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Generalized Treatment of Point Reactor Kinetics Driven by Random Reactivity Fluctuations Via the Wiener-Hermite Functional Method

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Abstract

In a recent paper by Behringer et al. (1990), the Wiener-Hermite Functional (WHF) method has been applied to point reactor kinetics excited by Gaussian random reactivity noise under stationary conditions, in order to calculate the neutron steady-state value and the neutron power spectral density (PSD) in a second-order (WHF-2) approximation. For simplicity, delayed neutrons and any feedback effects have been disregarded. The present study is a straightforward continuation of the previous one, treating the problem more generally by including any number of delayed neutron groups. For the case of white reactivity noise, the accuracy of the approach is determined by comparison with the exact solution available from the Fokker-Planck method. In the numerical comparisons, the first-order (WHF-1) approximation of the PSD is also considered.

Keywords: Noise analysis, neutron noise, point reactor kinetics, Wiener–Hermite functional method.

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I Introduction

In a recent paper by Behringer et al. (1990), which we would like to cite as P1, the Wiener-Hermite Functional (WHF) method (Cameron and Martin, 1947; Wiener, 1958; Barrett, 1964; Imamura et al., 1965) was applied to point reactor kinetics excited by Gaussian random reactivity noise under stationary conditions. The study given in P1 may be regarded as a first assessment of the efficiency of the WHF method to solve approximately a parametric random excitation problem in neutron kinetics. For simplicity, delayed neutrons and any feedback effects have been neglected. The neutron steady-state value and the neutron power spectral density (PSD) have been calculated in a second-order (WHF-2) approximation. Two cases have been considered: In the first case the noise source has been assumed to be white, while in the second case the source was low-pass white noise. The accuracy of the approaches has been determined by comparison with exact solutions of the problem. The exact solution in the first case is available by applying either the Fokker-Planck (FP) method or the direct integration (DI) method to the stochastic differential equation. In the second case, the exact solution is obtainable only by the DI method.

Due to the promising results obtained by the WHF method, the present paper (P2) will be a straightforward continuation of P1 by including the delayed neutrons. We well generally derive and treat the system equations in the WHF-2 approximation for coloured Gaussian reactivity noise input, and determine the neutron steady-state value and the neutron PSD. But we have to emphasize the treatment for white noise input, because only in this case is an exact solution of the problem available from the FP method for comparison. The DI method is no longer applicable when delayed neutrons are present. In this comparison we will also consider the WHF-1 approximation and its application range.

In Section II, the stochastic model is explained. Stability criteria available from the literature for a single group of delayed neutrons under the white noise input assumption (Williams, 1969; Akcasu and Karasulu, 1976) are discussed. The application of the WHF method to our model in the WHF-2 approximation is given in Section III. The general treatment of the resulting system equations for coloured reactivity noise input is shifted to Appendix A, because we will not make explicit use of it in our present study. In the following two sections we concentrate on the case of white noise input. In Section IV the analytical results of the WHF-2 approximation and the exact solution following from the FP method are represented and compared. The comparison reveals that the neutron PSD obtained from the WHF-2 approximation is essentially a first-order expansion of the exact PSD by the stability parameter of the second neutron moment. A general derivation of this stability parameter from the FP equation for any number of delayed neutron groups is given in Appendix B. Numerical results and comparisons including the WHF-1 approximation of the PSD are represented in Section V. In these numerical evaluations, only one group of delayed neutrons is included for simplicity. The conclusions of the investigations are summarized in Section VI.

II The Stochastic Neutron Model

The point model equations of reactor kinetics with N groups of delayed neutrons may be written as

$$\dot{X}(t) = (F(t) - \beta_o)X(t) + \sum_{n=1}^{N} \lambda_n C_n(t) + S_o \tag{II.1a}$$

$$\dot{C}_n(t) = \beta_n X(t) - \lambda_n C_n(t); n = 1, \ldots, N \tag{II.1b}$$

$$\beta_o = \sum_{n=1}^{N} \beta_n \tag{II.1c}$$

$X(t)$ is the neutron density. It is the system response function of interest. $F(t)$ is the driving reactivity term and is the forcing function. It is expressed in units of the neutron generation time Λ. S_o is a time-independent constant source term. $C_n(t)$ is the precursor density of the delayed neutrons in group n. β_n is the fraction of the delayed neutrons in units of Λ, and λ_n is their decay constant in group n.

In the present study we assume that $F(t)$ is a random function in time. In particular, we assume that its fluctuations are stationary Gaussian random noise. We are, therefore, concerned with the problem of evaluating the stochastic properties of $X(t)$ from those given for $F(t)$. Here, the PSD of the fluctuating component of $X(t)$ is of special interest. We require a stationary solution of the system equations (II.1), and decompose $X(t)$, $C_n(t)$ and $F(t)$ into the steady-state values X_o, C_{on} and F_o, and the fluctuating components $x(t)$, $c_n(t)$ and $f(t)$, respectively, by

$$X(t) = X_o + x(t) \tag{II.2a}$$

$$C_n(t) = C_{on} + c_n(t); n = 1, \ldots, N \tag{II.2b}$$

$$F(t) = -F_o + f(t) \tag{II.2c}$$

with the conditions that the ensemble averages, denoted by the brackets $< \ldots >$ are

$$< x(t) >= 0 \tag{II.3a}$$

$$< c_n(t) >= 0; n = 1, \ldots, N \tag{II.3b}$$

$$< f(t) >= 0 \tag{II.3c}$$

Between X_o and C_{on} we have the relationship

$$C_{on} = \beta_n X_o / \lambda_n; n = 1, \ldots, N \tag{II.4}$$

F_o in equation (II.2c) is assumed to be a positive quantity and represents the amount of (static) reactivity by which the system is subcritical in the absence of a reactivity perturber. It might be recalled that there is no power reactor which is operated at a critical state in the strict sense. There is always an inherent neutron source present.

Inserting equations (II.2a, b, c) into equations (II.1a, b), using equation (II.4), and eliminating the precursor densities, leads to the stochastic integro-differential equation between the fluctuating components $x(t)$ and $f(t)$ alone:

$$\dot{x}(t) = -F_o X_o + S_o - (F_o + \beta_o)x(t) + \sum_{n=1}^{N} \beta_n \lambda_n \int_o^\infty du x(t-u) e^{-\lambda_n u} + X_o f(t) + f(t)x(t) \tag{II.5}$$

Equation (II.5) shows that the neutron kinetics model can be formulated as a single-input/single-output system. This form is convenient for the later application of the WHF method. Despite the fact that equation (II.5) is linear in the usual sense, there is the cross-term $f(t)x(t)$ which effects additionally a parametric (multiplicative) excitation. The approximative treatment of the contribution of this cross-term via the WHF method is the main goal of our study. The usual approach is to consider this cross-term as negligibly small. We then have the "linearized" solution for X_o and the PSD of $x(t)$, denoted by $X_{o(lin)}$ and $S_{xx(lin)}$:

$$X_{o(lin)} = S_o / F_o \tag{II.6}$$

$$S_{xx(lin)}(\omega) = X_{o(lin)}^2 \, |Q(\omega)|^2 \, S_{ff}(\omega) \tag{II.7}$$

S_{ff} is the PSD of $f(t)$, ω is the angular frequency, and Q is the reactivity transfer function.

$$Q(\omega) = \left[i\omega \left(1 + \sum_{n=1}^{N} \frac{\beta_n}{\lambda_n + i\omega} \right) + F_o \right]^{-1} \tag{II.8}$$

In this "linearized" approach, the Gaussian assumption for $f(t)$ is not required. A PSD does not contain information about the amplitude distribution characteristics of the noise.

If the cross-term $f(t)x(t)$ becomes larger, then, due to the nonlinear excitation effect appearing, $x(t)$ will not be Gaussian in any case, even if $f(t)$ is Gaussian. The Gaussian assumption for $f(t)$ is not a necessary condition for the approximative treatment of equation (II.5) by the WHF method. We maintain this assumption for the special case when $f(t)$ is Gaussian white noise. In this case we have available exact values for N_o and the PSD of $x(t)$ from the FP method for comparison. Furthermore, stability criteria can be established from the FP method. They are an important problem and can obviously not be extracted from an approximative method. It was shown in P1 that, in the absence of delayed neutrons, the stability of the exact PSD for low-pass white noise input is governed by the same stability condition as for white noise input, despite the fact that the PSD resulting from the WHF-2 approximation seems to be stable over a wider range. This leads to expect that in the present, more general treatment, which includes delayed neutrons, exact stability criteria based on the white noise input assumption may also be valid for coloured noise input. When one applies them, one should be on the safe side.

