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FIELD THEORY APPROACH TO QUANTUM HALL EFFECT *

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ABSTRACT

The Fradkin's formulation of statistical field theory is applied to the Coulomb interacting electron gas in a magnetic field. The electrons are confined to a plane in normal 3D-space and also interact with the physical 3D-electromagnetic field. The magnetic translation group (MTG) Ward identities are derived. Using them it is shown that the exact electron propagator is diagonalized in the basis of the the wave functions of the free electron in a magnetic field whenever the MTG is unbroken. The general tensor structure of the polarization operator is obtained and used to show that the Chern-Simons action always describes the Hall effect properties of the system. A general proof of the Strêda formula for the Hall conductivity is presented. It follows that the coefficient of the Chern-Simons terms in the long-wavelength approximation is exactly given by this relation. Such a formula, expressing the Hall conductivity as a simple derivative, in combination with diagonal form of the full propagator allows to obtain a simple expressions for the filling factor and the Hall conductivity. Indeed, these results, after assuming that the chemical potential lies in a gap of the density of states, lead to the conclusion that the Hall conductivity is given without corrections by

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

where ν is the filling factor. In addition it follows that the filling factor is independent of the magnetic field if the chemical potential remains in the gap.

I. Introduction

The integer quantum Hall effect (IQHE) and the fractional one (FQHE) have been the subject of the most active research in the field of condensed matter physics in the recent ten years [1]. The interest has also extended to the field of QFT theorist [2]. In this connection it has been stressed that the advance of the research would be supported in great measure by a more close collaboration between the field theory and condensed matter theorist [1]. Most of the theoretical activity in this field was developed in the framework of the many particle QM [3, 4]. A relatively minor quantity of works have been devoted to develope and interacting field theory treatment [5, 6].

The present work intends to apply the Fradkin's functional approach to quantum statistics to the study of these effects [7]. The general aim is to exploit the generality of those methods to investigate some exact properties of the non-relativistic coulomb interacting electron gas confined to a plane in the physical 3D space [8].

The main conclusions of the work are organized in order to show that the Hall conductivity is exactly given by the product of the filling factor with $\frac{e^2}{h}$ whenever the fermi level is in a gap of the density of states. The effects of impurities are completely disregarded in the present approach.

The diagonalization property of the exact one-particle propagator, shown in a recent paper [9] and rederived here, helps in simplifying the discussion. The Strêda formula for the Hall conductivity and its equivalence with the coefficient of the Chern-Simons action is obtained in the context of the statistical QFT for the interacting electron gas [10,11]. As the Hall conductivity is given by that relation as a simple derivative of the density with respect to the magnetic field, and the density is also expressed in a simple way thanks to the diagonalization property, closed expressions for the filling factor and Hall conductivity are obtained. Finally it is argued that when the fermi level lies in a gap of the density of states, the Hall conductivity formula

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

where ν is the filling factor, is an exact one. In addition the filling ratio ν is independent of the magnetic field if the fermi level remains inside the gap.

In the second section the functional approach is presented. Section III is devoted to give an sketched derivation of the diagonalization property of the propagator in the functional formalism. The tensor structure of the polarization operator is obtained in section IV. It serves to the derivation of the Strêda formula from the finite temperature QFT in section V. The filling factor and Hall conductivity expression are obtained in section VI. The proof of the proportionality with the filling factor of the Hall conductivity is given in section VII as well as the independence of the filling factor on B when the Fermi energy lies in a gap is shown.

II. Functional approach

As mentioned in the introduction in this paper the analysis of the QHE is performed by using the Fradkin's functional formulation for statistics as restricted for non-relativistic systems [7]. We take the quantum 2D-electron plasma innbedded in a real 3D-plane as described by the following temperature Green function generating functional

$$Z = \int D\psi^* D\psi \exp[S] , \tag{1}$$

where the action S is given by

$$\begin{aligned} S = \frac{1}{\hbar c}\int dx_4\{\int \psi^*[-c\hbar\partial_4 - (\vec{p}_\sigma - \frac{e}{c}\vec{A}^t_\sigma)^2/(2m) + \mu + ieA^t_4]\psi d^2x \\ -\frac{\lambda}{2}\int \psi^*(x)\psi^*(x')U(\vec{x}-\vec{x}')\psi(x')\psi(x)\}d^2x d^2x' + \lambda\int \psi^*(x)\psi(x)U(\vec{x}-\vec{x}')n_0 d^2x d^2x'\} \\ +\frac{1}{\hbar c}\int d^3x dx_4\{\ -\tfrac{1}{16\pi}[(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \tfrac{2}{\alpha}(\partial_\mu A_\mu)^2]\ \} \\ +\int(\psi^*(x)\eta(x) + \eta^*(x)\psi(x))d^2x dx_4 . \end{aligned} \tag{2}$$

In (2) the fermion fields ψ and ψ^* are functions of the coordinates x_1 and x_2 of the points in the electron gas confinement plane $x_3 = 0$. The electromagnetic fields A_μ are functions of all the 3-space coordinates $\vec{x}$. The x_4 arguments for all the fields are real numbers as corresponds to the Matsubara theory. The external fermion sources η, η^* are grassmanian functions of the same kind as its corresponding fields [7]. The parameters e, m and μ are the electron charge, mass and the chemical potential respectively. The Gaussian units are used through the work. Other definitions that are needed in (2) and will be required are the following ones:

$$\begin{aligned} d^2x &= dx_1\, dx_2 , \\ d^3x &= dx_1 dx_2 dx_3 , \quad x_4 \in (0,\beta) , \\ p_\mu &= -i\hbar\frac{\partial}{\partial x_\mu} = -i\hbar\partial_\mu , \\ \vec{p}_\sigma &= (p_1, p_2, 0, 0) , \\ \vec{A}^t_\sigma &= (A^t_1, A^t_2, 0, 0) , \quad \vec{n} = (0,0,1) , \end{aligned} \tag{3}$$

