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The α_S^3 corrections to the Bjorken sum rule for polarized electroproduction and to the Gross-Llewellyn Smith sum rule

S.A. Larin [1], and J.A.M. Vermaseren

NIKHEF-H, P.O. Box 41882,
1009 DB, Amsterdam

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[1] On leave from Institute for Nuclear Research (INR) of the USSR Academy of Sciences, Moscow 117312.

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One of the most important experimental tests of perturbative QCD is the test of universality of Λ_{QCD}. This universality implies that the value of Λ_{QCD} extracted from the fits of different physical processes should be the same. From the experimental side this test needs of course a large amount of precise data at high energies which are assumed to be dominated by perturbative QCD. From the theoretical side we need at least three leading perturbative terms in the strong coupling constant α_S for each process to ensure a reliable extraction of Λ. We need at least three powers because first of all the truncated perturbative series depends on the renormalization scheme. The leading term (normally $\mathcal{O}(\alpha_S)$) is absolutely insufficient for extracting Λ since only the next-to-leading term fixes Λ properly for the given scheme. Because the perturbative series is an asymptotic series we can only hope that the error of the truncated QCD series is determined by the first discarded (or last included) term. Hence the next-next-to-leading term should be assumed to estimate the theoretical uncertainty of the prediction.

In the present paper we obtain the next-next-to-leading (three loop) corrections to the Bjorken sum rule for polarized deep inelastic electroproduction [1] and to the Gross-Llewellyn Smith sum rule for deep inelastic neutrino-nucleon scattering [2]. The leading $\mathcal{O}(\alpha_S)$ correction to the Gross-Llewellyn Smith sum rule was calculated in [3], while the same was done for the polarized Bjorken sum rule in [4]. The next-to-leading $\mathcal{O}(\alpha_S^2)$ corrections to both sum rules were obtained in [5]. Higher twist contributions were studied in [6].

First let us consider the Gross-Llewellyn Smith sum rule. Deep inelastic processes – a review of the the status of perturbative QCD for these processes is given in [7]– for unpolarized scattering are described by the hadronic tensor

$$\begin{aligned} W_{\mu\nu}(p,q) &= \int d^4z\, e^{iqz} < p \mid J_\mu^+(z) J_\nu(0) \mid p > \\ &= (-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}) F_1(x,Q^2) + (p_\mu - \frac{p\cdot q}{q^2} q_\mu)(p_\nu - \frac{p\cdot q}{q^2} q_\nu) \frac{1}{\nu} F_2(x,Q^2) \\ &\quad -i\varepsilon_{\mu\nu\kappa\lambda} p^\kappa q^\lambda \frac{1}{\nu} F_3(x,Q^2) \end{aligned} \tag{1}$$

where J_μ is the weak (or electromagnetic) quark current, $Q^2 = -q^2$, $\nu = p\cdot q$, $x = \frac{Q^2}{2p\cdot q}$ and $\mid p >$ is the hadron state (spin averaging is assumed here). The parton model predicts the sum rules for neutrino-nucleon scattering [8] [2] which receive α_S corrections in QCD:

$$\int_0^1 dx (F_1^{\bar{\nu}p}(x,Q^2) - F_1^{\nu p}(x,Q^2)) = 1 \tag{2}$$

$$\int_0^1 dx (F_3^{\bar{\nu}p}(x,Q^2) + F_3^{\nu p}(x,Q^2)) = 6 \tag{3}$$

The $\mathcal{O}(\alpha_S^3)$ correction to the Bjorken sum rule for neutrino-nucleon scattering (eq. 2) has been calculated in [9]. Here we will obtain the $\mathcal{O}(\alpha_S^3)$ correction to the Gross-Llewellyn

Smith sum rule (eq. 3). We will only consider the leading twist approximation, assuming the quarks to be massless.