Williams (1969) has given an exact treatment of the system equations (II.1) by the FP method, including a single group of delayed neutrons ($N = 1$). He considered (generally nonstationary time-dependent) first and second moments under the assumption that $f(t)$ is discrete Gaussian white noise, i.e. the "derivative" of the jump-like Wiener process. This "derivative" noise may be characterized as a random sequence of Gaussian needle pulses (Dirac-Delta pulses with normally distributed random weights). This assumption is well justified with respect to the micro-stochastics occurring basically in a nuclear reactor. The treatment requires the formulation of the system equations (II.1) as stochastic difference equations rather than as differential equations (Akcasu and Karasulu, 1976). The model reproduces the ordinary reactor kinetic equations as ensemble averages. In power applications, however, the insertion or appearance of a random reactivity perturber is a macrostochastic phenomenon which will give rise to effects which are ordinarily several orders of magnitude larger than those due to microstochastic events. Thus, so far we interpret $X(t)$ and $C_n(t)$ in equation (II.1) as ensemble averages in the microstochastic sense. We follow the approach by Akcasu und Karasulu (1976) who treated the problem by the FP method more generally for any number of delayed neutron groups, in particular with respect to the FP method, Gaussian white noise of the continuous type. This assumption is also consistent with the treatment of equation (II.5) by the WHF method. When we briefly speak of white noise, then we mean continuous Gaussian white noise. The FP method can handle both the discrete and the continuous cases. In a formal respect, the macrostochastic FP equation is obtained from its microstochastic formulation simply by replacing F_o by $F_o - \epsilon^2/2$, where ϵ^2 is the PSD of the white noise input ($S_{ff}(\omega) = \epsilon^2$) (Williams, 1974). We introduce the quantity F_1, where

$$F_1 = F_o - \epsilon^2/2 \tag{II.9}$$

As a first consequence, we have for the exact neutron steady-state value, instead of equation (II.6),

$$X_{o(\text{exact})} = S_o/F_1 \tag{II.10}$$

which indicates that the insertion of a white noise reactivity perturber lowers the given subcritical reactivity F_o and increases the reactor power. This result is valid for any number of delayed neutron groups and is independent of delayed neutron parameters. Under stationary conditions, the delayed neutrons act simply as a stationary bypass in the neutron field, and in the exciting white noise all frequency components are uniformly distributed. However, equation (II.10) is expected to change when the reactivity noise is coloured. From equation (II.10), we immediately obtain the stability condition for the first neutron moment:

$$F_1 > 0 \tag{II.11}$$

Williams (1969) formulated, for $N = 1$, the stability conditions for the second moments $< X^2 >$, $< XC >$ and $< C^2 >$ by a cubic eigenvalue equation. Among the three conditions for stability there is one stringent condition. We found that the two other conditions are automatically fulfilled if this stringent condition is satisfied. Hence, one has effectively one stability condition which is of the form in the modified version where F_o is replaced by F_1 and it is translated into our notation:

$$0 < \xi(N = 1) < 1 \tag{II.12}$$

with $\xi(N = 1)$ given by

$$\xi(N = 1) = \frac{\epsilon^2}{2} \frac{F_1 + \lambda}{F_1(F_1 + \beta_o + \lambda)} \tag{II.13}$$

$(\beta_1 = \beta_o; \lambda_1 = \lambda)$

By setting $\xi = 1$ we can define an upper bound $\epsilon^2_{\max}$:

$$\epsilon^2_{\max} = \frac{1}{2}\left[3F_o + \beta_o + 2\lambda - \sqrt{(F_o + \beta_o + 2\lambda)^2 - 4\beta_o F_o}\right] \tag{II.14}$$

$\epsilon^2_{\max}$ is always smaller than $2F_o$, which includes the stability of the first moment X_o. Fig. 1 shows a plot of $\epsilon^2_{\max}/F_o$ as a function of F_o/β_o (F_o in \$ units) for typical light water reactor (LWR) data. Since $\lambda \ll \beta_o, \epsilon^2_{\max}$ is very slightly below $2F_o$ as long as $F_o < \beta_o$. When the reactor is more subcritical, one can observe a transition region at $F_o \geq \beta_o$, where $\epsilon^2_{\max}$ drops rapidly, tending to the value F_o for $F_o \gg \beta_o$. In a strongly subcritical reactor, however, the delayed neutrons lose their importance relative to the

prompt neutrons. $\epsilon^2_{max} = F_o$ is the upper bound for the stability of the second moment $< X^2 >$ in the absence of delayed neutrons ($\beta_o = 0$). This case was considered in P1. The clear break of ϵ^2_{max} at 1 \$ subcriticality is an interesting feature, which is revealed here by neutron noise analysis. It is reminiscent of the opposite situation when a nuclear reactor is running out of control due to delayed neutrons at 1 \$ supercriticality.

Equation (II.14) allows the introduction of the excitation ratio

$$r = \epsilon^2/\epsilon^2_{max} \qquad (0 < r < 1) \tag{II.15}$$

This definition extends now to the inclusion of one group of delayed neutrons. In Section IV we shall generalize this definition and the quantity ξ for any arbitrary number of delayed neutron groups. Since ξ will govern the stability of the exact neutron PSD for white noise input, we will call ξ the stability parameter. (Higher moments will not be considered here). The stability of the PSD includes the stability of X_o. It will be shown that it is not necessary to consider the stability conditions for the second moments via a kinetic eigenvalue equation which is, of course, of a more general form. The question of stability reduces to an amplitude problem in the PSD of $x(t)$.

III System Equations in the WHF-2 Approximation

Under certain conditions, a noise record $g(t)$ of a stationary random process which in general may be non-Gaussian, can be modelled by the Volterra series

$$g(t) = \sum_{\nu=1}^{\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} dt_1 \ldots dt_\nu G^{(\nu)}(t - t_1, \ldots, t - t_\nu) H^{(\nu)}(t_1, \ldots, t_\nu) \tag{III.1}$$

assuming that $g(t)$ has zero-mean. The $G^{(\nu)}$'s are deterministic kernels which, under the assumption of process stationarity, depend only on time lags. They have to vanish for negative time arguments due to the postulate of causality. The higher order kernels are symmetric functions with respect to the exchange of the time arguments. The $H^{(\nu)}$'s are the Wiener-Hermite functionals (WHF's), based on the statistical properties of Gaussian white noise. They perform a complete orthogonal set in the statistical sense. The first term in the representation by equation (III.1) is the Gaussian part, while the remaining terms are the non-Gaussian contributions to $g(t)$. The expansion of a stochastic signal function into WHF's may be thought of as an expansion into "harmonics of the white noise" (Barrett, 1964).

The application of the WHF method to the analysis of our stochastic integro-differential equation (II.5) consists of expanding the forcing function $f(t)$ and the response function $x(t)$ in terms of the same WHF set.

$$f(t) = \int_{-\infty}^{+\infty} dt_1 F^{(1)}(t - t_1) H^{(1)}(t_1) \tag{III.2a}$$

$$x(t) = \int_{-\infty}^{+\infty} dt_1 X^{(1)}(t - t_1) H^{(1)}(t_1) + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dt_1 dt_2 X^{(2)}(t - t_1, t - t_2) H^{(2)}(t_1, t_2) \tag{III.2b}$$

Since the forcing function $f(t)$ is assumed to be Gaussian, there is only the Gaussian term with the given kernel $F^{(1)}$. The expansion of the response function $x(t)$ is truncated after the second term (the WHF-2 approximation).

In this Section, $f(t)$ must not necessarily be white noise. It can also be coloured Gaussian random noise, which is assumed to be obtained by passing white noise through a linear filter with the transfer function $\varphi(\omega)$, so that the PSD of $f(t)$ is given by

$$S_{ff}(\omega) = \epsilon^2 |\varphi(\omega)|^2 \tag{III.3}$$

φ may be regarded as a dimensionless function of frequency, normalized to unit gain at maximum magnitude.

The further treatment of equation (II.5) by the WHF method follows the scheme described in P1 and will not be repeated. We will give here only the resulting system of equations in the frequency domain, expressing all kernels by their Fourier transform, and writing $F^{(1)}(\omega) = \epsilon\varphi(\omega)$:

$$F_o X_o = S_o + \frac{\epsilon}{2\pi} \int_{-\infty}^{+\infty} d\omega \varphi^*(\omega) X^{(1)}(\omega) \tag{III.4}$$

$$X^{(1)}(\omega) = \epsilon Q(\omega) \left[X_o \varphi(\omega) + \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega' \varphi^*(\omega') X^{(2)}(\omega, \omega') \right] \tag{III.5}$$

$$X^{(2)}(\omega, \omega') = \frac{\epsilon}{2} Q(\omega + \omega') \left[\varphi(\omega) X^{(1)}(\omega') + \varphi(\omega') X^{(1)}(\omega) \right] \tag{III.6}$$

$Q(\omega)$ is defined by equation (II.8). The quantities to be determined are the constant X_o and the kernels $X^{(1)}(\omega)$ and $X^{(2)}(\omega, \omega')$.

