H_j + \xi^i_1\xi^j_2 R^\alpha_{ij}G_\alpha & (5.7)\\
\{\xi^i H_i,\xi H\} &= (\xi^i\partial_i\xi - \xi\partial_i\xi^i)H + 2f_{\alpha\beta\gamma}\xi^i\pi^{j\alpha}R^\beta_{ij}G^\gamma & (5.8)\\
\{\xi_1 H,\xi_2 H\} &= 2\xi_{[1}\partial_i\xi_{2]}\pi^i_\alpha\pi^{j\alpha}H_j & (5.9)\\
\{\xi^i H_i,\xi C\} &= -(\xi^i\partial_i\xi - \xi\partial_i\xi^i)C - 3\xi\ xi^i\epsilon_{ijk}f_{\alpha\beta\gamma}\pi^{j\alpha}\pi^{k\beta}G^\gamma & (5.10)\\
\{\xi_1 H,\xi_2 C\} &= 0 & (5.11)\\
\{\xi^{ij}C_{ij},\xi^k H_k\} &= (\xi^k\partial_k\xi^{ij} - \partial_k\xi^k\xi^{ij} + 2\partial_k\xi^i\xi^{kj})C_{ij} & (5.12)\\
\{\xi^{ij}C_{ij},\xi H\} &\approx 0\,. & (5.13)
\end{aligned}$$

As was shown in the discussion abcve, the last Poisson bracket (5.13) vanishes only weakly on the constraint surface. This is enough to state, that the constraints form a closed algebra. However, it is not enough to give explicitly the structure of the constraint algebra. We do not have an expression for the right hand side of (5.13) in terms of constraints which holds off the constraint surface.

To complete the proof of equivalence of the constraints in Ashtekar's form (4.19,4.20) with ours, we show that the constraints (4.19)-(4.23) lead to the action (3.8). We start from the action

$$\begin{aligned}
S &= \int d^4x\ \pi^i_\alpha\dot\omega^\alpha_i - \int dt\ H\\
&= \int d^4x\ \pi^i_\alpha\dot\omega^\alpha_i + \omega^\alpha_0 D_i\pi^i_\alpha + N f_{\alpha\beta\gamma}\pi^{i\alpha}\pi^{j\beta}R^\gamma_{ij} - 2N^i\pi^j_\alpha R^\alpha_{ij} - MC - M^{ij}C_{ij}\ ,
\end{aligned} \quad (5.14)$$

where the Hamiltonian is taken to be a linear combination of constraints. Note that the Lagrange multiplier N for the Hamiltonian constraint is not completely arbitrary, but it is restricted to $N > 0$, because this constraint generates translations along the $\mathbb{R}$-direction of the $\mathbb{R} \times \Sigma$-topolgy. The sign of N determines therefore the signature of 4-geometry. The two terms corresponding to the Ashtekar constraints can now be simplified by introducing a new field

$$\tilde{T}^{ij}_\alpha = N f_{\alpha\beta\gamma} \pi^{i\beta} \pi^{j\gamma} - N^{[i} \pi^{j]}_\alpha \ . \tag{5.15}$$

This definition has to be added to the action as a new contraint and one obtains:

$$\begin{aligned} S = \int d^4x \quad & \pi^i_\alpha \dot{\omega}^\alpha_i + \omega^\alpha_0 D_i \pi^i_\alpha + \tilde{T}^{ij}_\alpha R^\alpha_{ij} \\ - \quad & M^\alpha_{ij} (\tilde{T}^{ij}_\alpha - N f_{\alpha\beta\gamma} \pi^{i\beta} \pi^{j\gamma} - N^{[i} \pi^{j]}_\alpha) - MC - M^{ij} C_{ij} \ . \end{aligned} \tag{5.16}$$