In QCD the first moment (eq. 3) of the isospin singlet structure function $F_3^{\bar{\nu}p+\nu p}$ is proportional to the coefficient function $C^{(V)}$ of the flavor singlet vector current in the following short-distance OPE (operator product expansion):

$$i\int dz\; e^{iqz}T\{A_\mu^a(z)V_\nu^b(0)\} \overset{Q^2\to\infty}{\simeq} \delta^{ab}\epsilon_{\mu\nu\kappa\lambda}\frac{q_\lambda}{q^2}C^{(V)}(\frac{\mu^2}{Q^2},\alpha_S)V_\kappa(0)+\ldots \qquad (4)$$

where $V_\kappa=\bar{\psi}\gamma_\kappa\psi$ is a flavor singlet quark current, $V_\mu^a=\bar{\psi}\gamma_\mu t^a\psi$ and $A_\mu^a=\bar{\psi}\gamma_\mu\gamma_5 t^a\psi$ are vector and axial vector nonsinglet quark currents, the t^a being generators of the flavor group SU(f). The axial and vector parts of the charged weak current can be restored from A_μ^a and V_ν^b by proper projections in the flavor indices a and b.

To obtain $C^{(V)}$ we use the method of [10] which reduces the calculation of coefficient functions of OPE's to the calculation of diagrams of the propagator type only. This method relies heavily on the use of dimensional regularization [11] and the minimal subtraction scheme [12]. The method gives

$$\delta^{ab}\epsilon_{\mu\nu\kappa\lambda}\frac{q_\lambda}{q^2}C^{(V)}(\frac{\mu^2}{Q^2},\alpha_S) = \frac{1}{12n_f}R_{\overline{MS}}\int i\,d^4z\; e^{iqz} < 0\mid T\{\bar{\psi}(p)\gamma_\kappa\psi(p)A_\mu^a(z)V_\nu^b(0)\}\mid 0>\mid_{p=0}^{amputated} \qquad (5)$$

where some remarks are in order. $\psi(p)$ is the Fourier transform of the quark field carrying the momentum p. n_f is the number of flavors. $R_{\overline{MS}}$ is the ultraviolet R-operation in the standard $\overline{MS}$ scheme [3] which we use throughout this paper. In the r.h.s. of eq. 5 only the diagrams which cannot be disconnected by cutting a single line carrying only the momentum p should be taken into account. This assumes that the external fermion legs are amputated. In the dimensional regularization scheme all massless vacuum diagrams are equal to zero. So in the OPE of the r.h.s. of eq. 5 only the operators which produce tree graphs survive after the nullification of p. In our case the only operator of this type is $V_\kappa(0)$.

Two typical diagrams to be calculated are shown in fig 1:

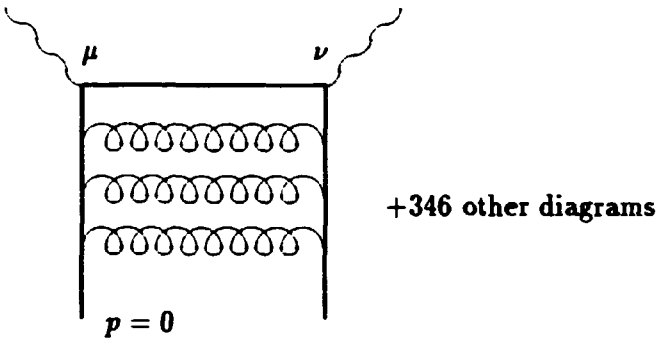

Figure 1a

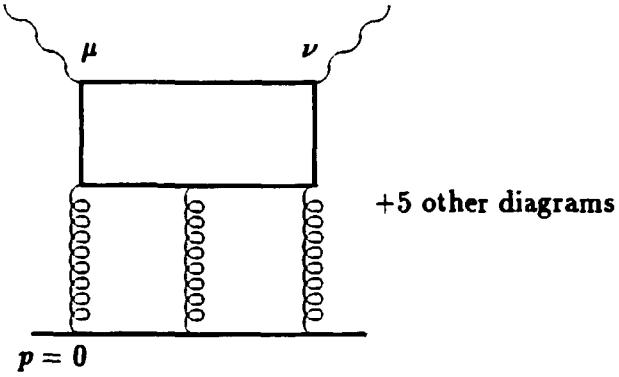

Figure 1b

The diagrams in figure 1a have a through going fermion line to which the external vector and axial vector vertices are connected. The diagrams in figure 1b have the vector and axial vector vertices in a closed line.

In total we had to calculate 347 + 6 diagrams. This count excludes symmetries. Diagrams with one or two loop insertions are counted as single diagrams. After nullifying the momentum p many diagrams contain infrared divergences which are to be removed according to [10] by the ultraviolet renormalization constant of the operator. But for the V_κ this constant is one due to current conservation, so all infrared divergences should cancel in the sum of all diagrams. This is a good check of the calculation.