When these quantities are determined, the PSD of $x(t)$ then follows in the WHF-2 approximation from

$$S_{xx(WHF-2)}(\omega) = S_{xx}^{(1)}(\omega) + S_{xx}^{(2)}(\omega) \tag{III.7}$$

where

$$S_{xx}^{(1)}(\omega) = \left|X^{(1)}(\omega)\right|^2 \tag{III.8}$$

$$S_{xx}^{(2)}(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega' \left|X^{(2)}(\omega', \omega - \omega')\right|^2 \tag{III.9}$$

Equations (III.4 - 6) reduce to the equations given in P1 for $\beta_o = 0$. If one inserts the kernel $X^{(2)}$ from equation (III.6) into equation (III.5), one obtains

$$X^{(1)}(\omega) = \epsilon \frac{Q(\omega)\varphi(\omega)}{A(\omega)}\left[X_o + \frac{\epsilon}{2\pi}\int_{-\infty}^{+\infty} d\omega' Q(\omega+\omega')\varphi^*(\omega')X^{(1)}(\omega')\right] \tag{III.10}$$

where $A(\omega)$ is defined by

$$A(\omega) = 1 - \frac{\epsilon^2}{2\pi}Q(\omega)\int_{-\infty}^{+\infty} d\omega' Q(\omega+\omega')\left|\varphi(\omega')\right|^2 \tag{III.11}$$

Equations (III.4) and (III.10) represent a coupled system of two Fredholm integral equations for the unknowns X_o and $X^{(1)}(\omega)$. As soon as $X^{(1)}(\omega)$ has been determined, $X^{(2)}(\omega, \omega')$ is also known, according to equation (III.6).

$Q(\omega)$ is a rational function of frequency and can be represented by a partial fraction series

$$Q(\omega) = \frac{\prod_{n=1}^{N}(\lambda_n + i\omega)}{\prod_{n=1}^{N+1}(\omega_n + i\omega)} = \sum_{n=1}^{N+1}\frac{q_n}{\omega_n + i\omega} \tag{III.12}$$

The q_n are real partial fraction coefficients. There are $N+1$ (negatively scaled) roots ω_n. It is wellknown in neutron kinetics (Glasstone and Edlund, 1952) that these roots are real positive quantities for any subcritical reactor state $F_o > 0$, and are different from each other with respect to the usual data of delayed neutron parameters. The roots follow from the equations

$$1 - \frac{F_o}{\omega_j} + \sum_{n=1}^{N}\frac{\beta_n}{\lambda_n - \omega_j} = 0; j = 1, \ldots, N+1 \tag{III.13}$$

and can be determined iteratively (Patry, 1962). When we consider Q in the Laplace transform as $Q(s)$ ($s = i\omega$), the partial fraction coefficients are obtained from

$$\begin{aligned}\frac{1}{q_j} &= \left[\frac{d}{ds}\frac{1}{Q(s)}\right]_{s=-\omega_j} \\ &= 1 + \sum_{n=1}^{N}\frac{\beta_n\lambda_n}{(\lambda_n - \omega_j)^2}\end{aligned} \tag{III.14}$$

There are a few useful relationships between the partial fraction coefficients and the roots, which can be derived from expressing Q in the time domain as the pulse response function $Q(t)$:

$$\left.\begin{aligned} Q(t) &= \textstyle\sum_{n=1}^{N+1} q_n e^{-\omega_n t} \quad ; \quad t \geq 0 \\ &= 0 \qquad\qquad\qquad\quad ; \quad t < 0 \end{aligned}\right\} \tag{III.15}$$

and using the initial value theorem and the end value theorem of the Laplace transform:

$$\begin{aligned} Q(t=0) &= \lim_{s\to\infty}[sQ(s)] \\ \to \sum_{n=1}^{N+1} q_n &= 1 \end{aligned} \tag{III.16}$$

$$\begin{aligned} \dot{Q}(t=0) &= \lim_{s\to\infty}[s(sQ(s) - Q(t=0))] \\ \to \sum_{n=1}^{N+1} \omega_n q_n &= F_o + \beta_o \end{aligned} \tag{III.17}$$

$$\begin{aligned} \int_0^\infty dt Q(t) &= \lim_{s\to 0}[Q(s)] \\ \to \sum_{n=1}^{N+1} q_n/\omega_n &= 1/F_o \end{aligned} \tag{III.18}$$

Equations (III.17) and (III.18) are in particular helpful as cross-checks in the numerical determination of the partial fraction coefficients and the roots.

The further treatment now requires explicit assumptions for $\varphi(\omega)$. In Section IV, following, the case of white noise input is considered and the resulting approximative neutron PSD is compared with the exact one, which is available from the FP method. The more general case of coloured noise input is treated in Appendix A. As will be shown there, it contains the white noise input as a limiting case. This limiting case, however, appears as a case of degeneration for which the results directly derived for white noise input give worthwile insight.

IV Analytical Results for White Noise Input

For white noise input we have $\varphi(\omega) = 1.A(\omega)$, defined by equation (III.11), reduces to

$$A(\omega) = 1 - \frac{\epsilon^2}{2} Q(\omega) \tag{IV.1}$$

We introduce the function $P(\omega)$ by

$$P(\omega) = \frac{Q(\omega)}{A(\omega)} = \frac{1}{\frac{1}{Q(\omega)} - \frac{\epsilon^2}{2}} = Q(\omega, F_o \rightarrow F_1) \tag{IV.2}$$

$P(\omega)$ has the same form as $Q(\omega)$, except that the reactivity value F_o is to be replaced by the lower value F_1. As long as $F_1 > 0$, which is the condition for the existence of the first moment $< X >= X_o$, we have the same representation of $P(\omega)$ by a partial fraction series as for $Q(\omega)$, but with other (real positive) roots and other partial fraction coefficients. We denote these new roots by Ω_n and the partial fraction coefficients by p_n. This gives the analogue relationships:

$$P(\omega) = \frac{\prod_{n=1}^{N}(\lambda_n + i\omega)}{\prod_{n=1}^{N+1}(\Omega_n + i\omega)} = \sum_{n=1}^{N+1} \frac{p_n}{\omega_n + i\omega} \tag{IV.3}$$

$$1 - \frac{F_1}{\Omega_j} + \sum_{n=1}^{N} \frac{\beta_n}{\lambda_n - \Omega_j} = 0; j = 1, \ldots, N+1 \tag{IV.4}$$

$$\frac{1}{p_j} = 1 + \sum_{n=1}^{N} \frac{\beta_n \lambda_n}{(\lambda_n - \Omega_j)^2} \tag{IV.5}$$

$$\sum_{n=1}^{N+1} p_n = 1 \tag{IV.6}$$

$$\sum_{n=1}^{N+1} \Omega_n p_n = F_1 + \beta_o \tag{IV.7}$$

$$\sum_{n=1}^{N+1} p_n/\Omega_n = 1/F_1 \tag{IV.8}$$

Using equations (IV.2) and (III.12), equation (III.10) can then be written as

$$X^{(1)}(\omega) = \epsilon P(\omega) \left[X_o + \frac{\epsilon}{2\pi} \sum_{n=1}^{N+1} q_n \Psi_n(\omega) \right] \tag{IV.9}$$

where $\Psi_n(\omega)$ is defined by

$$\Psi_n(\omega) = \int_{-\infty}^{+\infty} d\omega' \frac{X^{(1)}(\omega')}{\omega_n + i(\omega + \omega')}; n = 1, \ldots, N+1 \tag{IV.10}$$

For the determination of $\Psi_n(\omega)$ we perform the operation of dividing equation (IV.9) by $\omega_j + i(\omega" + \omega)$ and integrating over ω, thereby interchanging the order of integration:

$$\Psi_j(\omega") = \epsilon X_o I_j(\omega") + \frac{\epsilon^2}{2\pi} \sum_{n=1}^{N+1} q_n \int_{-\infty}^{+\infty} d\omega' X^{(1)}(\omega') J_{nj}(\omega', \omega");$$
$$j = 1, \ldots, N+1 \qquad \text{(IV.11)}$$

where

$$I_j(\omega") = \int_{-\infty}^{\infty} d\omega \frac{P(\omega)}{\omega_j + i(\omega" + \omega)} = 0 \qquad \text{(IV.12)}$$

$$J_{nj}(\omega', \omega") = \int_{-\infty}^{+\infty} d\omega \frac{P(\omega)}{[\omega_n + i(\omega' + \omega)][\omega_j + i(\omega" + \omega)]} = 0 \qquad \text{(IV.13)}$$

It follows that $\Psi_j(\omega") = 0$, which also implies that $\Psi_n(\omega) = 0$. Equation (IV.9) reduces to the form

$$X^{(1)}(\omega) = \epsilon X_o P(\omega) \qquad \text{(IV.14)}$$

For the determination of X_o we use equation (III.4) and obtain

$$X_{o(\text{WHF}-2)} = X_{o(\text{exact})} \qquad \text{(IV.15)}$$

where $X_{o(\text{exact})}$ is given by equation (II.10).