where $\beta = \frac{c\hbar}{kT}$ (k is the Boltzmann constant and T the temperature). The total electromagnetic field A^t_μ is defined by

$$A^t_\mu = A^e_\mu(x) + A_\mu(x) , \tag{4}$$

in which A^e_μ is the vector potential of the homogenous magnetic field in the symmetrical gauge

$$\vec{A}^e = \frac{1}{2}B\vec{n}\times\vec{x},\ A^e_4 = 0 . \tag{5}$$

The $A_\mu(x)$ field in (4) is the mean value of the quantum electromagnetic field. It need to be stated that in (1) the electromagnetic field functional integral is absent because it was exactly performed after the static approximation was assumed. This approximation results in the Coulomb interaction four fermion term in (2). However the mean field A_μ remains as a dynamical quantity having its own equations of motion [7]. This field will play an important role in the following discussion. Finally the term in (2) containing the arbitrary parameter α corresponds to the fixing of the Lorentz gauge condition in the quantization procedure [7]. Here the value $\alpha = 1$ will be selected which simplifies the photon propagator to be the inverse of the d'Alembertian operator. The parameter λ is the coupling constant of the coulomb interaction.

In (2) the factor n_0 represents the homogeneous background of charges (Jellium) which compensates the electron charge density at equilibrium. As usual the coulomb interaction potential is damped at large distances in the way

$$U(\vec{r}) = \frac{\lambda}{|\vec{r}|}\exp(-\mu|\vec{r}|) ,$$

in order to have convergence in the calculations. At the end the limit $\mu \to 0$ must be kept. The coupling constant value $\lambda = e^2$ must be fixed for concrete calculations. It is worth to remark that some of the conclusions in this paper correspond only to the zero temperature limit. In each case the validity conditions will be stated explicitely.

III. Magnetic translation group and diagonalization of the mass operator

It is well known that the translational invariance properties of electron in a homogenous magnetic field are mathematically described by the so-called magnetic translation group (MTG) [12]. In reference [9] it was also argued that the generators of this group, when represented in the space of states, commute with the coulombic interaction Hamiltonian. Then it follows that the theory described by the generating functional (1) should retain this symmetry if the ground state does not break it. In this section the Ward identities stemming from the magnetic translation group will be obtained in their functional formulation. By using these relations it will be also shown the exact diagonalization of the mass operator. This property occurs in the representation determined by a complete set of solution of free electron problem in the magnetic field. Such a result was also derived in ref. [9]. It should be stated here that the diagonality of the mass operator in QED was proven by Ritus [13]. Then our result constitutes the extension of this conclusion to the non-relativistic context.

The infinitesimal magnetic translation group transformation, leaving the action S in (2) invariant when all the sources vanish, are given as [9]

$$\psi(x) \to \psi(x) + \frac{1}{i\hbar}b_j(p_j + \frac{e}{c}A_j^e)\psi , \qquad (6)$$

$$\psi^*(x) \to \psi(x) + \frac{1}{i\hbar}b_j(p_j - \frac{e}{c}A^e)\psi^*(x) , \qquad (7)$$

$$A_\mu(x) = A_\mu(x) , \qquad (8)$$

with b_j, $j = 1,2$ being infinitesimal parameters.

After performing the change of variables (6) - (8) in (1) the wanted Ward indentities may be obtained as

$$\int d^2x dx_4 \left\{ \frac{1}{i\hbar}(p_j - \frac{e}{c}A_j^e)\frac{\delta Z}{\delta\eta_s(x)}\eta_s(x) + \eta_s^*(x)\frac{1}{i\hbar}(p_j + \frac{e}{c}A_j^e)\frac{\delta Z}{\delta\eta_s^*(x)} \right\}$$
$$- \int d^3x dx_4 \left\{ \frac{\delta Z}{\delta A_\mu(x)}\partial_i A_\mu(x) \right\} = 0 . \qquad (9)$$

In order to arrive to (9) the translational invariance of the free electromagnetic action term in (2) was employed. The Grassman functional derivatives over η and η^* in (9) are of the "right" and "left" type respectively [14]. In terms of the generating functional of the connected Green functions

$$W = ln\ Z , \qquad (10)$$

the relation (9) is transformed in

$$\int d^2x dx_4 \left\{ -\frac{1}{i\hbar}\Big[G_j^*(x)\frac{\delta W}{\delta\eta_s(x)}\Big]\eta_s(x) + \frac{1}{i\hbar}\eta^*(x)\Big[G_j(x)\frac{\delta W}{\delta\eta_s^*(x)}\Big] \right\}$$
$$- \int d^3x dx_4 \left\{ \frac{\delta W}{\delta A_\mu(x)}\partial_j A_\mu(x) \right\} = 0 , \qquad (11)$$

where it has been introduced the following notation for the generators of the MTG

$$G_j(x) = p_j + \frac{e}{c}A_j^e , \quad G_j^*(x) = -p_j + \frac{e}{c}A_j^e . \qquad (12)$$

Let's now pass to apply the Ward identities (11) to the proof of the diagonalization of the mass operator in the basis of the free electron eigenfunctions in the magnetic field [9].

After derivating (11) over two functional arguments $\eta^*(x)$, $\eta(x')$ and making vanish all the sources and the field A_μ, the Ward identity for the one-particle propagator is received in the form

$$G_j(x)G_{rs}(x,x') = G_{rs}(x,x')\overleftarrow{G}_j^*(x') , \qquad (13)$$

where the arrow means that the derivative is acting on the left.

