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INFINITE-PARAMETRIC EXTENSION OF THE CONFORMAL ALGEBRA IN D>2 SPACE-TIME DIMENSION *

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ABSTRACT

On the basis of the analytic continuations of semisimple Lie algebras discovered recently by us we construct manifestly quasiconformal infinite-dimensional algebras AC(so(4,1)) and PAC(so(3,2)) extending the conformal algebras in three-dimensional Euclidean and Minkowski space-time like the Virasoro algebra extends so(2,1). Their higher spin generalizations are also constructed. A counterpart of the central extension for $D > 2$ and possible applications in exactly solvable conformal quantum field models in $D > 2$ are discussed.

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1. Introduction

The Virasoro algebra [1] is an infinite–dimensional (inf–dim) Lie algebra containing sl_2 as its maximal finite–dimensional (fin–dim) simple subalgebra. Under the natural representation [sl_2, Vir] of sl_2 the Virasoro algebra is a non–decomposable sl_2 elementary module (plus one–dimensional centre).

It is of curiosity that so far the Virasoro algebra has been considered separately but not as a member of some class of inf–dim algebras. From the mathematical point of view it is natural to ask:

what is an analog of the Virasoro algebra when the maximal finite–dimensional subalgebra is arbitrary semisimple Lie algebra on the place of sl_2?

The answer for this problem has been given in our paper Ref.[2] with the help of so–called theory of the analytical continuation of semisimple Lie algebras formulated therein.

On the other hand, physically the Virasoro algebra is a conformal algebra in one dimension and Vir⊕Vir is a conformal one in two–dimensions. It plays a central role in the algebraic theory of 2D exactly solvable conformal models.

At the same time the functional theory of exactly solvable ~~conformal~~ models based on the closed set of equations for the Green functions has been formulated in unique manner for any dimension of space–time in Refs.[3–5]. The construction of [3–5] looks like the representation theory of a certain inf–dim algebra. However the very conformal algebra in D>2 is finite–dimensional. In this way the following physical problem arises:

what is an infinite–parametric symmetry in D>2 like the Virasoro one in D=2?

Naturally the answer might be based on the answer for our first mathematical problem. Exactly, there is an inf–dim Virasoro–like generalization of the conformal algebra in D>2. In this paper we give a manifest construction of the 3D conformal algebra extension.

2. Analytic Continuation of Arbitrary Semisimple Lie Algebra

Peculiar features of the Virasoro algebra are (in the sections 2–4 we will consider the **centreless case**):

1) it contains sl_2 as a maximal fin–dim subalgebra;

2) under the representation $[L_{0,\pm1}, L_n]$ of sl_2 the Virasoro algebra is a non–decomposable elementary module containing the sl_2 **adjoint** representation in its invariant subspace (we call such a representation as a quasiadjoint one or an analytic continuation of the **adjoint** rep);

3) the Virasoro commutation relations

$$[L_n, L_m] = (m-n)\, L_{n+m} \tag{1}$$

are just the same as in sl_2, and only the region of the definition of parameters is different ($n \in \mathbf{Z}$ for Vir and $n \in \mathbf{Z}$, $|n| \leq 1$ for sl_2).

The structure constants of Vir are in fact the Clebsh–Gordan coefficients $C_{qad(sl_2),qad(sl_2)}{}^{qad(sl_2)}$ for the tensor product of two quasiadjoint representations $qad(sl_2) \otimes qad(sl_2) = qad(sl_2) + \ldots$. They can be obtained by the straightforward analytic continuation of the sl_2 structure constants from $|n| \leq 1$ to all integer n.

What will happen when instead sl_2 one takes arbitrary semisimple complex or non–compact real algebra g? For any g we have defined an inf–dim algebra AC(g) (analytic continuation of g) with the following defining properties [2]:

1) AC(g) contains g as a maximal fin–dim subalgebra;

2) under [g, AC(g)] AC(g) is a non–decomposable elementary g–module with the **adjoint** representation of g in its invariant subspace (this is a quasiadjoint rep of g);

3) the structure constants of AC(g) are the Clebsh–Gordan coefficients $C_{qadg,qadg}{}^{qadg}$ for $qadg \otimes qadg = qadg + \ldots$. They can be obtained by straightforward analytic continuation

from the structure constants of g (of course in **some** suitable basis).

Remarkably for any complex or non–compact real semisimple Lie algebra g there exist one and only one quasiadjoint representation. Generally for any irreducible finite–dimensional representation of g there exists an unique inf–dim non–decomposable **elementary** representation containing it in the invariant subspace. It follows from the results of Refs.[6,7]. (About elementary representations or the basic series of representations see refs.[6–10].) Therefore to any fin–dim semisimple Lie algebra g we put in the correspondence unique inf–dim algebra AC(g), its analytic continuation [2]. The (**centreless**) Virasoro algebra in our terms is an analytic continuation of sl_2, Vir $\simeq$ AC(sl_2). This is a simplest member in the above class of inf–dim algebras.