Using equations (III.6) and (IV.14) we obtain for the kernel $X^{(2)}$:

$$X^{(2)}(\omega, \omega') = \frac{\epsilon^2}{2} X_o Q(\omega + \omega') [P(\omega) + P(\omega')] \qquad \text{(IV.16)}$$

The neutron PSD follows from equation (III.7), where we have to insert the kernel $X^{(1)}$ from equation (IV.14) into equation (III.8), and the kernel $X^{(2)}$ from equation (IV.16) into equation (III.9). The second term $S_{xx}^{(2)}(\omega)$ can be represented as

$$S_{xx}^{(2)}(\omega) = \epsilon^2 X_o^2 \xi' |Q(\omega)|^2 \qquad \text{(IV.17)}$$

where we have introduced the quantity ξ' by

$$\xi' = \frac{\epsilon^2}{2\pi} \int_{-\infty}^{+\infty} d\omega |P(\omega)|^2 = \epsilon^2 \sum_{n=1}^{N+1} p_n \frac{\prod_{j=1}^{N} (\lambda_j + \Omega_n)}{\prod_{j=1}^{N+1} (\Omega_j + \Omega_n)} \qquad \text{(IV.18)}$$

We then obtain the approximative PSD in the form:

$$S_{xx(\mathrm{WHF-2})}(\omega) = \epsilon^2 X_o^2 |P(\omega)|^2 \left(1 + \xi' \frac{|Q(\omega)|^2}{|P(\omega)|^2}\right) \tag{IV.19}$$

We will now compare this approach with the exact solution resulting from the FP method. We state that the exact PSD can be represented by

$$S_{xx(\mathrm{exact})}(\omega) = \frac{\epsilon^2 X_o^2}{1-\xi} |P(\omega)|^2 \tag{IV.20}$$

This representation exhibits a new result. ξ is the general stability parameter which we have to derive from the FP equation. Furthermore, we state that ξ' (equation (IV.18)) = ξ. The PSD in the WHF-2 approximation is then essentially the first-order expansion of the exact PSD by the stability parameter ξ (when $|Q(\omega)|$ and $|P(\omega)|$ are of the same order of magnitude).

Williams (1969) has derived the neutron autocovariance function (ACOF) by the FP method, considering a single group of delayed neutrons ($N = 1, \beta_1 = \beta_o, \lambda_1 = \lambda$) and discrete white noise input. The PSD is obtained from the Fourier transform of the ACOF. In the modified version (F_o replaced by F_1) and in our notation it becomes

$$S_{xx(\mathrm{exact})}(\omega) = \frac{\epsilon^2 X_o^2}{(1-\xi)} \frac{\lambda^2 + \omega^2}{(\Omega_1^2 + \omega^2)(\Omega_2^2 + \omega^2)} \tag{IV.21}$$

where ξ is given by equation (II.13). Ω_1 and Ω_2 follow explicitly from the equation

$$\Omega^2 - (F_1 + \beta_o + \lambda)\Omega + \lambda F_1 = 0 \tag{IV.22}$$

When one evaluates ξ' from equation (IV.18), using equations (IV.6) and (IV.8) and the relationships of the roots $\Omega_1 + \Omega_2 = F_1 + \beta_o + \lambda, \Omega_1\Omega_2 = \lambda F_1$, one finds that $\xi'(N = 1) = \xi(N = 1)$ exactly. Obviously, this is also true in the absence of delayed neutrons ($\beta_o = 0$) : $\xi'(N = 0) = \xi(N = 0)$.

Akcasu and Karasulu (1976) derived the PSD via the FP method in the more general form, considering any number of delayed neutron groups and continuous white noise input, but they gave the PSD only in relative units. The spectral dependence of the exact PSD, given by the function $|P(\omega)|^2$, is correctly represented. With regard to the amplitude, we integrate equation (IV.20) and get the variance. Under the assumption that $\xi' = \xi$, we can relate the second neutron moment to the first one by

$$< X^2 >= \frac{X_o^2}{1-\xi} \tag{IV.23}$$

In Appendix B it is shown that equation (IV.23) can be obtained in this form from the FP equation. ξ appears as

$$\xi = \frac{\epsilon^2}{2F_1} \cdot \frac{D}{\left(1 + \sum_{j=1}^{N} \frac{\beta_j}{\lambda_j}\right) D - F_1 \sum_{j=1}^{N} D^{(j)}} \tag{IV.24}$$

where D and $D^{(j)}$ are determinants with $N * N$ elements.

$$d_{ii} = F_1 + \lambda_i \left(1 + \sum_{n \neq i} \frac{\beta_n}{\lambda_i + \lambda_n}\right) \tag{IV.25a}$$

$$d_{ik} = -\beta_i \frac{\lambda_k}{\lambda_i + \lambda_k}; i \neq k \tag{IV.25b}$$

$$d_{ij}^{(j)} = \frac{\beta_i}{\lambda_i} \tag{IV.25c}$$

$$d_{ik}^{(j)} = d_{ik}; k \neq j \tag{IV.25d}$$

Equation (IV.24) correctly gives, for $N = 0$ ($\beta_j = 0$), $\xi = \epsilon^2/2F_1$. For $N = 1$, equation (II.13) is reproduced. This is a rather simple case, because only the second moments $< X^2 >, < XC >$, and $< C^2 >$ are involved (apart from the first moments). In the next case, $N = 2$, the cross-moment $< C_1 C_2 >$ also contributes. The evaluation of ξ in this case gives explicitly

$$\xi(N = 2) = \frac{\epsilon^2}{2F_1} \frac{\lambda_1(F_1 + \lambda_2)(F_1 + \beta_2 + \lambda_1) + \lambda_2(F_1 + \lambda_1)(F_1 + \beta_1 + \lambda_2)}{\lambda_1(F_1 + \beta_2 + \lambda_2)(F_1 + \beta_o + \lambda_1) + \lambda_2(F_1 + \beta_1 + \lambda_1)(F_1 + \beta_o + \lambda_2)} \tag{IV.26}$$

Equation (IV.26) covers the case $N = 0$ as well. It is symmetric with regard to the exchange of the two groups of delayed neutrons, and reduces to equation (II.13) for $N = 1$ either, by setting $\beta_1 = 0$ or $\beta_2 = 0$, or by letting the two groups concide ($\lambda_1 = \lambda_2$). On the other hand, the evaluation of equation (IV.18), using equations (IV.6 - 8) and the relationships between the 3 roots $\Omega_1, \Omega_2, \Omega_3$ and the coefficients of the corresponding cubic polynomial, leads to $\xi'(N = 2) = \xi(N = 2)$.

Since explicitly $\xi' = \xi$ for $N = 0, 1$ and 2, we inductively conjecture that this relation is also generally true for any number $N > 2$. Equations (IV.18) and (IV.24) can be considered as two equivalent representations of the stability parameter. We therefore generalize the stability condition for the second neutron moment by

$$0 < \xi < 1 \tag{IV.27}$$

excluding the lower limit to be zero since in this case the excitation is zero and the PSD vanishes. We can also generalize the definition of the excitation ratio introduced by equation (II.15) for any number of delayed neutron groups. ϵ^2_{max} is the smallest positive value which results from setting $\xi = 1$. Condition (IV.27) must in general include $F_1 > 0$. Our process is weakly ergodic. $X_o^2 \leq < X^2 >$ follows from the positive definiteness of the process probability density function.

V Numerical Comparisons

In our numerical investigations, under the white noise input assumption we will also include the WHF-1 approximation. The WHF-1 approximation is obtained by disregarding equation (III.6) and the kernel $X^{(2)}$ in equation (III.5). We have (with $\varphi(\omega) = 1$):

$$X^{(1)}_{(\mathrm{WHF}-1)}(\omega) = \epsilon X_o Q(\omega) \tag{V.1}$$

From equation (III.4) we obtain for X_o

$$X_{o(\mathrm{WHF}-1)} = X_{o(\mathrm{exact})} \tag{V.2}$$

The neutron steady-state value in addition corresponds in the WHF-1 approximation to the exact value given by equation (II.10). For the PSD of the neutron response we have simply

$$S_{xx(\mathrm{WHF}-1)}(\omega) = \epsilon^2 X_o^2 |Q(\omega)|^2 \tag{V.3}$$

The frequency dependence of the PSD in the WHF-1 approximation is the same as of the "linearized" result given by equation (II.7). The difference lies only in the amplitude. $X^{(1)}_{(\mathrm{WHF}-1)}(\omega)$ is different from $X^{(1)}_{(\mathrm{WHF}-2)}(\omega)$. It should be noted that, in contrast to the characteristics of classical perturbation theory (Poincaré), the approximative treatment of nonlinear stochastic equations by the WHF method requires recalculation of all terms when one goes a step further on to the next higher-order approximation. Here is the special result that $X_{o(\mathrm{WHF}-1)} = X_{o(\mathrm{WHF}-2)}$.