Relation (13) expresses that the generator of the magnetic translation group commutes with the exact Green function. It may be also shown by acting with $G_j(x)$ on (13) that the following relation is valid

$$G^2(x)G_{rs}(x,x') = G_{rs}(x,x')(\overleftarrow{G}^2(x'))^* , \qquad (14)$$

where it has been defined

$$G^2(x) = G_i(x)G_i(x) . \qquad (15)$$

Let us introduce now the operator

$$H = (p_i - \frac{e}{c}A_i^e)(p_i - \frac{e}{c}A_i^e)\ , \tag{16}$$

which is proportional to the one-particle Hamiltonian and also defines the expression for the free-electron Green propagator. By using the definition of G_j in (12) and the explicit expression (5) for the vector potential A_i^e, the following equation can be obtained

$$H = G^2 - \frac{2eB}{c}L_3\ , \tag{17}$$

in which L_3 is the third component of the angular momentum operator for a free-particle

$$L_3 = \epsilon^{ij3}x_i p_j\ . \tag{18}$$

Among the three operator H, G^2 and L_3 the following commutation relations may be obtained

$$[G^2, H] = 0\ , \tag{19}$$

$$[G^2, L_3] = 0\ , \tag{20}$$

$$[H, L_3] = 0\ . \tag{21}$$

It also follows that the rotational invariance of system allows to show, through the use of its corresponding Ward identities, the additional relation [9]

$$L_3(x)G_{rs}(x,x') = G_{rs}(x,x')\ \overleftarrow{L_3}\ (x')\ . \tag{22}$$

Therefore after using (19) - (22) and (14) the following relation is obtained

$$H(x)G_{rs}(x,x') = G_{rs}(x,x')\ \overleftarrow{H}\ (x')\ . \tag{23}$$

The identities (22) and (23) imply that the eigenfunctions of the Green function kernel can be selected as the common set of eigenfunctions of the free-Hamiltonian and the angular momentum operators. More details about this result can be found in [9]. This property of the exact one-particle Green function, as mentioned before, becomes the extension to the non-relativistic (and statistical) framework of the analogous result derived by Ritus for QED [13]. It must be remarked that this conclusion is in no way restricted to the 2D-electron gas. The argumentation works equally well for the 3D-electron system.

From the commutativity of the inverse of the free-propagator of the system with H and L_3 and (23) directly follows the diagonalization of the exact mass operator in the

basis of common eigenfunctions of H and L_3. The explicit form for the propagator in the temporal Fourier representation takes the form [9]

$$G_{\alpha\beta}(\vec{x},\vec{x}',k_4) = \delta_{\alpha\beta}\sum_{n=0}^{\infty} G_n(k_4)\varphi_n^0(0)\varphi_n^{0*}(\vec{x}'-\vec{x}) \exp\left[\frac{ie\vec{A}^e(\vec{x})}{\hbar c}\cdot(\vec{x}-\vec{x}')\right]\ , \tag{24}$$

where $\varphi_n^m(\vec{x})$ are the normalized eigenfunctions of the free electron problem. In arriving to (24) the sum over the angular momentum eigenvalues was explicitely calculated by mean of the formula [15]

$$\sum_{m=-\infty}^{n} \varphi_n^m(\vec{x})\varphi_n^{m*}(\vec{x}') = \varphi_n^0(0)\varphi_n^{0*}(\vec{x}'-\vec{x})\exp\left[\frac{ie}{\hbar c}\vec{A}^e(\vec{x})\cdot(\vec{x}-\vec{x}')\right] \tag{25}$$

where the sum runs up to $-\infty$ because we have considered the magnetic field in the positive x_3 axis direction and the electric charge $e < 0$.

The diagonal form (24) is a greatly simplifying result. It expresses the fact that the spatial dependence of the propagator is kinematically fixed. That is, in a similar way as the translational invariance in the absence of magnetic field allows the Fourier decomposition of the propagator, the MTG when the field is present, determines the spatial dependence of the propagator in terms of the Laguerre functions. Formula (24) is also a generalization of the results of Girvin and Mc. Donald for one-particle density matrices [16,17]. The generalization of (24) to the case of crossed electric and magnetic fields has been presented in ref. [18].

IV. Linear response and the Chern-Simons term

In ref. [8] the general tensor structure of the polarization operator caracterizing the linear response properties of the electromagnetic field was calculated. This result allowed there to point out the relevance of the Chern-Simons action for the description of QHE. The argumentation was performed in the one-loop approximation. It may be considered that one of the central aims of the present work is to present the generalization of the above conclusions of ref. [8] to all orders in perturbation theory for IQHE and FQHE.

In this section the expression of the polarization tensor Π in the functional approach is presented. After passing to the Fourier representations and using the tranversality property, the general tensor structure is obtained in terms of the characteristic vectors of the problem. Then when the long-wavelengh approximation is considered, it shows that the Chern-Simons action always describe the Hall effect properties of the system. A formula for the Hall conductivity (or what is the same the coefficient of the Chern-Simons terms) is also obtained. It serves in the next section to obtain a derivation of the Strêda formula for the CS term coefficient in the contex of statistical QFT [11]. The equation of motion for the mean electromagnetic field $A_\mu(x)$ is given in the Fradkin's approach by [7]

$$\frac{\delta W}{\delta A_\mu(x)}[\eta^*,\eta,A]\Big|_{\eta^*,\eta=0} = \frac{ien_0}{c\hbar}u_\mu \ ,$$
$$W = \ln\ Z\ ,\ u_\mu = (0,0,0,1)\ , \tag{26}$$

where the fermion external sources vanish. The relation (26) is a highly non-linear one. The corresponding equations for the small perturbations of the background magnetic field are obtained by performing a functional expansion in A_μ and retaining the linear terms. Then for the expansion of W up to guadratic terms we have

$$\begin{aligned} W[0,0,A] &= W[0,0,0] + \int \frac{\delta W[0,0,A]}{\delta A_\mu}\Big|_{A=0} A_\mu(x)d^4x \\ &+ \frac{1}{2}\int A_\mu(x)\frac{\delta^2 W}{\delta A_\mu(x)\delta A_\nu(y)}\Big|_{A=0} A_\nu(y)d^4xd^4y \\ &+ 0(A^3)\ . \end{aligned} \tag{27}$$