To use formula 5 we must define the γ_5-matrix within dimensional regularization. The most practical definition for our purpose, and the only one known to be selfconsistent, is due to 't Hooft and Veltman [11]:

$$\gamma_5 = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \tag{6}$$

in which the ε-tensor is unavoidably a four-dimensional object. It should however be taken

outside the R_{MS}-operation. Inside the R_{MS}-operation the indices $\mu \ldots \sigma$ of the gamma matrices should be taken to be d-dimensional. This definition of γ_5 can be applied straight forwardly in computer algebra. To save much computer time it is better to use an equivalent definition [13] for the axial current:

$$A^a_\mu \equiv \bar{\psi}\gamma_\mu\gamma_5 t^a \psi = \frac{1}{6}\varepsilon_{\mu\rho\sigma\tau}\bar{\psi}\gamma_\rho\gamma_\sigma\gamma_\tau t^a\psi \tag{7}$$

Both definitions however violate the axial Ward identity. In particular the renormalization constant Z_A of the axial current in eq. 7 is not equal to one within the MS-scheme (see for instance [14]). Hence we had to compute the three loop approximation for Z_A:

$$\begin{aligned} Z_A &= 1 + (\frac{\alpha_S}{\pi})^2(11C_FC_A - 2C_Fn_f)/(24\epsilon) + (\frac{\alpha_S}{\pi})^3(\frac{C_F}{432\epsilon^2}(-121C_A^2 + 44C_An_f - 4n_f^2) \\ &+ \frac{C_F}{2592\epsilon}(-1386C_FC_A + 1789C_A^2 + 144C_Fn_f - 416C_An_f + 4n_f^2)) \end{aligned} \tag{8}$$

where C_F and C_A are the Casimir operators of the defining and the adjoint representations of the color group. The dimension of space-time is defined as $d = 4 - 2\epsilon$. We used the convention that the relation between the renormalized and the bare operators is $O_R = Z\, O_B$.

To restore the Ward identity one may perform [15] a finite renormalization of the axial current, or in other words introduce the finite "axial charge" $Z_5(\alpha_S)$ which is initially equal to one

$$A^a_\mu \to Z_5 A^a_\mu \equiv \tilde{A}^a_\mu \tag{9}$$

This "charge" Z_5 can be obtained from the relation

$$(R_{\overline{MS}}V^a_\mu)\gamma_5 = Z_5 R_{\overline{MS}}A^a_\mu \tag{10}$$

where A^a_μ is defined in eq. 7. This relation means that the anticommutativity of γ_5 is effectively restored. To three loops in QCD our result is

$$\begin{aligned} Z_5 &= 1 - \frac{\alpha_S}{\pi}C_F + (\frac{\alpha_S}{\pi})^2(\frac{11}{8}C_F^2 - \frac{107}{144}C_FC_A + \frac{1}{72}C_Fn_f) \\ &+ (\frac{\alpha_S}{\pi})^3(C_F^3(-\frac{185}{96} + \frac{3}{2}\zeta_3) + C_F^2C_A(\frac{2917}{864} - \frac{5}{2}\zeta_3) \\ &+ C_FC_A^2(-\frac{2147}{1728} + \frac{7}{8}\zeta_3) + C_F^2n_f(-\frac{31}{864} - \frac{1}{6}\zeta_3) \\ &+ C_FC_An_f(\frac{89}{1296} + \frac{1}{6}\zeta_3) + \frac{13}{1296}C_Fn_f^2) \end{aligned} \tag{11}$$

Note that both Z_A and Z_5 are gauge independent quantities. Now we can use eq. 5 with the axial current $\tilde{A}^a_\mu$ to calculate the Gross-Llewellyn Smith sum rule

$$\int_0^1 dx\; F_3^{\bar{\nu}p+\nu p} = 6\,C^{(V)}(\frac{\mu^2}{Q^2}, \alpha_S(\mu^2/\Lambda^2)) = 6\,C^{(V)}(1, \alpha_S(Q^2/\Lambda^2)) \tag{12}$$