For simplicity, we restrict the numerical work to the inclusion of one group of delayed neutrons only. We use typical LWR parameter data, $\beta_o = 100 sec^{-1}, \lambda = 8x10^{-2} sec^{-1}$, assuming a neutron generation time of $\Lambda = 8x10^{-5} sec$. The neutron generation time remains, to a good approximation, constant when the reactivity changes are accomplished by changes in absorption. The value of λ is an effective one, obtained from condensing the usual 6 groups of delayed neutrons into one. The considered frequency range extends ordinarily from $10^{-4} rad/sec$, which covers sufficiently the frequency region where the

delayed neutron effects are dominant, up to about 5 times of $F_o + \beta_o$ which is the break point of the plateau in $|Q(\omega)|$.

The accuracies of the PSD's of the WHF-1 and WHF-2 approximations are characterized by the relative deviation

$$\Delta_{1,2}(\omega) = (S_{xx(\text{WHF}-1,2)}(\omega) - S_{xx(\text{exact})}(\omega))/S_{xx(\text{exact})}(\omega) \qquad \text{(V.4)}$$

for given values F_o and r. $\Delta_{1,2}(\omega)$ is always negative for all frequencies, with the general inequalities

$$S_{xx(\text{WHF}-1)}(\omega) < S_{xx(\text{WHF}-2)}(\omega) < S_{xx(\text{exact})}(\omega) \qquad \text{(V.5)}$$

Fig. 2 shows plots of the normalized PSD's (NPSD = PSD/X_o^2) of the exact solution and of the WHF-2 approximation at 1 \$ subcriticality. The curves refer to three excitation ratio values, $r = 20, 40$ and 60%. The WHF-2 results exhibit a very similar approximation behaviour to that given in P1 for the same excitation ratio values, but neglecting the delayed neutrons. In Fig. 3, $\Delta_2(\omega)$ is plotted for the values $r = 20, 30$ and 40%, at 1 \$ and 10 cts subriticalities. If one accepts the allowance for a maximum absolute deviation of 10 % from the exact PSD, then r should not exceed the value of 30 %, as shown by the curves denoted by a. This statement refers to a subcritical state of 1 \$, which is the worst case in the WHF-2 approximation accuracy. The value of ξ (equation (II.13)) is, in this case, a maximum of about 17.6 %. If the reactor is more subcritical, so that the delayed neutrons become ineffective, the 10 % level of accuracy is reached for r values tending to 35 %, which is the result obtained in P1 for $\beta_o = 0$. On the other hand, when the reactor is subcritical below 1 \$, the approximation accuracy improves relatively. This is shown by the curves denoted by b for 10 cts subcriticality and the same values of r. For subcritical states $F_o/\beta_o \leq 1$ ct, $|\Delta_2(\omega)|$ becomes of the order of magnitude of $^o/_{oo}$ and less for $r \leq 30\%$. The reason lies in the behaviour of ξ. For very weakly subcritical states, equation (II.13) is roughly given by

$$\xi(F_o \ll \beta_o) \cong r(F_o(1-r) + \lambda)/(\beta_o(1-r)) \qquad \text{(V.6)}$$

In order to reach values for ξ of, let us say, about 10 %, extremely high excitation ratio values of the order of magnitude of 90 % or even higher are needed.

The WHF-1 approximation of the PSD shows more or less similar behaviour, but at a much lower admissible range of excitation. In Fig. 4, plots of $\Delta_1(\omega)$ are represented for $r = 2, 4$ and 6%, and subcriticalities at 1 \$, 10 cts and 0.1 cts. The worst case in the approximation accuracy also occurs here, at 1 \$ subcriticality (curves denoted by a). At the level of 10 % accuracy, the value of r should be less than about 4 %. For a less subcritical reactor, the approximation accuracy also improves relatively, but more slowly than in the case of the WHF-2 approximation. There is still an appreciable deviation in

the low frequency region at 0.1 cts subcriticality (curves denoted by c), which is no longer perceptible in $\Delta_2(\omega)$. It is questionable whether one can permit values of r higher than 4 % in the most interesting range of very weakly subcritical states, because the PSD in the WHF-1 approximation is much easier obtainable than in the WHF-2 approximation. $\Delta_1(\omega)$ shows in this case a significant missmatch in the form of peaking at zero frequency. This can be immediately recognized from

$$\Delta_1(\omega = 0; F_o \ll \beta_o) \cong -r(2-r); r < 0.8 \qquad (V.7)$$

Such peaking does not occur significantly in $\Delta_2(\omega)$ at zero frequency.

It should be noted that in our concept the reactivity value refers to the reactor state in the absence of the random reactivity perturber. It may happen in practical experiments that the reactivity loss introduced by the insertion of the reactivity perturber, which increases the reactor power, is partially or fully compensated by control rod movements. Our considerations are independent of such a boundary condition. However, such a boundary condition will alter the parameteric representation of the PSD's.

VI Conclusions

The application of the WHF method to approximately solving a closure problem in neutron kinetics, when the neutron fluctuations are excited by a parametrically (multiplicatively) acting coloured Gaussian random reactivity perturber, reveals the interesting feature that the required kernels, from which the neutron steady-state value and the neutron PSD are derived, can be obtained through linear relationships. In the previous paper (P1), and in the present paper (P2), we have followed the concept of selecting as many cases as possible where an exact solution of the problem is available for determining the accuracy of the approaches and for establishing system stability criteria. When delayed neutrons are present, an exact solution is only obtainable by the application of the FP method with white noise input. The white noise input may be considered as a special case, because this noise is idealized. On the other hand, it is the most general noise. In this case, allowing 10 % maximum deviation, the worst case analysis (at 1 $ subcriticality) shows that the PSD in the WHF-1 approximation is well represented for excitation ratios of up to 4 %. In the WHF-2 approximation, the admissible range of the excitation ratio extends up to 30 %. These results are based on numerical calculations with one group of delayed neutrons. The PSD in the WHF-1 approximation corresponds essentially to the approach of linearizing the system equations. The PSD in the WHF-2 approximation appears to be mainly the first-order expansion of the exact PSD by the stability parameter for the second neutron moment. In both approximations the neutron steady-state value is given by the exact one. Our more general investigations, by including the delayed neutrons, reveal again that the WHF method is a powerful approximative tool. Its application potential lies here in modelling the neutron PSD in the WHF-2 approximation, when a strong non-white

reactivity noise source is present for which the Gaussian assumption may be sufficient. For such cases, the use of the given general stability criterion and the introduced excitation ratio, both based on the white noise input assumption, is suggested as a guide line.

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APPENDIX A:
General Solution of the WHF-2 System Equations

We have essentially to solve the coupled system of the integral equations (III.4) and (III.10) for the unknown constant X_o and the unknown kernel $X^{(1)}(\omega)$:

$$F_o X_o = S_o + \frac{\epsilon}{2\pi} \int_{-\infty}^{+\infty} d\omega \varphi^*(\omega) X^{(1)}(\omega) \tag{A.1}$$

$$X^{(1)}(\omega) = \epsilon \frac{Q(\omega)\varphi(\omega)}{A(\omega)} \left[X_o + \frac{\epsilon}{2\pi} \int_{-\infty}^{+\infty} d\omega' Q(\omega + \omega') \varphi^*(\omega') X^{(1)}(\omega') \right] \tag{A.2}$$

where $A(\omega)$ is given by equation (III.11) as

$$A(\omega) = 1 - \frac{\epsilon^2}{2\pi} Q(\omega) \int_{-\infty}^{+\infty} d\omega' Q(\omega + \omega') |\varphi(\omega')|^2 \tag{A.3}$$

We assume that the dimensionless filter function $\varphi(\omega)$, throug which we generate coloured reactivity noise from white noise, is a rational function of $i\omega$ with a denominator polynomial of degree M ($M \geq 1$) and a nominator polynomial of degree M', $0 \leq M' \leq M - 1$. $\varphi(\omega)$ can then be represented by the partial fraction series

$$\varphi(\omega) = \sum_{m=1}^{M} a_m / (\alpha_m + i\omega) \tag{A.4}$$

The coefficients a_m are real quantities. The α_m must be real positive quantities to have a stable filter. They are assumed to be different from each other. A multiple pole can always be separated into different single poles clustered around its central value, so that $\varphi(\omega)$ is approximated with sufficient accuracy. It is convenient to introduce the normalization

$$|\varphi(\omega_o)| = 1 \tag{A.5}$$

where ω_o is the frequency point where the filter function has maximum gain. At that frequency the PSD of $f(t)$ then has the value $S_{ff}(\omega_o) = \epsilon^2$. Equation (A.5) is a suggestion which comes from the case of low-pass white noise input considered in P1, in order to avoid renormalization of the excitation parameter with respect to system stability examination.