After substituting in (26) it follows

$$\left[\frac{\delta W[0,0,A]}{\delta A_\mu(x)}\right]_{A=0} + \int\left[\frac{\delta^2 W[0,0,A]}{\delta A_\mu(x)\delta A_\nu(y)}\right]_{A=0} A_\nu(y)d^4y$$
$$+0(A^2) = \frac{ien_o}{c\hbar}u_\mu\ . \tag{28}$$

Thus under the assumption of no-spontaneous breaking of the symmetry of the external magnetic field it follows

$$\left[\frac{\delta W}{\delta A_\mu(x)}\right]_{\eta,\eta^*,A=0} = \frac{ien_0}{c\hbar}u_\mu\ . \tag{29}$$

In physical words this condition expresses the assumption that the system does not developes any internal electromagnetic field in addition to the constant magnetic field. That is the zero field $A_\mu = 0$ must be a solution of the quantum equation of motion (26).

The linear Maxwell equations coming from (28) after performing the functional derivativies of W take the form

$$\frac{1}{4\pi\hbar c}\partial^2 A_\mu(x) + \int \Pi_{\mu\nu}(x,x')A_\nu(x')d^4x' = 0\ . \tag{30}$$

In arriving at (30) the value $\alpha = 1$ was substituted for the gauge parameter. The polarization tensor $\Pi_{\mu\nu}$ in (30) takes the explicit form

$$\begin{aligned} \Pi_{\mu\nu}(x,x') &= \frac{e^2}{\hbar mc^3}\delta(x_3)\delta(x_3')\delta^{(3)}(x-x')\frac{P_{\mu\nu}\delta^2 Z}{Z\delta\eta(x^+)\delta\eta^*(x)}\Big|_{\eta,\eta^*,\ A=0} \\ &- \delta(x_3)\Big\{\frac{ieu_\nu}{\hbar c} + \frac{eP_{\mu\alpha}}{2\hbar mc^2}\Big[p_\alpha(x) - p_\alpha(x^+) - \frac{2e}{c}A^e_\alpha(x)\Big]\Big\} \\ &\cdot\ \frac{1}{Z}\ \frac{\delta^3 Z}{\delta\eta_s(x^+)\delta\eta_s^*(x)\delta A_\nu(x')}\Big|_{\eta,\eta,\ A=0}\ , \end{aligned} \tag{31}$$

where the electron gas four-velocity [7] (as given in (26)),

$$u_\mu = (0,0,0,1) = \delta_{\mu 4}\ , \tag{32}$$

has been introduced in addition with the projection on the gas plane Lorentz tensor

$$P_{\mu\nu} = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix} = \delta_{\mu 1}\delta_{\nu 1} + \delta_{\mu 2}\delta_{\nu 2}\ . \tag{33}$$

In (31) the three-demensional Dirac's delta function is defined by

$$\delta^{(3)}(x-x') = \delta(x_1 - x_1')\delta(x_2 - x_2')\delta(x_4 - x_4')\ . \tag{34}$$

The special delta functions in the x_3 coordinates reflect the loss of translational invariance implied by the confinement of the gas to the plane $x_3 = 0$. Finally the x^+ four-vector is defined as

$$x_\mu^+ = x_\mu + \delta u_\mu\ ,\quad \delta > 0\ , \tag{35}$$

where the limit $\delta \to 0$ is implicitly considered in (31). This variable takes care of the correct ordering of the operators to which the functional derivatives are associated. It may be useful to remember that the spatial dependence of the fermion variables are always on the variables x_1, x_2, x_4 with x_3 excluded.

The tensor $\Pi_{\mu\nu}$ obey the so-called transversality condition [7]. It can be directly deduced from the Ward identity associated to the gauge invariance and has a close relation with charge conservation. In the coordinate representation it writes

$$\partial_\mu \Pi_{\mu\nu}(x,x') = 0\ . \tag{36}$$

Relation (36) will play an important role in fixing a close form for the tensor structure of $\Pi_{\mu\nu}$.

Let's introduce now the momenta representation in the variables of the plane x_1, x_2 and the Matsubara variable x_4 in the following way

$$\Pi_{\mu\nu}(k_j x_3, x_3') = \int d^2x dx_4 e^{-ik_\alpha(x_\alpha - x_\alpha')}\Pi_{\mu\nu}(x,x')\ , \tag{37}$$

$$\Pi_{\mu\nu}(x,x') = \int \frac{d^2kdk_4}{(2\pi)^3}e^{ik_\alpha x_\alpha}\Pi_{\mu\nu}(k,x_3,x_3')\ , \tag{38}$$

where the four-momenta k is defined as $k_\mu = (k_1,k_2,0,k_4)$ and the translational invariance in the plane of the gas has been considered in (37) by supposing that Π only depends on the difference of the variables $x_\alpha - x_\alpha'$, $\alpha = 1,2$ and 4. In the x_3 variable there is no such an invariance.

The explicit dependence of $\Pi_{\mu\nu}$ on the x_3 variable may be exactly obtained. For this purpose it can be noticed that performing the functional derivative over A in (31) gives rise to a x' and ν dependences which are symmetrical to the x and μ ones. Thus a global factor $\delta(x_3)\delta(x_3')$ appears which completely defines the dependence of Π on x_3 and x_3'. Furthermore, it also follows that

$$\begin{aligned}\Pi_{3\alpha}(x,x') &= \Pi_{\alpha 3}(x,x') = 0 , \\ \alpha &= 1,2,3,4 .\end{aligned} \tag{39}$$

Thus the polarization tensor takes the form

$$\Pi_{\mu\nu}(k,x_3,x_3') = \Pi_{\mu\nu}(k)\delta(x_3)\delta(x_3') , \tag{40}$$

with

$$n_\alpha \Pi_{\alpha\mu}(k) = \Pi_{\mu\alpha}(k) n_\alpha = 0 , \tag{41}$$

expressing the vanishing of all the components with an index equal to three in terms of the four-vector normal to the gas plane

$$n_\mu = (0,0,1,0) . \tag{42}$$

The tranversatily property (36) takes the form

$$k_\alpha \Pi_{\alpha\mu}(k) = \Pi_{\mu\alpha}(k) k_\alpha = 0 . \tag{43}$$

The conclusion arises that the linear response properties of this problem are described by a special tensor $\Pi_{\mu\nu}(k)$. It has the same basic properties that the one corresponding to a purely two-dimensional electron gas interacting with an also 2D-electromagnetic field [19, 20].

