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# Trapped Surfaces due to Concentration of Gravitational Radiation

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**Abstract**

Sequences of global, asymptotically flat solutions to the time-symmetric initial value constraints of general relativity in vacuo are constructed which develop outer trapped surfaces for large values of the argument. Thus all such configurations must gravitationally collapse. A new proof of the positivity of mass in the strong-field regime is also found.

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Christodoulou and Klainerman have recently proven [1] that there exist global solutions to the vacuum Einstein equations which are asymptotically flat (in particular: all of whose null geodesics have infinite affine length), provided the initial data are sufficiently close to ones for Minkowski space. On the other hand, it is known from a theorem by Penrose [2] that spacetimes with a non-compact Cauchy surface must have incomplete null geodesics if this, or any other, initial slice has a closed trapped surface [3]. Thus the result of Ref. [1] cannot continue to hold for 'large' data, provided these have trapped surfaces (TS's). In this paper we present a construction of global, singularity-free initial data *in vacuo*, for simplicity in the time-symmetric case, which are asymptotically flat at spatial infinity and contain closed TS's so that Penrose's theorem applies [4]. (Partial results of a more local nature have been obtained in Refs. [5]. For trapped surfaces due to concentration of matter rather than gravitational radiation, see Refs. [6].) Our argument recovers the fact that the constructed data have positive ADM mass [7]. Thus we have a new positive-energy proof which holds in the strong-field regime, far away from flat spacetime.

The standard way of constructing solutions to the (time-symmetric) initial value constraints of general relativity (see e.g. Choquet-Bruhat and York [8]) is to pick a Riemannian metric $\tilde{g}$ on a three-manifold $\widetilde{M}$ having fast decay at infinity. One now looks for a positive function $\phi$, tending to one at infinity, so that $g' = \phi^4\tilde{g}$ has zero scalar curvature, i.e.

$$R[g'] = 0 \Longleftrightarrow L_{\tilde{g}}\phi \equiv \left(-\Delta_{\tilde{g}} + \frac{1}{8}R[\tilde{g}]\right)\phi = 0, \qquad \phi > 0, \qquad \phi \to 1 \text{ at } \infty, \tag{1}$$

where $R$ is the scalar curvature, $\Delta_{\tilde{g}} = \tilde{g}^{ab}\widetilde{D}_a\widetilde{D}_b$ is the Laplacian and $L_{\tilde{g}}$ is (minus) the conformal Laplacian of a metric $\tilde{g}$. The (conformal) metric $\tilde{g}$, called the 'free data', is in fact not completely free, but has to satisfy a global inequality in order for (1) to have a solution. We now consider an infinite sequence of metrics $\tilde{g}_n$, on which we can solve (1), but tending as $n \to \infty$ to a metric $\tilde{g}_\infty$ which violates this global inequality. We call this a *critical sequence* (CS). We show that the mass $m_n$, which is basically the monopole (Schwarzschild) part of the solution $\phi_n$, grows unboundedly as $n$ increases. (In particular, it must be positive for large $n$.) In addition we prove that the higher-order multipoles of $\phi_n$ diverge no more rapidly than $m_n$, so that, loosely speaking, the geometry at fixed large radii is dominated by its Schwarzschildian contribution even in the limit as $n \to \infty$. Consequently, along any CS for large $n$, the existence of TS's can be inferred from the properties of the maximally extended Schwarzschild geometry, namely that the mean curvature of the surfaces of constant radius on a $t$ = constant slice changes sign at the throat.

For technical reasons we work on a 'conformally compactified' manifold $M$. Let $M$ be a compact, orientable three-manifold, and $g$ a positive definite metric on $M$. Let $\Lambda$ be a point of $M$, the 'point at infinity', kept fixed throughout. For $L_g$ as in Equ. (1), we look for a positive function $G$ which is smooth on $\widetilde{M} = M \setminus \Lambda$, but blows up at $\Lambda$ in such a way that

$$\int_M (L_g f) G \, dV_g = 4\pi f|_\Lambda, \qquad G > 0 \tag{2}$$

for all smooth functions $f$ on $M$. In particular, $L_g G = 0$ outside $\Lambda$. In other words, $G$ is a global Green function for $L_g$ w.r. to the point $\Lambda$. Let $\Omega^{1/2}$ be an *asymptotic distance function* (ADF) near $\Lambda$, i.e. $\Omega$ satisfies ($\Omega_a \equiv D_a\Omega$, $\Omega_{ab} \equiv D_aD_b\Omega$)