The representation of $\varphi(\omega)$ by equation (A.4) does not include the case of pure high-pass filtering, but high-pass filtering in combination with low-pass filtering, e.g. the simple band-pass filter

$$\begin{aligned}\varphi(\omega) &= \frac{i\omega(\omega_L+\omega_H)}{(\omega_L+i\omega)(\omega_H+i\omega)} \\ &= \frac{\omega_L+\omega_H}{\omega_H-\omega_L}\left(-\frac{\omega_L}{\omega_L+i\omega}+\frac{\omega_H}{\omega_H+i\omega}\right);\omega_H>\omega_L \end{aligned} \tag{A.6}$$

where ω_L is the low-cutoff frequency and ω_H is the high-cutoff frequency. In modelling of reactivity noise, mainly low-pass filtering is required. Power reactivity noise is mostly dominant in the region of very low frequencies, where delayed neutrons affect the system behaviour.

Our treatment must contain the case of white noise input and reproduce the results outlined in Section IV. As a test procedure one can insert the low-pass filter ($M = 1$)

$$\varphi(\omega) = \frac{\omega_H}{\omega_H+i\omega} \tag{A.7}$$

and let $\omega_H \to \infty$.

The insertion of equation (A.4) into equation (A.3) leads to

$$A(\omega) = 1-\epsilon^2 Q(\omega)\sum_{n=1}^{N+1}\sum_{m=1}^{M}\varphi(s=\alpha_m)\frac{q_n a_m}{\omega_n+\alpha_m+i\omega} \tag{A.8}$$

where we represent $Q(\omega)$ by the partial fraction series of equation (III.12) and use the notation of φ in the Laplace transform as $\varphi(s)$ ($s = i\omega$). We introduce the function $G(\omega)$ by

$$G(\omega) = \frac{Q(\omega)}{A(\omega)} = \left[i\omega\left(1+\sum_{n=1}^{N}\frac{\beta_n}{\lambda_n+i\omega}\right)+F_o-\epsilon^2\sum_{n=1}^{N+1}\sum_{m=1}^{M}\varphi(s=\alpha_m)\frac{q_n a_m}{\omega_n+\alpha_m+i\omega}\right]^{-1} \tag{A.9}$$

$G(\omega)$ can be regarded as an extended reactivity transfer function. Low-pass filtered excitation, in particular, lowers the break characteristics being stronger for low frequencies than for high frequencies.

There are $(N+1)(M+1)$ poles. The (negatively scaled) roots Γ_j follow from

$$1-\frac{F_o}{\Gamma_j}+\sum_{n=1}^{N}\frac{\beta_n}{\lambda_n-\Gamma_j} + \frac{\epsilon^2}{\Gamma_j}\sum_{n=1}^{N+1}\sum_{m=1}^{M}\varphi(s=\alpha_m)\frac{q_n a_m}{\omega_n+\alpha_m-\Gamma_j} = 0; \quad j=1,\ldots,(N+1)(M+1) \tag{A.10}$$

For a physically acceptable solution, all Γ's (including possible multiple roots) must have real positive values. This condition, however, depends on the value of ϵ^2. We then have the representation of $G(\omega)$ by the partial fraction series

$$G(\omega) = \sum_{n=1}^{(N+1)(M+1)} \frac{g_n}{\Gamma_n + i\omega} \tag{A.11}$$

The partial fraction coefficients are real. For simplicity, the roots are assumed to be different from each other.

$$\frac{1}{g_j} = 1 + \sum_{n=1}^{N} \frac{\beta_n \lambda_n}{(\lambda_n - \Gamma_j)^2} + \epsilon^2 \sum_{n=1}^{N+1} \sum_{m=1}^{M} \varphi(s = \alpha_m) \frac{q_n a_m}{(\omega_n + \alpha_m - \Gamma_j)^2};$$
$$j = 1, \ldots, (N+1)(M+1) \tag{A.12}$$

$$\sum_{n=1}^{(N+1)(M+1)} g_n = 1 \tag{A.13}$$

Furthermore, one can establish similar relationships between the partial fraction coefficients and the roots as given by equations (IV.7) and (IV.8). One has to note that, in the limiting case of white noise input, equation (A.11) degenerates to a series of $(N+1)$ terms.

We can write

$$Q(\omega + \omega')\varphi^*(\omega) = \sum_{M=1}^{N+1} \sum_{m=1}^{M} \frac{q_n a_m}{\omega_n + \alpha_m + i\omega} \left[\frac{1}{\omega_n + i(\omega + \omega')} + \frac{1}{\alpha_m - i\omega'} \right] \tag{A.14}$$

Equation (A.2) can then be brought into the form

$$\begin{aligned} X^{(1)}(\omega) &= \epsilon X_o G(\omega)\varphi(\omega) \left[1 + \frac{\epsilon^2}{2} \sum_{n=1}^{N+1} \sum_{m=1}^{M} \frac{q_n a_m \eta_m}{\omega_n + \alpha_m + i\omega} \right] \\ &\quad + \frac{\epsilon^2}{2\pi} G(\omega)\varphi(\omega) \sum_{n=1}^{N+1} \sum_{m=1}^{M} \frac{q_n a_m}{\omega_n + \alpha_m + i\omega} \Psi_n(\omega) \end{aligned} \tag{A.15}$$

where

$$\eta_m = \frac{1}{\epsilon X_o \pi} \int_{-\infty}^{+\infty} d\omega' \frac{X^{(1)}(\omega')}{\alpha_m - i\omega'}; m = 1, \ldots, M \tag{A.16}$$

$$\Psi_n(\omega) = \int_{-\infty}^{+\infty} d\omega' \frac{X^{(1)}(\omega')}{\omega_n + i(\omega + \omega')} = 0; n = 1, \ldots, N+1 \qquad \text{(A.17)}$$

The procedure for proving the validity of equation (A.17) is the same as outlined for equation (IV.9). $X^{(1)}(\omega)$ reduces to

$$X^{(1)}(\omega) = \epsilon X_o G(\omega)\varphi(\omega)\left[1 + \frac{\epsilon^2}{2}\sum_{n=1}^{N+1}\sum_{m=1}^{M}\frac{q_n a_m \eta_m}{\omega_n + \alpha_m + i\omega}\right] \qquad \text{(A.18)}$$

In order to determine the values η_m, we divide equation (A.18) by $\epsilon X_o \pi(\alpha_k - i\omega)$ and integrate over ω, thereby using the partial fraction representations of $G(\omega)$ and $\varphi(\omega)$, and equation (A.16). We obtain the system of M equations

$$\begin{aligned} \eta_k &= 2G(s = \alpha_k)\varphi(s = \alpha_k)\left[1 + \frac{\epsilon^2}{2}\sum_{m=1}^{M} Q(s = \alpha_m + \alpha_k)a_m\eta_m\right]; \\ & \quad k = 1, \ldots, M \end{aligned} \qquad \text{(A.19)}$$

where we additionally introduce the notation of the functions Q and G in the Laplace transform. When these M values have been determined, we can determine X_o using equation (A.1), again with the partial fraction representation of $\varphi(\omega)$ (equation (A.4)) and equation (A.16).

$$X_{o(\mathrm{WHF-2})} = S_o / \left(F_o - \frac{\epsilon^2}{2}\sum_{m=1}^{M} a_m \eta_m\right) \qquad \text{(A.20)}$$

The kernel $X^{(1)}(\omega)$ of equation (A.18) is then completely determined by equations (A.19) and (A.20). For the case of low-pass white noise input with the filter function of equation (A.7) and the absence of delayed neutrons ($N = 0, \beta_o = 0$), the formulae given in P1 for X_o and $X^{(1)}(\omega)$ are reproduced. When we consider the limiting case of white noise input, using the filter function of equation (A.7) and letting $\omega_H \to \infty$, for any number of delayed neutron groups we have: $G(\omega) \to P(\omega)$ (equation (IV.2)), $\eta_1 \to 0$, and $a_1\eta_1 \to 1$, reproducing equations (IV.14) and (IV.15).

The kernel $X^{(2)}$ is given by the algebraic equation (III.6) with the kernel $X^{(1)}$ from equation (A.18). With regard to the determination of the PSD, the integration to be performed to obtain $S_{xx}^{(2)}(\omega)$ from equation (III.9) can, in principle, be treated analytically by applying partial fraction techniques, but it is a tedious procedure. We suggest numerical integration (such as using the IMSL routing QDAGI).