The remaining part of tensor structure of $\Pi_{\mu\nu}(k)$ may be expressed in terms of three scalar functions by using the transversality property (43). After some algebraic operations the following result may be received

$$\begin{aligned}\Pi_{\mu\nu}(k) &= \frac{(\pi_1(k)+\pi_2(k))}{k_4^2}\ell_\mu^{(1)}\ell_\nu^{(1)} \\ &+ \pi_1(k)\ell_\mu^{(2)}\ell_\nu^{(2)} + \frac{\pi_3(k)}{k_4}\epsilon^{\mu\alpha\nu\beta} n_\alpha k_\beta ,\end{aligned} \tag{44}$$

where the newly defined four-vectors are given in the rest frame of the gas by the expressions

$$\ell_\mu^{(1)} = \left(\frac{k_4 \vec{k}}{|\vec{k}|}, -|\vec{k}|\right) , \tag{45}$$

$$\ell_\mu^{(2)} = \left(\frac{\vec{n}\times\vec{k}}{|\vec{k}|}, 0\right) , \tag{46}$$

where

$$\vec{n} = (0,0,1) , \tag{47}$$

$$\vec{k} = (k_1,k_2,0) . \tag{48}$$

The scalar functions in (44) satisfy the following relations

$$\pi_1(k) = \pi_1(-k) , \tag{49}$$

$$\pi_2(k) = \pi_2(-k) , \tag{50}$$

$$\pi_3(k) = -\pi_3(-k) . \tag{51}$$

The main information in (44) is that the last contribution in the sum which breaks the space-time inversion as implied by (51), is exactly the Chern-Simons term when if π_3/k_4 is taken in the zero momenta limit. Therefore the conclusion arises that for 2D-electron system the Chern-Simons action describes the Hall effect properties, no matter of being at the quantized or normal regimen. It is also interesting that the Chern-Simons appearance of the parity breaking term in (44) is a direct consequence of the gauge invariance as expressed by the transversality relations (43). All the discussion in this section is valid at finite temperatures.

V. Strêda formula for interacting electrons as the coefficient of the *CS* action

The main objective of this section will be the derivation of the Strêda formula for the static value of the Hall conductivity for an interacting electron gas in the framework of statistical QFT.

The formula for the conductivity tensor in terms of the polarization operator is given by [7].

$$\sigma_{ij}(k) = \frac{\hbar c^2}{k_4}\Pi_{ij}(k) , \tag{52}$$

in which the δ-function structure in the x_3 variables as being common to all the $\Pi_{\mu\nu}$ components is not considered. Such a dependence only expresses the physical fact that all the internal currents and charges are confined to the plane $x_3 = 0$.

The interest here is in the static value of the Hall conductivity tensor which is determined in the zero momenta limit (after the analytical continuation in the k_4 variable [7]) by

$$\begin{aligned}\sigma_{ij}^{(H)} &= \lim_{|\vec{k}|\to 0}\{\lim_{k_4\to 0}[\frac{\pi_3(k)}{k_4^2}\epsilon^{i\alpha j\beta} n_\alpha k_\beta]\}\hbar c^2 \\ &= \lim_{|\vec{k}|\to 0}\{\lim_{k_4\to 0}[\frac{\pi_3(k)}{k_4}]\}\epsilon^{i3j}\hbar c^2 \\ &= \sigma_{xy}\epsilon^{ij3} .\end{aligned} \tag{53}$$

After expressing π_3 in terms of the Π_{ij} components we have

$$\begin{aligned}\frac{\pi_3(k)}{k_4} &= -\frac{1}{|\vec{k}|k_4}\cdot\ell_i^{(2)}\Pi_{ij}k_j \\ &= -\frac{1}{\vec{k}^2k_4}\epsilon^{i\ell m}n_\ell k_m\Pi_{ij}k_j \\ &= \frac{1}{\vec{k}^2}\epsilon^{i\ell m}n_\ell k_m\Pi_{i4}\ ,\end{aligned} \tag{54}$$

where relation (46) for $\ell_i^{(2)}$ and the transversality condition have been used. After derivating the transversality relation over the spatial momenta and taking the limit $k_4\to 0$ the following formula is obtained

$$\Pi_{i4} = -k_\ell\frac{\partial\Pi_{\ell 4}}{\partial k_i}\bigg|_{k_4\to 0}\ , \tag{55}$$

which when substituted in (53) and using

$$\Pi_{i4}(k) = \int d^2r dr_4 e^{-i\vec{k}\vec{r}}\Pi_{i4}(r)\ , \tag{56}$$

allows to write the relation

$$\sigma_{xy} = -\hbar c^2\lim_{|\vec{k}|\to 0}\left\{\int d^2r dr_4(e^{-i\vec{k}\vec{r}})\frac{n_\ell k_m k_j}{\vec{k}^2}(-ir_i)\Pi_{j4}(r)\epsilon^{i\ell m}\right\}\ , \tag{57}$$

in which according to (31) and the translation invariance

$$\begin{aligned}\Pi_{j4}(r) &= -\frac{e}{2\hbar mc^2}[p_j(r)-p_j(r^+)-2\frac{e}{c}A_j^e(r)] \\ &\cdot\int\frac{1}{Z}\frac{\delta^3 Z}{\delta\eta_s(r^+)\delta\eta_s^*(r)\delta A_4(0,x_3')}dx_3' \\ &= \left\{\int\frac{\delta}{\delta A_j(r,x_3)}\left[\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}\right]dx_3\,dx_3'\right\}_{\eta,\eta^*,\ A=0}.\end{aligned} \tag{58}$$