$$\Omega|_\Lambda = 0 = \Omega_a|_\Lambda, \qquad (\Omega_{ab} - 2g_{ab})|_\Lambda = 0, \qquad \Omega_{abc}|_\Lambda = 0 \tag{3}$$

and is extended as a smooth, positive but otherwise arbitrary function to all of $M$. Note that [9]

$$G = \Omega^{-1/2} + \frac{m}{2} + O(\Omega^{1/2}), \qquad m = \text{constant}, \tag{4}$$

where $m$ does not depend on the choice of ADF. Next recall that under conformal transformations $\bar{g} = \omega^2 g$ ($\omega > 0$) we have the operator equation

$$L_{\bar{g}} \circ \omega^{-1/2} = \omega^{-5/2} \circ L_g \tag{5}$$

so that, given an initial data set $(g, G)$, there exists a whole class, given for each $\omega$ by

$$\bar{g} = \omega^2 g, \qquad \bar{G} = \omega^{-1/2} G, \qquad \Rightarrow \bar{m} = m, \tag{6}$$

provided that the 'scaling' is fixed, i.e. $\omega|_\Lambda = 1$. The connection of the present setting with the standard one is as follows: The manifold $(\widetilde{M}, \tilde{g})$ with $\tilde{g} = \Omega^{-2} g$, $\Omega^{1/2}$ being some ADF, is asymptotically flat near $\Lambda$ with zero ADM mass [10]. However the metric $g' = G^4 g$, due to Equ. (4), is asymptotic to the Schwarzschild metric of mass $m$ [11]. We also have

$$R[g'] = G^{-4}(R[g] - 8G^{-1}\Delta_g G) = 0 \qquad \text{on} \widetilde{M} \tag{7}$$

and $\phi = \Omega^{1/2} G$ satisfies the Lichnerowicz equation (1) w.r. to $\tilde{g}$ on the 'physical' manifold $\widetilde{M}$. Note that the physical metric $g'$ depends only on e conformal class of $g$ provided that $\omega|_\Lambda = 1$.

Now recall that $L_g$, defined on $C^\infty(M)$, is an essentially self-adjoint operator on $L^2(M)$ (with the standard inner product). Its spectrum consists of real isolated eigenvalues which are bounded below. The lowest eigenvalue $\lambda_1(g)$ is known [12] to be non-degenerate and the associated eigenfunction nowhere zero.

The following result has, in the physical setting and in the special case when $\widetilde{M} \cong \mathbf{R}^3$, been effectively proven by Cantor and Brill [13].

**Theorem:** There exists a unique positive solution $G$ of Equ. (2) if and only if $\lambda_1(g) > 0$. (This restricts the possible topologies of $M$ [14], $M \cong S^3$, $\widetilde{M} \cong \mathbf{R}^3$ is permitted.)

**Proof:** The 'if'-direction is essentially in Lee and Parker [9]. Conversely, let there be a positive $G$ satisfying (1). Applying Equ. (1) to a solution $f_1(g)$ of $L_g f_1 = \lambda_1(g) f_1$, chosen to be positive, we have that

$$\lambda_1 \int_M f_1 G dV_g = 4\pi\, f_1|_\Lambda > 0. \tag{8}$$

Since $f_1 > 0$ and $G > 0$ it follows that $\lambda_1(g) > 0$.

We should point out that the sign of $\lambda_1(g)$ coincides with that of the conformally invariant Yamabe number of $g$ [15], whereas the *value* of $\lambda_1(g)$, due to its lack of conformal invariance, is physically uninteresting.

A critical sequence $\{g_n\}$ is now defined as a sequence of metrics with $\lambda_1(g_n) > 0$ and converging uniformly, with derivatives of at least 3 orders, to a metric $g_\infty$ for which $\lambda_1(g_\infty) = 0$. We also assume the CS to be such that the metrics $g_n$ and connections $\partial g_n$ all coincide at the point $\Lambda$, so that an ADF $\Omega^{1/2}$ satisfying (3) for $g = g_n$ can be chosen independently of $n$.