APPENDIX B:
Determination of the Parameter ξ from the Fokker-Planck Equation

Using the Fokker-Planck equation, as given by Akcasu and Karasulu (1976), for any number N of delayed neutron groups, the time-dependent conditional second moments $< X^2 >, < XC_j >$ and $< C_iC_k >$ at time $t > 0$, provided they have given values at time $t = 0$, follow in our notation from the equations:

$$\frac{d}{dt} < X^2 > = 2\left[-(F_1 + \beta_o - \frac{\epsilon^2}{2}) < X^2 > + \sum_{n=1}^{N} \lambda_n < XC_n > + S_o < X >\right] \quad \text{(B.1)}$$

$$\frac{d}{dt} < XC_j > = \beta_j < X^2 > -(F_1 + \beta_o + \lambda_j) < XC_j > + \sum_{n=1}^{N} \lambda_n < C_nC_j > + S_o < C_j >; j = 1, \ldots, N \quad \text{(B.2)}$$

$$\frac{d}{dt} < C_iC_k > = -(\lambda_i + \lambda_k) < C_iC_k > + \beta_i < XC_k > + \beta_k < XC > ; \quad i, k = 1, \ldots, N \quad \text{(B.3)}$$

Under stationary conditions, the time derivatives on the left-hand side of these equations must be zero. Inserting the auto- and cross-terms $< C_iC_k >$ from equation (B.3) into equation (B.2), observing that under stationary conditions $< X >= X_o$ and $< C_j >= \beta_j X_o/\lambda_j$, and expressing the source S_o by F_1X_o, we obtain the following system of $N+1$ equations after some algebraic manipulation:

$$\left(F_1 - \frac{\epsilon^2}{2}\right) < X^2 > + F_1 \sum_{n=1}^{N} < XC_n >= F_1X_o^2 \left(1 + \sum_{n=1}^{N} \frac{\beta_n}{\lambda_n}\right) \quad \text{(B.4)}$$

$$\left[F_1 + \lambda_j \left(1 + \sum_{n \neq j}^{N} \frac{\beta_n}{\lambda_j + \lambda_n}\right)\right] < XC_j > -\beta_j \sum_{n \neq j}^{N} \frac{\lambda_n}{\lambda_j + \lambda_n} < XC_n > \quad \text{(B.5)}$$

$$= \beta_j < X^2 > + \frac{\beta_j}{\lambda_j} F_1X_o^2; j = 1, \ldots, N$$

If we introduce

$$\xi = 1 - X_o^2/ < X^2 > \quad \text{(B.6)}$$

and the variables

$$y_j = < XC_j > / < X^2 > -\beta_j/\lambda_j \tag{B.7}$$

we then obtain equations (B.4) and (B.5) in the form:

$$\xi\left(1+\sum_{n=1}^{N}\frac{\beta_n}{\lambda_n}\right)+\sum_{n=1}^{N}y_n=\frac{\epsilon^2}{2F_1} \tag{B.8}$$

$$\left[F_1+\lambda_j\left(1+\sum_{n\neq j}^{N}\frac{\beta_n}{\lambda_j+\lambda_n}\right)\right]y_i-\beta_j\sum_{n\neq j}^{N}\frac{\lambda_n}{\lambda_j+\lambda_n}y_n = -\frac{\beta_j}{\lambda_j}F_1\xi; \quad j=1,\ldots,N \tag{B.9}$$

Equations (B.8) and (B.9) indicate that ξ can be expressed as

$$\xi=\frac{\epsilon^2}{2F_1}(\ldots) \tag{B.10}$$

The explicit solution for ξ leads to the result given by equation (IV.24) in conventional determinant notation.

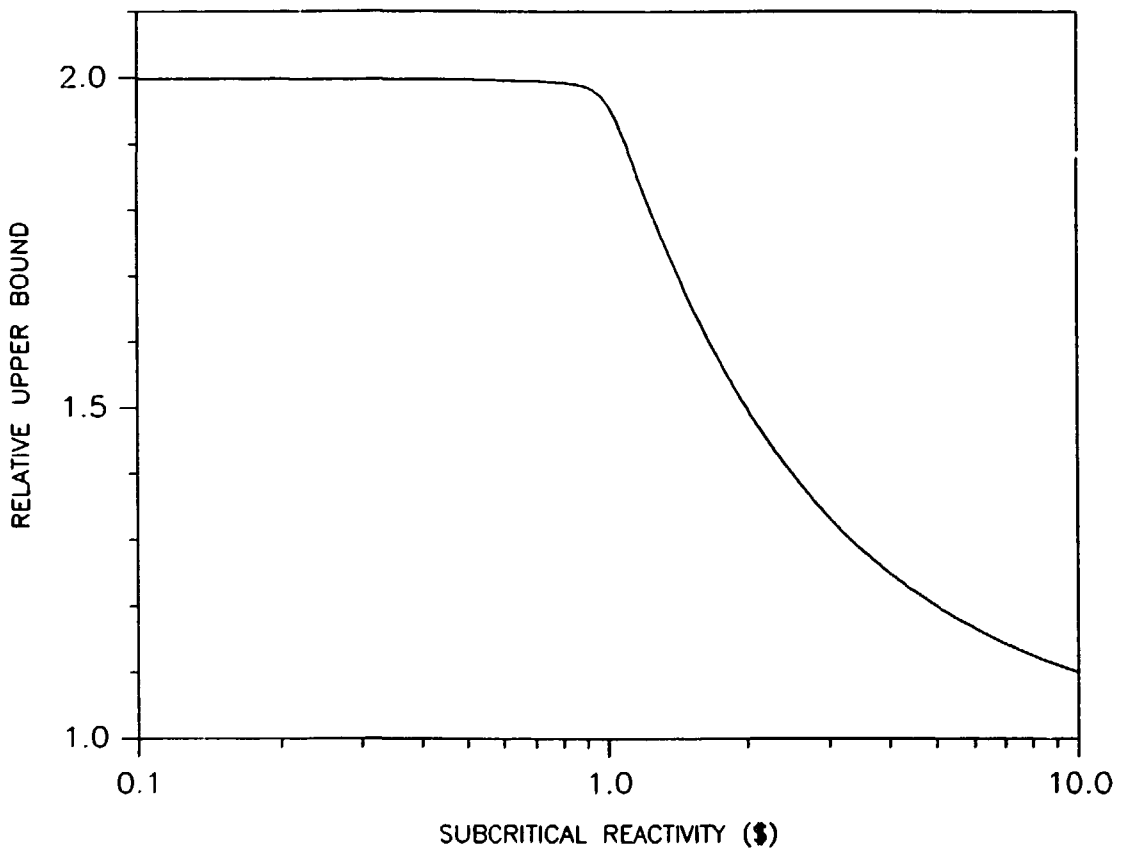

Fig. 1: Relative upper bound $\epsilon^2_{\max}/F_o$ as function of the subcritical reactivity F_o/β_o.