Before continuing let us note that the limit in (57) may be taken by fixing an arbitrary direction for the vector $\vec{k}$ in the plane. Then we may define two orthogonal unit vectors $t_i^{(1)}$ and $t_i^{(2)}$, which consequently obey

$$t_i^{(1)}t_j^{(1)} + t_i^{(2)}t_j^{(2)} = \delta_{ij}\ . \tag{59}$$

By considering the limit (57) for $\vec{k}$ direction along each of the unit vectors $t^{(i)}$, $i=1,2$, and performing the semisum of both expressions it follows

$$\sigma_{xy} = -\frac{\hbar c^2}{2}\lim_{|\vec{k}|\to 0}\left\{\int d^2r dr_4 e^{-i\vec{k}\cdot\vec{r}}(-i)\epsilon^{i\ell j}n_\ell r_i\Pi_{j4}(r)\right\}\ . \tag{60}$$

The substitution of (58) in (60) gives

$$\begin{aligned}\sigma_{xy} &= -\frac{\hbar c^2}{2}\lim_{|\vec{k}|\to 0}\int d^2r dr_4 e^{-i\vec{k}\cdot\vec{r}}(-i)\epsilon^{i\ell j}n_\ell r_i \\ &\cdot\left\{\int\frac{\delta}{\delta A_j(r,x_3)}\left[\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}\right]dx_3\,dx_3'\right\}_{\eta,\eta^*,A=0}\ ,\end{aligned} \tag{61}$$

which after considering

$$\frac{dA_i^e(r)}{dB} = \frac{1}{2}\epsilon^{i\ell j}n_\ell r_j\ , \tag{62}$$

takes the form

$$\begin{aligned}\sigma_{xy} &= -\hbar c^2 i\int d^2r dr_4\frac{dA_i^e}{dB}(r)\left\{\frac{\delta}{\delta A_i(r,x_3)}\left[\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}\right]\right\}_{\eta,\eta^*,A=0} \\ &= -\hbar c^2 i\int\frac{d}{dB}\left[\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')}\right]_{\eta,\eta^*,A=0}dx_3'\ .\end{aligned} \tag{63}$$

But

$$\frac{1}{Z}\frac{\delta Z}{\delta A_4(0,x_3')} = -\frac{ie}{\hbar c}\frac{1}{Z}\frac{\delta^2 Z\delta(x_3')}{\delta\eta_s(0^+)\delta\eta_s^*(0)} = i\frac{e}{\hbar c}n(0)\delta(x_3')\ , \tag{64}$$

where $n(0)$ is the density of particles. Thus, the substitution of (64) in (63) gives the Strêda formula

$$\sigma_{xy} = ec\frac{dn}{dB}\bigg|_{\mu,T=const}\ . \tag{65}$$

The relation (65) expresses the Hall conductivity as a simple derivative of the density of particles. But actually the density also has a simple expression in terms of the fermion Green function. Thus the here also obtained diagonalization property of the exact propagator further can simplify the discussion of the Hall conductivity. The result (65) is valid at $T\neq 0$.

VI. The filling factor formula and other relations

According to (24) we have for the fermion Green function

$$\begin{aligned}G_{\alpha\beta}(x,x') &= \frac{1}{Z}\frac{\delta^2 Z}{\delta\eta_\alpha^*(x)\delta\eta_\beta(x')} \\ &= \delta_{\alpha\beta}\sum_{n=0}G_n(x_4-x_4')\varphi_n^0(0)\cdot\varphi_n^{0*}(\vec{x}-\vec{x}')\exp i\frac{e\vec{A}^e(\vec{x})}{\hbar c}\cdot(\vec{x}-\vec{x}')\ .\end{aligned} \tag{66}$$

Therefore for the density of particles results

$$\begin{aligned}n &= -\frac{1}{Z}\frac{\delta^2 Z}{\delta\eta_s(0^+)\delta\eta_s^*(0)} \\ &= -2\sum_{n=0}^{\infty}G_n(-\delta)|\varphi_n^0(0)|^2\bigg|_{\delta\to 0^+}\ ,\end{aligned} \tag{67}$$

where in the zero temperature limit

$$\lim_{\delta\to 0^+} G_n(-\delta) = \lim_{\delta\to 0^+} \int \frac{dk_4}{2\pi} \frac{e^{-ik_4\delta}\hbar c}{ic\hbar k_4 + \epsilon_n - \mu + \sigma_n(k_4)} \quad . \tag{68}$$

In (68) the $\sigma_n(k_4)$ are the eigenvalues of the mass-operator being associated to the particular eigenfunction φ_n^m. The independence of σ_n of m can be shown by using the results of the section III, [9].

After considering that

$$|\varphi_n^0(0)|^2 = \frac{1}{2\pi r_0^2} = \frac{|eB|}{hc} ,$$

the density n writes as follows:

$$\begin{aligned} n &= -2\frac{|eB|}{hc}\sum_{n=0}^{\infty}\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\frac{dk_4}{2\pi}\frac{\hbar c\, e^{-ik_4\delta}}{ic\hbar k_4 + \epsilon_n - \mu + \sigma_n(k_4)} \\ &\equiv \frac{|eB|}{hc}\nu , \end{aligned} \tag{69}$$

in which the filling factor ν is defined.

As a matter of checking it is possible to disregard the $\sigma_n(k_4)$ in (69). In such a case the integral in it may be readily calculated to give (if μ lies in the gap between two Landau levels)

$$n = \frac{2|eB|}{hc}\sum_{n=0}^{\infty}\theta(\mu - \epsilon_n) , \tag{70}$$

in which $\theta(x)$ is the Heaviside function. The result (70) is the expected one in the three approximation for the Green function.