For the actual calculation we have to consider now 347 + 6 diagrams of the types shown in figure 1. Through three loops the propagator type massless diagrams can be analytically calculated by means of the method outlined in [16]. This method is at its best when computerized symbolic manipulation is used. First of all each diagram has a nontrivial trace (or gluon vertex) structure. This nontriviality is enhanced by the introduction of the gauge parameter ξ in the gluon propagator which then becomes $(g_{\mu\nu} - \xi\frac{q_\mu q_\nu}{q^2})/q^2$. We used this form throughout to have some extra tests on the results. Each individual diagram depends on ξ, but the sum of all diagrams should be independent of it. After this there are three loops to be integrated over and each integral contains up to 8 different denominators. To simplify these integrals one can apply some recursion relations so that eventually each integral has at least one denominator fewer. There is one exception to this rule which is the most fundamental non-planar three loop integral. This integral is known so it causes no problems. The recursion can introduce many new terms. For many integrals it is then necessary to apply a second type of recursion to eliminate one more denominator. This gives then an enormous number of integrals each of which has a one loop subintegral that can be factored out. Sometimes the formula for a single diagram takes more than 10 megabytes during this procedure. The resulting two loop integrals can again be treated with some variations of the recursion relation after which it can be factored into two one loop integrals (and one "fundamental" two loop integral which is known to sufficient power in ϵ). The final answer of each diagram is a polynomial in ξ and (negative powers of) ϵ. The near equality of the results for the two sum rules was a powerful test. It will be commented on later. The computations were performed with the use of a set of procedures written in the language of the symbolic manipulation program Form [17]. The collective name of these procedures is Mincer [18] [19]. Even though these procedures have been optimized very much the total execution time of all computations in this paper was close to 100 hours CPU time on an Apollo DN10000 workstation.

We find for the Gross-Llewellyn Smith sum rule in the $\overline{MS}$-scheme

$$\begin{aligned}\int_0^1 dx\ F_3^{\bar{\nu}p+\nu p} &= 6(1 - \frac{3}{4}C_F(\frac{\alpha_s}{\pi}) + (\frac{\alpha_s}{\pi})^2 C_F(\frac{21}{32}C_F - \frac{23}{16}C_A + \frac{1}{4}n_f) \\ &+(\frac{\alpha_s}{\pi})^3(-\frac{3}{128}C_F^3 + C_F^2C_A(\frac{1241}{576} - \frac{11}{12}\zeta_3) \\ &+C_FC_A^2(-\frac{5437}{864} + \frac{55}{24}\zeta_5) + C_F^2n_f(-\frac{133}{1152} - \frac{5}{24}\zeta_3) \\ &+C_FC_An_f(\frac{3535}{1728} + \frac{3}{8}\zeta_3 - \frac{5}{12}\zeta_5) - \frac{115}{864}C_Fn_f^2 \\ &+n_f\frac{d^{abc}d^{abc}}{n_c}(-\frac{11}{192} + \frac{1}{8}\zeta_3))) \end{aligned} \tag{13}$$

For QCD the values of the Casimir operators are $C_F = \frac{4}{3}$ and $C_A = 3$. The number of

colors $n_c = 3$ and $d^{abc}d^{abc} = \frac{40}{3}$. This gives the formula

$$\begin{aligned}\int_0^1 dx\ F_3^{\bar{\nu}p+\nu p} &= 6(1-(\frac{\alpha_S}{\pi})+(\frac{\alpha_S}{\pi})^2(-\frac{55}{12}+\frac{1}{3}n_f)\\ &+(\frac{\alpha_S}{\pi})^3(-\frac{13841}{216}-\frac{44}{9}\zeta_3+\frac{55}{2}\zeta_5\\ &+n_f(\frac{10009}{1296}+\frac{91}{54}\zeta_3-\frac{5}{3}\zeta_5)-\frac{115}{648}n_f^2))\end{aligned} \tag{14}$$

Let us consider now the Bjorken sum rule for deep inelastic polarized electron-nucleon scattering. The hadronic tensor (1) has two more form factors, because now there is no spin averaging.