**Lemma (Ó Murchadha [16]):** $\limsup_{n\to\infty} \max_M \phi_n = \infty$, where $\phi_n = \Omega^{1/2} G_n$.

**Proof:** Again by (2) we have, writing $L_n = L_{g_n}$, etc.,

$$\int_M (L_n f_1) G_n dV_n = 4\pi f_1|_\Lambda \tag{9}$$

where we choose $f_1 = f_1(g_\infty)$, satisfying $L_\infty f_1 = \lambda_1(g_\infty) f_1 = 0$ and $f_1 > 0$. Thus

$$0 < 4\pi f_1|_\Lambda \le (\max_M \phi_n)(\int_M \Omega^{-1/2} dV_n) \sup_M |L_n f_1|. \tag{10}$$

The second factor on the right of (10) is bounded and the third one goes to zero as $n \to \infty$, which proves the lemma.

The next step shows that the point at which $\max \phi_n$ is achieved stays away uniformly from $\Lambda$. First note that the CS $\{g_n\}$ can be conformally rescaled, without changing its defining properties, so that $R[g_n] \ge 0$ in some $n$-independent neighbourhood of $\Lambda$. Having done so, consider the formula

$$\Omega^{-2} R[\tilde{g}] = R[g] + 2\Omega^{-2}(\Omega \Delta \Omega - 3\Omega_a \Omega^a) \equiv R[g] + 2\sigma. \tag{11}$$

From (3), $\sigma$ at $\Lambda$ is direction-dependent but finite. A change of ADF $\Omega' = \mu\Omega$ ($\mu|_\Lambda = 1$, $\mu_a|_\Lambda = 0$) can be made with the result that, for all $n$, and for all directions, $\lim_{\to\Lambda} \sigma_n \ge s > 0$. Then there is an $n$-independent punctured open neighbourhood $\tilde{N}$ of $\Lambda$, such that

$$R[\tilde{g}_n] \ge 0 \text{ in } \tilde{N}, \qquad \tilde{g}_n = \Omega^{-2} g_n. \tag{12}$$

Hence $L_{\tilde{g}_n} \phi_n = 0$ admits a maximum principle on $\tilde{N}$ (see Ref. [17]). Taking a value for $n$ large enough so that $\max_{\tilde{M}} \phi_n > 1$ it follows that this maximum may be obtained neither in $\tilde{N}$ nor at $\Lambda$ (where $\phi_n \to 1$). Thus, making $\tilde{N}$ smaller if necessary, $\max \phi_n$ is obtained in $C = \widetilde{M} \setminus \tilde{N}$ for large $n$ [18].

In the bounded set $C$ we can, since $\phi_n > 0$, invoke the *Harnack inequality* (see Ref. [17]) which implies that

$$\max_C \phi_n \le C_n \min_C \phi_n \le C_n \min_{\partial C} \phi_n, \qquad C_n > 0, \tag{13}$$

where $C_n$ is uniformly bounded when $\tilde{g}_n$, $\partial \tilde{g}_n$, $\partial\partial \tilde{g}_n$ are, so that $C_n$ can be chosen to be $n$-independent: $C_n \le C$. Consequently $\min_{\partial C} \phi_n$ blows up as $n \to \infty$. Hence $\min_{\partial C} G_n$ also blows up.

Now consider $G_n$. It is known that for an elliptic equation there always exists in a sufficiently small neighbourhood of $\Lambda$ a *local fundamental solution* $G^{loc}$ [19] of the form

$$G^{loc} = \Omega^{-1/2} V + W \tag{14}$$

where $V$, $W$ can be chosen so that

$$V = 1 + O(\Omega), \qquad W|_\Lambda = 0. \tag{15}$$

Doing this for each $n$ we obtain regular functions $F_n = G_n - G_n^{loc}$ in $N = \tilde{N} \cup \Lambda$ obeying

$$L_n F_n = 0 \text{ in } N. \tag{16}$$

Since $R[g_n] \ge 0$, $F_n$ satisfies a minimum principle. Equ. (16) tells us that an interior minimum of $F_n$, if it occurs, has to be positive. Since $G_n^{loc}$ remains finite and bounded on $\partial C$, whereas

$\min_{\partial C} G_n$ blows up as $n \to \infty$, we have that $\min_{\partial C} F_n$ must become positive for large $n$. Thus $F_n > 0$ for large $n$, so that $m_n = 2F_n|_\Lambda$ is also positive for large $n$. This is our positive-mass theorem.