In continuing let's examine the value of the conductivity predicted by the use of the Strêda formula (65) and the expression (69) for the density. Substituting (69) in (65)

$$\begin{aligned} \sigma_{xy} &= ec\frac{dn}{dB}|_{\mu,T=const} \\ &= ec\,\frac{d}{dB}\,(\nu\frac{eB}{hc}) \\ &= \frac{e^2}{h}\,\frac{d}{dB}\,(\nu B) \\ &= \frac{e^2}{h}\Big[\nu + B\frac{d\nu}{dB}\Big]_{\mu,T=const} . \end{aligned} \tag{71}$$

Hence, the Hall conductivity is exactly proportional to the filling factor ν if this magnitude does not depends on B.

Below a relation expressing $\frac{d\nu}{dB}$ in terms or the derivatives over μ and λ will be obtained. Note that under the changes of variables

$$\begin{aligned} q &= k_4\frac{c}{\omega_0} , \\ \mu' &= \mu/(\hbar\omega_0) , \qquad \omega_0 = \frac{|eB|}{mc} , \end{aligned} \tag{72}$$

the filling factor expresses as

$$\nu = -\sum_{n=0}^{\infty}\lim_{\delta'\to 0}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\frac{2e^{-iq\delta'}}{iq + n + \frac{1}{2} - \mu' + \sigma_n/(\hbar\omega_0)} , \tag{73}$$

where the mass operator eigenvalue satisfies with independence of m [9], the relation

$$\sigma_n(k_4) = \int\int \varphi_n^{m^*}(x)\Sigma(\vec{x},\vec{x}\,',k_4)\varphi_n^m(\vec{x}')d^2x\, d^2x' . \tag{74}$$

In (74) the trivial spinor structure of the wave function and Σ are not considered. That is

$$\Sigma_{\alpha\beta}(\vec{x},\vec{x}\,',k_4) = \delta_{\alpha\beta}\Sigma(\vec{x},\vec{x}\,',k_4) . \tag{75}$$

Let us consider now the generating functional (1) evaluated at zero electromagnetic field. In this case it generates the fermionic Green functions of the problem. In particular the one-electron one. Now, it is worth to introduce a new set of dimensionless integration fields and space-time variables according to

$$\vec{x} \to r_0\vec{z} , \tag{76}$$

$$x_4 \to \frac{c}{\omega_0}z_4 , \tag{77}$$

$$\psi(x) \to \psi(z)/r_0 , \tag{78}$$

$$\psi^*(x) \to \psi^*(z)/r_0 . \tag{79}$$

After that the generating functional in the new variables takes the form

$$\begin{aligned} Z[\xi,\xi^*] &= \int D\psi^*(z)D\psi(z)\exp S \\ S &= \int dz_4\Big\{\int \psi^*(z)\Big[-\frac{\partial}{\partial z_4} - \frac{1}{2}(-i\vec{\nabla} + \frac{\vec{n}\times\vec{z}}{2})^2 + \mu'\Big]\psi(z)d^2z \\ &- \frac{1}{2}\int \psi^*(z)\psi^*(z')\frac{\lambda'}{|z-z'|}\psi(z')\psi(z)d^2zd^2z' \\ &+ \int \psi^*(z)\psi(z)\frac{\lambda'}{|z-z'|}r_0^2n_0d^2zd^2z'\Big\} \\ &+ \int(\xi^*(z)\psi(z) + \psi^*(z)\xi(z))d^2zdz_4 , \end{aligned} \tag{80}$$

where taking into account (29) and (67),

$$r_0^2 n_0 = \left[\frac{1}{Z}\int D\psi^*(z)D\psi(z)\psi^*(0^+)\psi(0)\exp S\right]_{\xi,\xi^*=0} \tag{81}$$

and new auxiliary sources ξ and ξ^* have been introduced.

The parameter μ' and λ' are given by

$$\mu' = \frac{\mu}{\hbar\omega_0} \quad , \quad \lambda' = \frac{\lambda}{r_0\hbar\omega_0} \; . \tag{82}$$

Thus at $T = 0$ any fermionic Green function with p legs, after multiplied it by r_0^p is a function of the parameters B, μ and λ only through the adimensional constants λ' and μ'. It also follows that in (73) $\frac{\sigma_n}{\hbar\omega_0}$ is the eigenvalue of the mass operator calculated within the transformed generating functional description. Then $\sigma_n/\hbar\omega_0$ is a function of B only through μ' and λ'. This fact leads to the following relation

$$\begin{aligned} \left.\frac{d\nu}{dB}\right|_{\mu,\lambda=const} &= -\frac{1}{B}\left[\mu\frac{\partial\nu}{\partial\mu} + \frac{1}{2}\lambda\frac{\partial\nu}{\partial\lambda}\right] \\ &= -\frac{\hbar}{2B^2|e|}\left[\mu\frac{\partial n}{\partial\mu} + \frac{1}{2}\lambda\frac{\partial n}{\partial\lambda}\right] \; . \end{aligned} \tag{83}$$

Thus, the B derivative of ν expresses linearly in terms of the derivatives of n, respect to μ and λ. It is needed to stress that such a result assumes that the system satisfies the equilibrium equation (29) when calculating the derivatives. In physical terms, this means that the background of compensating charges mantains the system's neutrality upon variation of the parameters. It is apparent that this condition is strongly connected with the plateaus stability. We expect to address this problem in future work.