$$\Delta W_{\mu\nu} = i\varepsilon_{\mu\nu\rho\sigma}q_\rho(\frac{s_\sigma}{p\cdot q}g_1(x,Q^2)+\frac{s_\sigma p\cdot q - p_\sigma q\cdot s}{(p\cdot q)^2}g_2(x,Q^2)) \tag{15}$$

s_σ describes the spin of the hadron. Note that there are different normalizations used in the literature. The Bjorken polarized sum rule reads [1]:

$$\int_0^1 dx\ (g_1^{ep}(x,Q^2)-g_1^{en}(x,Q^2)) = \frac{1}{3}|\frac{g_A}{g_V}|C^{(A)} \tag{16}$$

in which g_A and g_V are the constants in neutron weak decay and $\frac{g_A}{g_V} = -1.26$. In QCD the first moment of the isospin nonsinglet structure function g_1^{ep-en} (eq. 16) is proportional to the coefficient function $C^{(A)}$ of the flavor nonsinglet axial vector current $\tilde{A}_\mu^a(0)$ in the following OPE

$$i\int dz\ e^{iqz}T\{V_\mu^a(z)V_\nu^b(0)\}\ \overset{Q^2\to\infty}{=}\ \varepsilon_{\mu\nu\rho\sigma}\frac{q_\sigma}{q^2}C^{(A)}(\frac{\mu^2}{Q^2},\alpha_S)d^{abc}\tilde{A}_\rho^c(0)+\ldots \tag{17}$$

Now the method as defined in [10] gives the formula for $C^{(A)}$:

$$\begin{aligned}d^{abc}C^{(A)} &= \frac{1}{216}\frac{1}{Z_5Z_A}R_{\overline{MS}}\int i\,dz\ e^{iqz}\\ &< 0\mid T\{\bar{\psi}(p)\gamma_{[\mu}\gamma_\nu\gamma_{\rho]}q_\rho t^c\psi(-p)V_\mu^a(z)V_\nu^b(0)\}\mid 0>|_{p=0}^{amputated}\end{aligned} \tag{18}$$

Note that now the infrared divergences that appear after the nullification of p are removed by the ultraviolet renormalization constant Z_A. Therefore the finiteness of $C^{(A)}$ is a strong check. This time only the diagrams of figure 1a give a contribution. Because we consider the coefficient function $C^{(A)}$ of the nonsinglet operator $\tilde{A}_\mu^a$ the diagrams of figure 1b don't contribute. Hence we obtain the Bjorken polarized sum rule

$$\int_0^1 dx\ g_1^{ep-en} = \frac{1}{3}|\frac{g_A}{g_V}|(1-\frac{3}{4}C_F(\frac{\alpha_S}{\pi})+(\frac{\alpha_S}{\pi})^2C_F(\frac{21}{32}C_F-\frac{23}{16}C_A+\frac{1}{4}n_f)$$

$$+(\frac{\alpha_S}{\pi})^3(-\frac{3}{128}C_F^3+C_F^2C_A(\frac{1241}{576}-\frac{11}{12}\zeta_3)$$
$$+C_FC_A^2(-\frac{5437}{864}+\frac{55}{24}\zeta_5)+C_F^2n_f(-\frac{133}{1152}-\frac{5}{24}\zeta_3)$$
$$+C_FC_An_f(\frac{3535}{1728}+\frac{3}{8}\zeta_3-\frac{5}{12}\zeta_5)-\frac{115}{864}C_Fn_f^2)) \qquad (19)$$

For the QCD values of the Casimir operators this gives the formula

$$\int_0^1 dx\ g_1^{ep-en} = \frac{1}{3}|\frac{g_A}{g_V}|(1-(\frac{\alpha_S}{\pi})+(\frac{\alpha_S}{\pi})^2(-\frac{55}{12}+\frac{1}{3}n_f)$$
$$+(\frac{\alpha_S}{\pi})^3(-\frac{13841}{216}-\frac{44}{9}\zeta_3+\frac{55}{2}\zeta_5$$
$$+n_f(\frac{10339}{1296}+\frac{61}{54}\zeta_3-\frac{5}{3}\zeta_5)-\frac{115}{648}n_f^2)) \qquad (20)$$

Note that the Gross-Llewellyn Smith sum rule and the Bjorken polarized sum rule coincide up to the terms proportional to $(d^{abc})^2$ which come from the diagrams in figure 1b. This isn't a simple coincidence. It is due to the axial Ward identity. In the massless limit and with the effectively restored anticommutativity of γ_5 (eq. 10) one can compare eq. 5 and eq. 18 and verify that up to the terms in $(d^{abc})^2$ one gets $C^{(A)}=C^{(V)}$.