It is of importance that each non–decomposable elementary module has a finite Jordan–Gelder series of invariant submodules (among them there may be only one fin–dim sub–module, all the other are inf–dim). This gives us a Jordan–Gelder series of subalgebras in AC(g) (the partial analytic continuations PAC(g) [2]).

It should be mentioned interconnections among analytic continuations of real and complex fin–dim algebras are not so simple as for the fin–dim ones themselves. To be exact, the complexification $AC(g)_{\mathbf{C}}$ of AC(g) for non–compact real g coincides with the analytic continuation of the complexification $g_{\mathbf{C}}$ of g only if g is a maximal non–compact form of $g_{\mathbf{C}}$. For other real forms g of $g_{\mathbf{C}}$ $AC(g)_{\mathbf{C}}$ is one of the Jordan–Gelder subalgebras of $AC(g_{\mathbf{C}})$. Note also that the compact real forms do not have any analytic continuation at all (all representations of the compact algebra are fin–dim).

An interesting problem consists in constructing the geometrical realizations of AC(g). The basic one is a realization in terms of vector fields on the group manifold of the Lie group G corresponding to the Lie algebra g. A number of realizations in terms of vector fields on the cosets G/H may be obtained from the basic one.

Now pass to the analytic continuations of the conformal algebras. The conformal algebra in D–dimensions is so(D+1,1) (in Euclidean case) or so(D,2) (in Minkowski case). so(D+1,1) is a minimally non–compact form of so(D+2;$\mathbb{C}$) and we have AC(so(D+1,1))$_{\mathbb{C}}$ ⊂ AC(so(D,2))$_{\mathbb{C}}$ ⊂ AC(so(D+2;$\mathbb{C}$)). The analytic continuation of the Minkowski algebra is larger than the Euclidean one. At the same time AC(so(D,2)) contains a Jordan–Gelder subalgebra PAC(so(D,2)) such that its complexification is isomorphic to the complexification of the Euclidean algebra AC(so(D+1,1)). The important difference between AC(so(D,2)) and its subalgebra (partial analytic continuation PAC(so(D,2))) obtained from the Euclidean algebra consists in the following. Let us consider the reduction

$$so(D,2) \longrightarrow so(D-1,1) \oplus so(1,1), \tag{2}$$

where so(D–1,1) and so(1,1) are the Lorentz and dilatation subalgebras respectively. The decomposition of AC(so(D,2)) into a direct sum of the Lorentz algebra modules contains inf–dim non–decomposable elementary modules of so (D–1,1), while PAC(so(D,2)) contains only the Lorentz invariant fin–dim irreducible subspace of them. The Euclidean algebra AC(so(D+1,1)) evidently contains only fin–dim irreducible modules of the Euclidean Lorentz algebra so(D).

For the physical applications in QFT we should have a manifest Lorentz invariance, and working in the Minkowski space we decide the algebra PAC(so(D,2)) instead of AC(so(D,2)). The algebras PAC(so(D,2)) and AC(so(D+1,1)) can be transformed one into the other by means of the Wick rotation. Taking into account that the above algebras naturally extend the conformal ones, we will also call them quasiconformal algebras.

3. The Conformal Algebras in D = 1, 2

The Virasoro algebra, analytic continuation of the little conformal algebra so(2,1) in D=1, has the following very simple Poisson–bracket realization (see Refs.[11,12] for higher spin and superextensions)

$$L_n = \frac{1}{2}(p)^{1+n}(q)^{1-n}, \quad n \in \mathbb{Z}, \tag{3}$$

$$[f, g]_{PB} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}. \tag{4}$$

This is a generalization of the Schwinger realization of sl_2 by the second order polynomials of the canonical variables. In (3) both positive and negative powers of q and p may appear ($n \in \mathbb{Z}$, but not $|n| \le 1$ as for sl_2).

The Euclidean conformal algebra AC(so(3,1)) in D=2 is an analytic continuation of the little conformal algebra so(3,1) ≃ sl(2;$\mathbb{C}$) and

$$AC(so(3,1)) \simeq AC(sl(2;\mathbb{C})) \simeq Vir \oplus \overline{Vir}. \tag{5}$$

The Minkowski conformal algebra is obtained by the Wick rotation and is isomorphic to PAC(so(2,2)) ≃ Vir ⊕ Vir. As we have discussed it is not isomorphic to AC(so(2,2)) but this is its subalgebra (AC(so(2,2)) itself is isomorphic to the algebra of vector fields on the torus T^2).