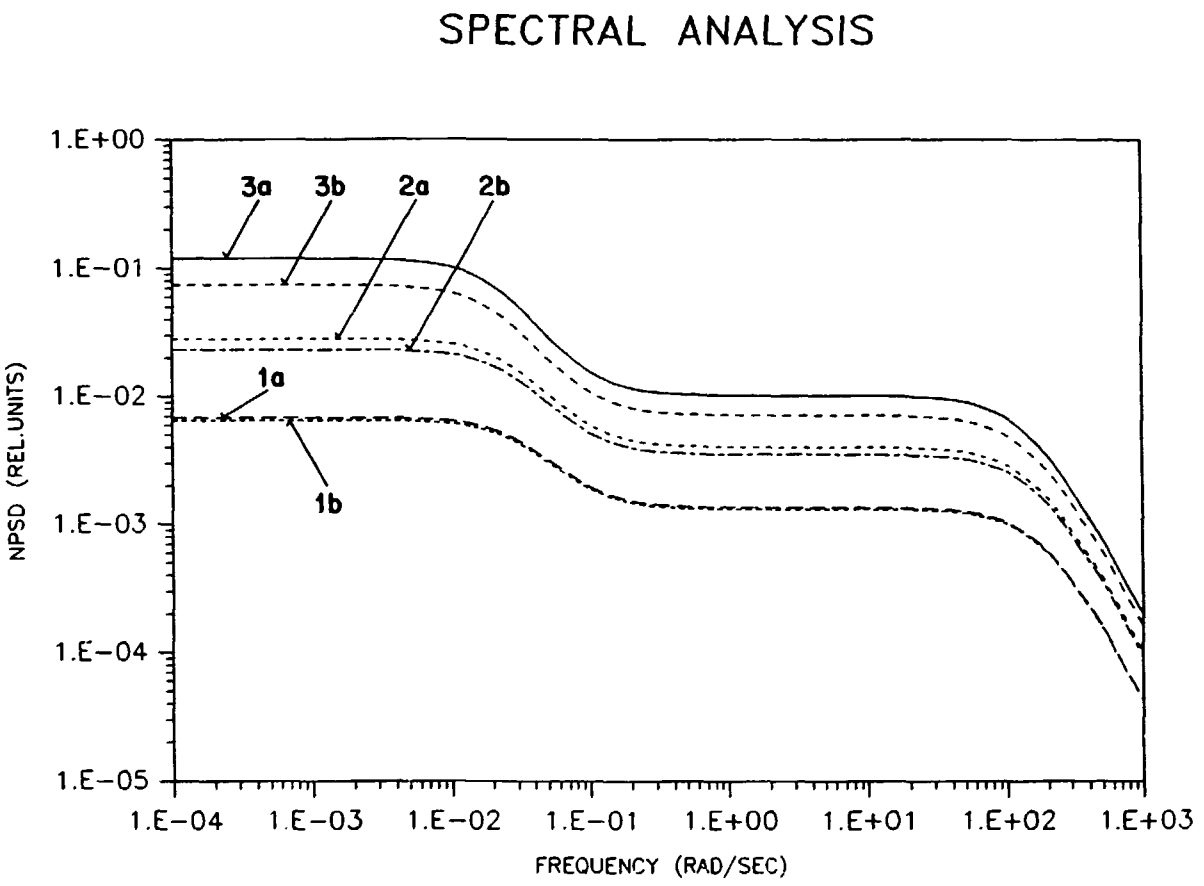

Fig. 2: Neutron NPSD's for white noise input at 1 $ subcriticality

a : exact solutions

b : WHF-2 approximations

1, 2, 3: r = 20, 40, 60 %

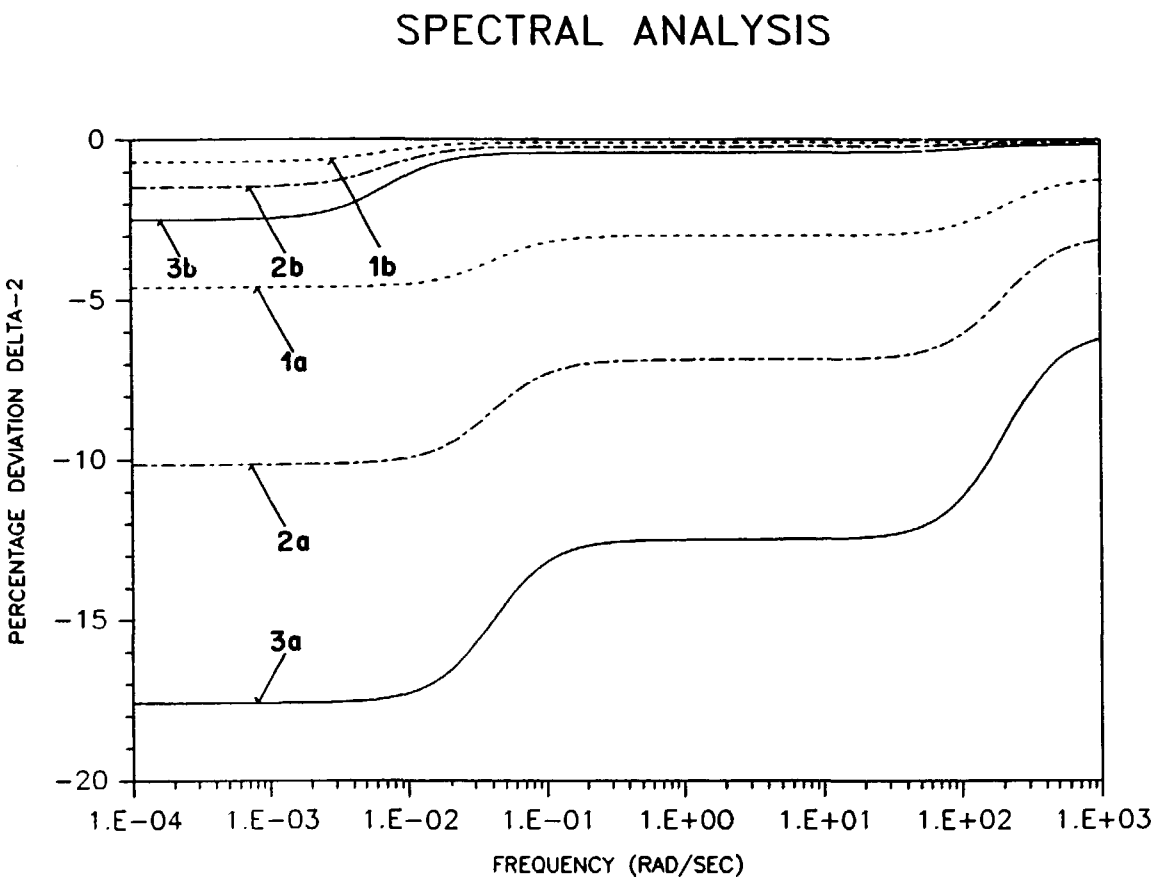

Fig. 3: Percentage PSD deviations $\Delta_2(\omega)$ (WHF-2 approximation)
1, 2, 3: r = 20, 30, 40 %
a : at 1 $ subcriticality
b : at 10 cts subcriticality

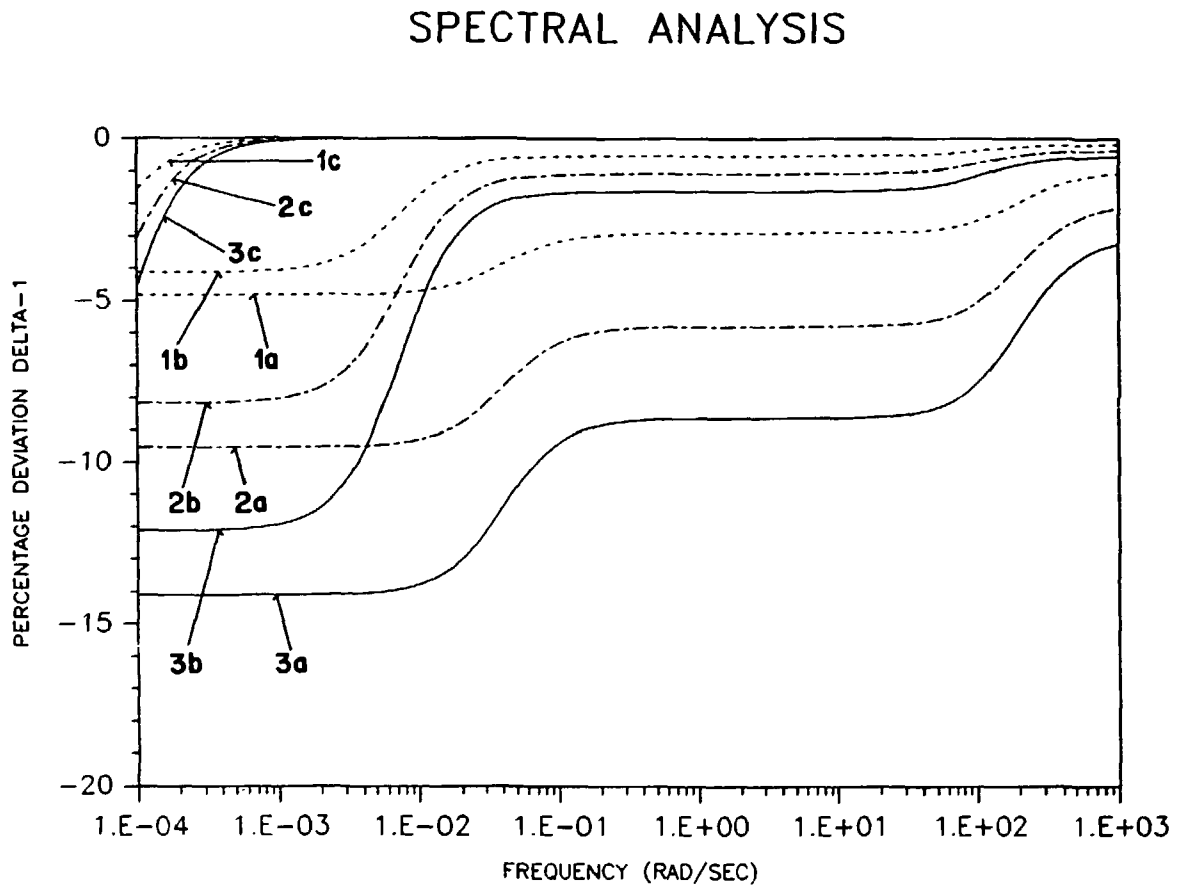

Fig. 4: Percentage PSD deviations $\Delta_1(\omega)$ WHF-1 approximation)

1, 2, 3: r = 2, 4, 6 %

a : at 1 $ subcriticality

b : at 10 cts subcriticality

c : at 0.1 cts subcriticality