The numerical values of the coefficients in the above results are functions of the number of active flavors. This depends of course on the energy range in which the measurements are done. We give the second and the third order coefficients for the various possibilities:

	Gross-Llewellyn Smith		Bjorken	
flavors	$(\frac{\alpha_S}{\pi})^2$	$(\frac{\alpha_S}{\pi})^3$	$(\frac{\alpha_S}{\pi})^2$	$(\frac{\alpha_S}{\pi})^3$
3	-3.5833	-18.9757	-3.5833	-20.2153
4	-3.2500	-12.1957	-3.2500	-13.8503
5	-2.9167	-5.7743	-2.9167	-7.8402
6	-2.5833	+0.2940	-2.5833	-2.1851

Let us present the obtained sum rules in a form suitable for extracting a value for Λ from a fit to experimental data. To this end we rewrite them in the effective scheme [20] where all higher order corrections are zero (sometimes this scheme is called "fastest apparent convergent"):

$$\int_0^1 dx\ F_3^{\bar{\nu}p+\nu p} = 6(1-\frac{1}{\pi}\alpha_{GLS}^{eff}(Q^2/(\Lambda_{GLS}^{eff})^2)) \qquad (21)$$

$$\int_0^1 dx\ g_1^{ep-en} = \frac{1}{3}|\frac{g_A}{g_V}|(1-\frac{1}{\pi}\alpha_{BjP}^{eff}(Q^2/(\Lambda_{BjP}^{eff})^2)) \qquad (22)$$

where the effective coupling constants are connected to α_S in the $\overline{MS}$-scheme by the rule

$$\frac{\alpha^{eff}}{\pi}=\frac{\alpha_S}{\pi}+r_1(\frac{\alpha_S}{\pi})^2+r_2(\frac{\alpha_S}{\pi})^3 \qquad (23)$$

in which r_1 and r_2 can be obtained by comparing the eqs. 21 and 22 with the eqs. 13 and 19 respectively. The running coupling constant obeys the standard renormalization group equation

$$\begin{aligned}\frac{\partial \alpha_S/\pi}{\partial ln\, Q^2} &= \beta(\frac{\alpha_S}{\pi}) \\ &= -\beta_0(\frac{\alpha_S}{\pi})^2 - \beta_1(\frac{\alpha_S}{\pi})^3 - \beta_2(\frac{\alpha_S}{\pi})^4 \\ &= -\beta_0(\frac{\alpha_S}{\pi})^2(1 + c_1(\frac{\alpha_S}{\pi}) + c_2(\frac{\alpha_S}{\pi})^2)\end{aligned} \tag{24}$$

The beta function in the MS-scheme has been calculated for QCD at the three loop level [21]:

$$\begin{aligned}\beta_0 &= \frac{1}{4}(11 - \frac{2}{3}n_f) \\ \beta_1 &= \frac{1}{16}(102 - \frac{38}{3}n_f) \\ \beta_2 &= \frac{1}{64}(\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2)\end{aligned} \tag{25}$$

The solution of eq. 24 for the effective coupling constant in the next-next-to-leading order has the standard form

$$\begin{aligned}\frac{\alpha_S^{eff}}{\pi} &= \frac{1}{\beta_0 ln(\frac{Q^2}{(\Lambda^{eff})^2})} - \frac{c_1}{\beta_0^2}\frac{ln\ ln(\frac{Q^2}{(\Lambda^{eff})^2})}{ln^2(\frac{Q^2}{(\Lambda^{eff})^2})} \\ &+ \frac{1}{\beta_0^3 ln^3(\frac{Q^2}{(\Lambda^{eff})^2})}(c_1^2 ln^2 ln(\frac{Q^2}{(\Lambda^{eff})^2}) - c_1^2 ln\ ln(\frac{Q^2}{(\Lambda^{eff})^2}) + c_2^{eff} - c_1^2)\end{aligned} \tag{26}$$

with

$$(\Lambda^{eff})^2 = \Lambda^2_{\overline{MS}} e^{r_1/\beta_0} \tag{27}$$

being renormalization scheme invariant but depending on the process. This relation is valid in all orders of perturbation theory [22]. The second coefficient of the effective beta function is given by $c_2^{eff} = c_2 + r_2 - c_1 r_1 - r_1^2$. The last expression is known to be renormalization scheme invariant [23] [24]. The effective quantities are again functions of the number of effective flavors. They are:

n_f	e^{r_1/β_0}	$c_{2_{GLS}}^{eff}$	$c_{2_{BLP}}^{eff}$
3	4.9165	4.2362	5.4757
4	4.7588	-0.3223	1.3304
5	4.5802	-4.9354	-2.8695
6	4.3762	-9.6566	-7.1775

An experimental fit can be made with the use of the eqs. 21 and 22. In these equations one has to substitute the eq. 26 first. The fit yields a value for Λ^{eff} which depends on the process. It can be converted into a process independent value for $\Lambda_{\overline{MS}}$ with the use of eq. 27.