VII. The exact formula $\sigma = \nu\frac{e^2}{h}$ when μ lies in a gap

In this section it will be argued that the both derivatives $\frac{\partial n}{\partial\mu}\Big|_{B,\lambda=const}$ and $\frac{\partial n}{\partial\lambda}\Big|_{B,\mu=const}$ vanish under the condition that the Fermi level lies in a gap of the density of states. If such is the case from (71) arises the exact result

$$\sigma_{xy} = \frac{e^2}{h}\nu = \frac{e^2}{h}\,\frac{n}{2\pi r_0^2} \; . \tag{84}$$

Then, let us analyze the expression (69) for the density at $T = 0$,

$$n = -2\frac{|eB|}{hc}\sum_{n=0}^{\infty}\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{e^{-iq\delta}}{iq-\mu'+n+\frac{1}{2}+\sigma_n^*} \; , \tag{85}$$

where σ_n^* depends on q and μ' in the way

$$\sigma_n^*(q,\mu',\lambda') = \sigma_n^*(iq-\mu',\lambda') \; , \tag{86}$$

as may be seen from the modified generating functional (80).

It can be noticed that, as the Lehmann representation predicts, the singularties of the integrand in (85) must correspond to a branch cut or a set of poles along the imaginary axis. But in statistical QFT the energy of the excitations is measured from the fermi energy. Therefore the condition for a gap is that a sufficiently small open interval of the imaginary axis including the origin, does not contain any polus or branch cut. But as may be seen from (85) and (86) a small change of μ (or μ') is equivalent to a paralel shift (because the integrand depends on $iq-\mu'$) of the integration contour. Thus, under the validity of the above condition for a gap, that deformation does not alter the result of the integral. In this way it follows that

$$\left.\frac{dn}{d\mu}\right|_{B,\lambda=const} = 0 \; . \tag{87}$$

Fig.1 graphically shows the above description.

Now, let us consider the $\frac{\partial n}{\partial\lambda}$ derivative. For this purpose the following expression for the density in terms of the thermodynamical potential will be used.

$$n = -\frac{1}{V}\left.\frac{\partial\Omega}{\partial\mu}\right|_{T,V,B=const} \; . \tag{88}$$

where Ω is defined by

$$\Omega = -kT\ln Z \; . \tag{89}$$

Thus, $\dfrac{\partial n}{\partial\lambda}$ may be writen as follows

$$\frac{\partial n}{\partial\lambda} = -\frac{1}{V}\frac{\partial}{\partial\mu}\left[\frac{\partial\Omega}{\partial\lambda}\right]_{T,V,B=const} \; . \tag{90}$$

For the λ derivative of Ω the following expression can be obtained by standard methods [21]

$$\frac{\partial\Omega}{\partial\lambda} = \frac{1}{2\lambda}\int d^2x \lim_{\vec{x}'\to\vec{x}}\lim_{x_4'\to x_4^+} \tag{91}$$

$$\left\{-c\hbar\frac{\partial}{\partial x_4} - (\vec{p} - \frac{e}{c}\vec{A}^e(x))^2/(2m) + \mu\right\}G_{\alpha\alpha}(x,x') \; . \tag{92}$$

After substituting the diagonal form (24) for the exact propagator the following expression can be obtained in the zero temperature limit

$$\frac{\partial n}{\partial\lambda} = c\frac{\partial}{\partial\mu}\left[\sum_{n=0}^{\infty}\lim_{\delta\to 0^+}\int_{-\infty}^{\infty}\frac{dq}{2\pi}\,\frac{(-iq+\mu'-n-1/2)e^{-iq\delta}}{-iq+\mu'-n-1/2-\sigma_n^*(q)}\right] \; , \tag{93}$$

where c is a constant. Relation (93) expresses $\dfrac{\partial n}{\partial\lambda}$ as a derivative over μ of an integral of the similar kind to (85). Therefore, the small changes of μ again are equivalent to parallel shifts of the integration axis in the complex q plane. Since in this process by assumption the axis is not passing over any singularity, the result of the integral is unchanged and thus $\frac{\partial n}{\partial\lambda}$ vanishes. Then, the independence of the filling factor from the magnetic field follows, as well as the exactness of the formula $\sigma_{xy} = \nu\frac{e^2}{h}$.

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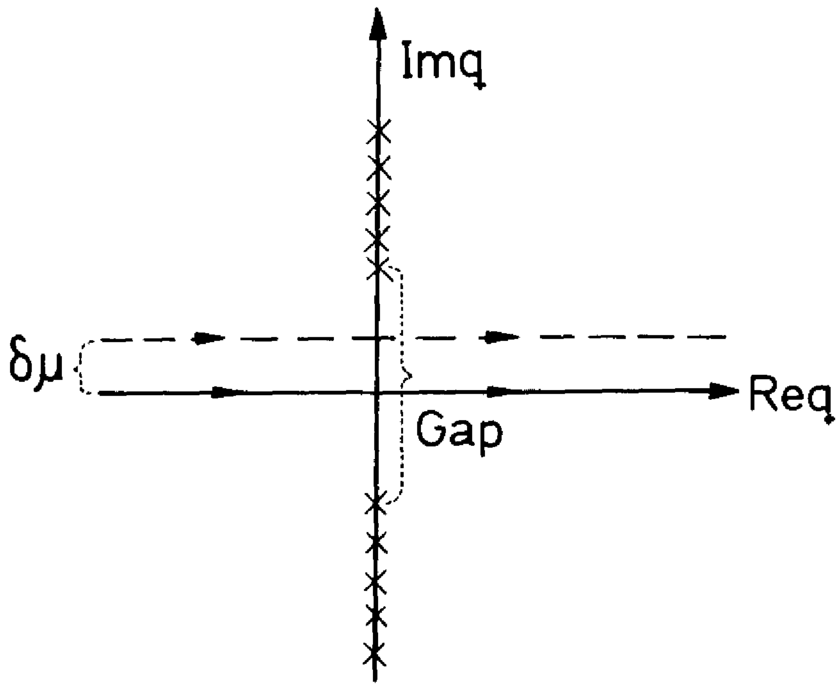

Fig. 1

Fig. 1. Chemical potential increment $\delta\mu$ as a shift of the integration contour in (85).

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