We can now apply the Harnack inequality to $F_n$ in $N$ with the result that

$$\max_{\partial C} F_n \le \sup_N F_n \le D \inf_N F_n, \qquad D > 0. \tag{17}$$

But if $\min_{\partial C} F_n \to \infty$, then clearly $\max_{\partial C} F_n \to \infty$ as $n \to \infty$. So, according to (17), $F_n$ blows up at all points of $N$ and in particular $m_n \to \infty$ as $n \to \infty$. Now define the *renormalized sequence* $\bar{F}_n$ in $N$ by

$$\bar{F}_n \equiv \frac{F_n}{F_n|_\Lambda} = \frac{2}{m_n} F_n. \tag{18}$$

Since $\bar{F}_n \le \sup_N F_n / \inf_N F_n$, (17) implies that $\bar{F}_n$ stays uniformly bounded as $n \to \infty$. But the *Schauder inequality* [17] shows that

$$\sup_N |\partial \bar{F}_n| + \sup_N |\partial\partial \bar{F}_n| \le E \sup_N \bar{F}_n \tag{19}$$

for some constant $E > 0$. In particular, $\partial \bar{F}_n$ and $\partial\partial \bar{F}_n$ are also uniformly bounded.

We now come to the issue of TS's. For sufficiently small $\Omega > 0$ the level sets of $\Omega$ are smooth surfaces $\cong S^2$, which enclose a compact region in $\widetilde{M}$. Since $g_n'$ are time-symmetric initial data on $\widetilde{M}$, the condition for $\Omega =$ constant to be outer trapped is that $H_n$, the trace of its extrinsic curvature w.r.t. to the metric $g_n' = G_n^4 g_n$ with the normal pointing towards infinity, be negative. A computation gives [20]

$$-H_n = \frac{G_n^{-3}\Omega^{-1/2}}{(\Omega_c\Omega^c)^{1/2}} \left[\Omega^{1/2} G_n \left(g_n^{ab} - \frac{\Omega^a\Omega^b}{\Omega_d\Omega^d}\right)\Omega_{ab} + 4\Omega^{1/2} G_n^a \Omega_a\right]. \tag{20}$$

From the previous analysis of $F_n$ and the Taylor theorem we have that $G_n$ satisfies

$$G_n = \Omega^{-1/2} + \frac{m_n}{2} + m_n O(\Omega^{1/2}), \qquad (G_n - \Omega^{-1/2})_a = m_n O(1), \tag{21}$$

where the constants involved in the $O$-symbols are understood to be independent of $n$. Using (21) and (3) in Equ. (20), we infer

$$H_n = \frac{4G_n^{-3}\Omega^{-1/2}}{(\Omega_d\Omega^d)^{1/2}} \left[1 - \frac{m_n\Omega^{1/2}}{2} + m_n O(\Omega)\right], \tag{22}$$

which immediately gives the

**Theorem:** Positive constants $\varepsilon$ and $\delta$ exist which are independent of $n$, so that the surfaces $\Omega = \Omega_0$ with

$$\varepsilon \le \Omega_0^{-1/2} \le \frac{m_n}{2} - \delta \tag{23}$$

are (outer) trapped for large $n$. Thus every sufficiently far-out surface of constant $\Omega$ gets trapped eventually and the area of the outermost trapped surface increases like $O(m_n^2)$.

To avoid confusion we point out that, although the quantities appearing in (23) depend on scale, i.e. change under replacing $g_n$ by $c^2 g_n$ ($c^2 =$ constant $\neq 1$), the theorem itself is true for any scaling. In conclusion, several methods are available in the literature [21] allowing one to obtain large classes of CS's. We wish to stress that this is the first rigorous proof of the fact that pure gravitational waves can undergo gravitational collapse.

It seems reasonable to conjecture that, in some suitable norm, the boundary of the set where $\lambda_1(g) > 0$ consists only of metrics with $\lambda_1(g) = 0$ . With such a result one could combine our proof of the strong-field positivity of mass with the weak field proof of Brill and Deser [22] to obtain a full proof of the positivity of mass. Details and generalizations of this work will be published elsewhere.

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[4] Strictly speaking, these surfaces are *outer trapped*, in that they span a compact region of the Cauchy slice and outgoing beams of light locally converge. Inspection of Penrose's proof shows that, with 'trapped' being replaced by 'outer trapped', the theorem goes through as before.

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