The total effect of our three loop corrections to the sum rules can be quantified in two examples. Let us first take a typical fixed target experimental situation. We assume that the number of active flavors is three and that the value of α_S is about 0.3. The other example is more relevant for HERA. We assume that the number of active flavors is five and that the value of α_S is about 0.2. For high Q^2 data this value may be smaller. We define the relative coefficients a_1 and a_2 by the relation

$$\frac{1}{6}\int_0^1 dx\ F_3^{\bar{\nu}p+\nu p} - 1 = -\frac{\alpha_s}{\pi}(1 + a_1 + a_2 + \ldots) \tag{28}$$

with $a_1 = r_1\frac{\alpha_S}{\pi}$ and $a_2 = r_2(\frac{\alpha_S}{\pi})^2$. We can define similar quantities for the Bjorken sum rule. The numbers are

	a_1	a_2
GLS low energy	0.342	0.173
GLS high energy	0.186	0.023
BjP low energy	0.342	0.184
BjP high energy	0.186	0.032

If we assume that the error in the truncated asymptotic series is given by the value of the last calculated term we see that at low energies the error is of the order of 20% while at high energies it is about 2%. The difference between the two sum rules is of the order of 1%.

As we have seen before the only difference between the results for the Gross-Llewellyn Smith and the Bjorken polarized sum rules is due to a restricted class of three loop graphs. This means that if it would be observed, a difference between the experimental values for these sum rules would be an indication of the relative importance of higher order QCD corrections. In other words, if the physics is well described by the leading orders in α_S, the Q^2 dependence of both sum rules (one in neutrino-nucleon scattering and one in polarized electron-nucleon scattering) should be the same.

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The α_S^3 correction to the Bjorken sum rule

S.A. Larin [1], F.V. Tkachov [1]
and J.A.M. Vermaseren

NIKHEF-H, P.O. Box 41882,
1009 DB, Amsterdam

Abstract

We have computed analytically within the framework of perturbative QCD the $O(a_S^3)$ corrections to the Bjorken sum rule for deep inelastic neutrino nucleon scattering. For five active flavors the correction is about $8.49(\frac{a_S}{\pi})^3$ times the zeroth order contribution within the $\overline{MS}$ scheme.

[1] On leave from Institute for Nuclear Research (INR) of the USSR Academy of Sciences, Moscow 117312.

A perturbative QCD prediction for an observable should incorporate three terms: a leading QCD term (normally, $\mathcal{O}(\alpha_S)$) which describes the magnitude of the effect, the next-to-leading correction ($\mathcal{O}(\alpha_S^2)$) which determines the size of the effective Λ and thus the region of applicability of the result, and the next-next-to-leading term which can be assumed to describe the theoretical uncertainty of the prediction. While the leading and next-to-leading corrections are known for most of the interesting cases, the next-next-to-leading ones are not. This of course is due to considerable technical difficulties.

In the present paper we consider the Bjorken sum rule for deep inelastic neutrino-nucleon scattering [1], which is the simplest problem of this kind in the family of deep inelastic computations. The leading QCD correction to it was calculated in [2], and the next-to-leading one, in [3]. Higher twist contributions were studied in [4].

The Bjorken sum rule we wish to consider reads in the leading twist approximation:

$$\int_0^1 dx (F_1^{\nu p} - F_1^{\bar{\nu} p}) = C(\alpha_S(\frac{Q^2}{\Lambda^2}))$$

where C is proportional to the coefficient function of the flavor non-singlet spin-one quark operator $\bar{\psi}\gamma^\mu t^a \psi$ in the short-distance operator-product expansion for the corresponding amplitude, $\bar{\alpha}_S$ is the running coupling constant.