The last three terms in the action correspond to the constraints (4.16)-(4.18), which are equivalent to the constraints (4.6)-(4.8) and may therefore be replaced by them. With this substitution and a little redefinition of the momenta and the $\tilde{T}^{ij}_\alpha$:

$$T^{0i}_\alpha = \frac{1}{2} \pi^i_\alpha \quad \text{and} \quad T^{ij}_\alpha = \frac{1}{2} \tilde{T}^{ij}_\alpha$$

the action can be written in a covariant form, which is identical with (3.8):

$$S = \int d^4x \, T^{\mu\nu}_\alpha R^\alpha_{\mu\nu} + M_{\mu\nu\rho\sigma} \left(T^{\mu\nu}_\alpha T^{\rho\sigma\,\alpha} - \frac{1}{2} (q^{\mu\sigma} q^{\nu\rho} - q^{\mu\rho} q^{\nu\sigma}) \right) \ . \tag{5.17}$$

This shows, that general relativity can be embedded in the phase space of a $SO(3,1)$-Yang-Mills theory with polynomial constraints, which are all first class. Complex variables are not needed.

Acknowledgements
It is a pleasure to thank J.-W. van Holten and J. Kowalski-Glikman for valuable discussions and helpful comments.

APPENDIX

A The symmetries of the action

In this appendix we study the symmetries of the action

$$S = \int d^4x\, T^{\mu\nu}_{\alpha} R^{\alpha}_{\mu\nu} + M_{\mu\nu\rho\sigma} \left({}^{\mu\nu}_{\alpha} T^{\rho\sigma\,\alpha} - \frac{1}{2}\left(q^{\mu\sigma} q^{\nu\rho} - q^{\mu\rho} q^{\nu\sigma}\right)\right) . \tag{A.1}$$

We first give the definition of the scalar product on the $SO(3,1)$-algebra. Because the fibre structure of the action with a fibre isomorphic to $\mathbb{R}^4$ is used frequently, it is natural to define the scalar-product on the representation of the $SO(3,1)$-algebra acting on this fibre i.e. the generators are represented by antisymmetric 4×4-matrices. The scalar product is therefore given by:

$$(A, B) = A^a_b B^b_a \quad , \quad A, B \in SO(3,1) . \tag{A.2}$$

This defines a scalar product with signature $(+,+,+,-,-,-)$. According to this definition the boost generators are positve definite and the rotation generators are negative definite. The symmetry group of this scalar product is by definition the 15-dimensional $SO(3,3)$. But on the other hand it easy to see that this definition of the scalar product is aiso invariant under $GL(4)$ transformations. The scale transformations of $GL(4)$ act trivially on $A \in SO(3,1)$ in the adjoint reprsentation, therfore only the $SL(4)$ acts nontrivially on A. Hence the $SO(3,3)$ and the $SL(4)$ algebra are equivalent.

By construction the action (A.1) is invariant under local $SO(3,1)$ transformations. Let now g be a local $GL(4)$ transformation 'outside' the $SO(3,1)$, i.e. $g \in GL(4)/SO(3,1)$. One has then the following transformation properties:

$$T^{\mu\nu} \;\rightarrow\; g T^{\mu\nu} g^{-1} \tag{A.3}$$

$$R_{\mu\nu} \;\rightarrow\; g R_{\mu\nu} g^{-1} + \partial_{[\mu} g\, \omega_{\nu]} g^{-1} + g\omega_{[\nu} \partial_{\nu]} g^{-1} \tag{A.4}$$

$$(T^{\mu\nu}, R_{\mu\nu}) \;\rightarrow\; (T^{\mu\nu}, R_{\mu\nu}) + 2(T^{\mu\nu}, g^{-1}\partial_\mu g\, \omega_\nu) + 2(T^{\mu\nu}, \omega_\nu \partial_\mu g^{-1} g) . \tag{A.5}$$

The last two terms in the transformation of the scalar product can be written as

$$\begin{aligned} (T^{\mu\nu}, g^{-1}\partial_\mu g\, \omega_\nu) + (T^{\mu\nu}, \omega_\nu \partial_\mu g^{-1} g) &= Tr(T^{\mu\nu} g^{-1} \partial_\mu g \omega_\nu - T^{\mu\nu} \omega_\nu g^{-1} \partial_\mu g) \\ &= -Tr([T^{\mu\nu}, \omega_\nu] g^{-1} \partial_\mu g) . \end{aligned} \tag{A.6}$$

Locally g may be written as $g = g_0 \exp(\xi_\alpha(x) Y^\alpha)$, where the Y^α are $GL(4)$ generators from the orthogonal complement of the $SO(3,1)$-algebra and g_0 is a constant $GL(4)$ transformation. From this one concludes, that the expression (A.6) is zero and the action (A.1) is therefore invariant under local $GL(4)$-transformations.