The state of the art in calculations of this kind is as follows. For intermediate regularization and UV renormalization the dimensional regularization [5] and the MS scheme [6] are used, respectively. Explicit expressions for the coefficient functions of operator product expansions are obtained using the recipe described in [7]. Explicit formulae for the Bjorken sum rule were presented in [3]. This recipe is backed by the general theory of euclidean asymptotic expansions [8].

The recipe of [7] reduces the problem to the calculation of propagator-type multiloop integrals. *Through three loops it has been shown that such integrals are analytically* calculable [9]. However, the corresponding expressions are very large and one has to use a computer algebra system. Hence the reduction of the integrals used the algorithms of the Mincer program as described in [10]. The algebraic system we used was a beta release of version 2 of the program Form [11]. The adaptation of the mincer program to Form will be described elsewhere [12].

In our case we had to consider 347 three loop W-boson-quark forward scattering diagrams. In the count of 347 diagrams each one and two loop gluon propagator insertion was counted as a single diagram only, and the diagrams with a zero color factor were not included at all. To ensure correctness of the result we used a gluon propagator of the form $g_{\mu\nu} - \xi q_\mu q_\nu / q \cdot q$. In the final answer the gauge parameter ξ has to cancel, which indeed it does. The complete computation took about 33 hours on an Apollo DN10000 workstation as compared to 137 sec for the two loop calculation. The largest intermediate expression for a single diagram was more than 40 Mbytes. This diagram took more than 2.5 hours of

computer time. Most diagrams are much simpler. Actually the most time consuming part of the calculation was the drawing and typing of the diagrams.

The result of the calculation is given in the $\overline{MS}$ scheme [2] by the formula

$$\int_0^1 dx F_1^{\gamma p,\,ep} = 1 - \frac{1}{2}C_F(\frac{\alpha_s}{\pi}) + (\frac{\alpha_s}{\pi})^2 C_F(\frac{11}{16}C_F - \frac{91}{72}C_A + \frac{2}{9}n_f)$$
$$+ (\frac{\alpha_s}{\pi})^3(C_F^3(-\frac{313}{64} - \frac{47}{4}\zeta_3 + \frac{35}{2}\zeta_5) + C_F^2 C_A(\frac{2731}{288} + \frac{91}{6}\zeta_3 - \frac{95}{4}\zeta_5)$$
$$+ C_F C_A^2(-\frac{8285}{1296} - \frac{5}{2}\zeta_3 + 5\zeta_5) + C_F^2 n_f(-\frac{335}{576} + \frac{1}{12}\zeta_3)$$
$$+ C_F C_A n_f(\frac{4235}{2592} - \frac{7}{12}\zeta_3 + \frac{5}{6}\zeta_5) - \frac{155}{1296}(C_F n_f^2))$$

ζ is the Riemann zeta function. C_F and C_A are the Casimir operators of the defining and the adjoint representation of the color group and n_f is the number of active flavors. For QCD the values are $C_F = \frac{4}{3}$ and $C_A = 3$. With these values the results are

$$\int_0^1 dx F_1^{\gamma p,\,ep} = 1 - \frac{2}{3}(\frac{\alpha_s}{\pi}) + (\frac{\alpha_s}{\pi})^2(-\frac{23}{6} + \frac{8}{27}n_f)$$
$$+ (\frac{\alpha_s}{\pi})^3(-\frac{4075}{108} + \frac{622}{27}\zeta_3 - \frac{680}{27}\zeta_5 + n_f(\frac{3565}{648} - \frac{59}{27}\zeta_3 + \frac{10}{3}\zeta_5) - \frac{155}{972}n_f^2)$$

At HERA energies with five active flavors the formula becomes

$$\int_0^1 dx F_1^{\gamma p,\,ep} = 1 - 0.6667\frac{\alpha_s}{\pi} - 2.3519(\frac{\alpha_s}{\pi})^2 - 8.1852(\frac{\alpha_s}{\pi})^3$$

In the low energy region with three active flavors the last two coefficients are -2.9111 and -18.5963 respectively. In the limit that there are six flavors the last two coefficients become -2.0556 and -3.9081 respectively.

It is clear that for values of α_s around 0.12 (the value at the mass of the Z) the series behaves rather well and the lower order terms describe the physics adequately if we assume that the error of the truncated asymptotic series is determined by the first discarded term.

From the technical point of view we have shown that calculations of this type are practically feasible and can be performed in a reasonable amount of time.

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