The expression $T^{\mu\nu}_{\alpha} D_\mu T^{\alpha}_{\rho\sigma}$ is also invariant under local $GL(4)$-transformation by the same argument.

B The sign of $\det \pi^i_\alpha \pi^{j\alpha}$

For the proof that $\det \pi^i_\alpha \pi^{j\alpha} \geq 0$ I use the fact that the $SO(3,1)$ algebra is equivalent to the $SU(2) \times \overline{SU(2)}$ algebra and the momenta may therefore be written as complex coefficients ${\pi_C}^i_\alpha$ of the $SU(2)$ generators. It is convenient to define a positive definite scalar product on the $SU(2)$ algebra, but one has to keep in mind, that this definition extended to the $SU(2) \times \overline{SU(2)}$ algebra differs from the scalar product of the previous section by a minus sign. It is useful to define

$$
\begin{aligned}
h_C^{ij} &= {\pi_C}^i_\alpha {\pi_C}^{j\alpha} && \text{(B.1)}\\
h_R^{ij} &= Re\, h_C^{ij} = \frac{1}{2}\left({\pi_C}^i_\alpha {\pi_C}^{j\alpha} + {\overline{\pi}_C}^i_\alpha {\overline{\pi}_C}^{j\alpha}\right) && \text{(B.2)}\\
h_I^{ij} &= Im\, h_C^{ij} = \frac{1}{2i}\left({\pi_C}^i_\alpha {\pi_C}^{j\alpha} - {\overline{\pi}_C}^i_\alpha {\overline{\pi}_C}^{j\alpha}\right)\ . && \text{(B.3)}
\end{aligned}
$$

Because of the additional minus sign in the definition of the scalar-product one now has to show that $\det h_R^{ij} \leq 0$.

For this consider the algebraic constraints in the $SU(2) \times \overline{SU(2)}$ basis:

$$
\begin{aligned}
C &= \epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\left({\pi_C}^{i\alpha}{\pi_C}^{j\beta}{\pi_C}^{k\gamma} + {\overline{\pi}_C}^{i\alpha}{\overline{\pi}_C}^{j\beta}{\overline{\pi}_C}^{k\gamma}\right) = 0 && \text{(B.4)}\\
&\Rightarrow \pi_C := \det {\pi_C}^i_\alpha \in i\mathbb{R} && \text{(B.5)}\\
&\Rightarrow h_C = \det h_C^{ij} = {\pi_C}^2 \in \mathbb{R} \quad \text{and} \quad h_C \leq 0\ . && \text{(B.6)}
\end{aligned}
$$

The second constraint reads:

$$
\begin{aligned}
C_{ij} &= \epsilon_{ikl}\epsilon_{jmn}\left({\pi_C}^k_\alpha {\pi_C}^{m\alpha} {\overline{\pi}_C}^l_\beta {\overline{\pi}_C}^{n\beta} + {\overline{\pi}_C}^k_\alpha {\overline{\pi}_C}^{m\alpha} {\pi_C}^l_\beta {\pi_C}^{n\beta}\right) && \text{(B.7)}\\
&= \epsilon_{ikl}\epsilon_{jmn}\left(h_C^{km}\overline{h}_C^{ln} + \overline{h}_C^{km} h_C^{ln}\right) = 0 && \text{(B.8)}\\
&\Rightarrow \epsilon_{ikl}\epsilon_{jmn}\left(h_R^{km}h_R^{ln} + h_I^{km}h_I^{ln}\right) = 0\ . && \text{(B.9)}
\end{aligned}
$$

Combining the above results one obtains:

$$
4 \det h_R^{ij} = h_C \leq 0\ , \qquad \text{(B.10)}
$$

which completes the proof.

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