We rewrite the two–dimensional conformal algebra in the manifestly Lorentz covariant basis (in tensor notations) to make a contact with the case D>2. The generators are

$$\left.\begin{array}{l} 1)\ D\ \text{(dilatation, so(1,1))} \\ 2)\ M_{\mu\nu}\ \text{(Lorentz)} \\ 3)\ P_\mu,\ K_\mu \end{array}\right\} \text{little conformal algebra} \tag{6}$$
$$4)\ P_{\mu(\ell)},\ K_{\mu(\ell)},\ \ell = 2, 3, \ldots,$$

where $\mu, \nu = 1, 2$, and the additional generators $P_{\mu(\ell)} = P_{\mu_1 \ldots \mu_\ell}$, $K_{\mu(\ell)} = K_{\mu_1 \ldots \mu_\ell}$ are completely symmetric rang–ℓ tracelles tensors (we use the short–hand notations for symmetric tensors as in Ref.[13]). The commutation relations in the covariant basis read as follows

$$
\begin{aligned}
&[P_{\mu(l)}, P_{\nu(l')}] = (l-l') P_{\mu(l)\nu(l')}, \quad [D, P_{\mu(l)}] = -P_{\mu(l)},\\
&[K_{\mu(l)}, K_{\nu(l')}] = (l'-l) K_{\mu(l)\nu(l')}, \quad [D, K_{\mu(l')}] = K_{\mu(l')},\\
&[M_{\rho\sigma}, P_{\mu(l)}] = l(\eta_{\rho\mu} P_{\mu(l-1)\sigma} - \eta_{\sigma\mu} P_{\mu(l-1)\rho}),\\
&[M_{\rho\sigma}, K_{\mu(l)}] = l(\eta_{\rho\mu} K_{\mu(l-1)\sigma} - \eta_{\sigma\mu} K_{\mu(l-1)\rho}),\\
&[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\rho\nu} M_{\sigma\mu} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\rho\mu},\\
&[P_{\mu(l)}, K_{\nu(l')}] = \begin{cases} (l+l')\eta_{\mu(l),\nu(l)} K_{\nu(l'-l)}, & l' > l,\\ l(\eta_{\mu(l),\nu(l)} D + \eta_{\mu(l-1),\nu(l-1)} M_{\mu\nu}), & l = l',\\ (l+l')\eta_{\mu(l'),\nu(l')} P_{\mu(l-l')}, & l > l', \end{cases}
\end{aligned}
\tag{7}
$$

where $\eta_{\mu(n),\nu(n)} = \eta_{\mu\nu} \ldots \eta_{\mu\nu}$ (n times) and $\eta_{\mu\nu}$ is the 2D metric (all the indices denotes by the same letter are supposed to be symmetrized).

In this notations the Virasoro algebra looks like the higher spin algebras [14–18]. But the principal difference is that the higher spin algebras consist of the tower of fin–dim irreducible representations of the maximal fin–dim subalgebra, but the Virasoro algebra consists of one non–decomposable inf–dim rep.

The usual generators $L_n, \bar{L}_n$ are in fact the generators (6) in the complex coordinates:

$$
\begin{aligned}
&L_n = K_{\underbrace{z\ldots z}_{n}}, \quad L_{-n} = P_{\underbrace{\bar{z}\ldots\bar{z}}_{n}},\\
&\bar{L}_{-n} = P_{\underbrace{z\ldots z}_{n}}, \quad \bar{L}_n = K_{\underbrace{\bar{z}\ldots\bar{z}}_{n}}, \qquad (n>0)\\
&L_0 = \tfrac{1}{2}(D + M_{\bar{z}z}), \quad \bar{L}_0 = \tfrac{1}{2}(D - M_{\bar{z}z}).
\end{aligned}
\tag{8}
$$

4. Infinite–Parametric Quasiconformal Extension of the Conformal Algebra in D = 3

Here we will construct manifestly the algebras AC(so(4,1)) and PAC(so(3,2)) which are Virasoro–like extensions of the 3D conformal algebras so(4,1) and so(3,2).

As we have seen in two dimensions there are two possible formalisms to work with the conformal algebra: tensor formalism (7) and complex coordinates (8). However in D>2 the tensor formalism for inf–dim algebras becomes cumbersome and the commutation relations like (7) have rather complicated form due to the presence of the tensors corresponding to different Young tableaux. Nevertheless in D=3 and 4 there exists a very convenient formalism to work with the conformal and higher spin algebras [14–18]. This is a two–component multispinorial formalism (see e.g. Ref.[19]). It was used in Ref.[17] and Ref[18] to construct higher spin generalizations of the conformal superalgebras in D = 2+1 and D = 3+1 respectively. So the generating elements of so(3,2) in Ref.[17] were chosen as follows

$$
a^{+}_{\alpha} = a_{\alpha}, \quad b^{+}_{\alpha} = b_{\alpha}, \quad \alpha = 1,2
$$
$$
[a_{\alpha}, b_{\beta}] = \epsilon_{\alpha\beta}, \tag{9}
$$

where α, β,... are the two–component so(2,1) spinorial indices raised and lowered by means of the symplectic metric $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, \epsilon^{12} = \epsilon_{12} = 1$ as $a^{\alpha} = \epsilon^{\alpha\beta} a_{\beta}$, $a_{\alpha} = \epsilon_{\beta\alpha} a^{\beta}$. The second order generators*)

---

*) We follow the two–component notations and conventions of Refs.[14–18,20–22]. A symmetrization is implied for any set of upper or lower spinorial indices denoted by the same letter. When all the necessary symmetrizations are carried out, the maximal possible set of upper and lower indices denoted by the same letter should be contracted. For instance

$$
A_{\alpha(2)} B^{\alpha} = \tfrac{1}{2}(A_{\alpha_1\alpha_2} + A_{\alpha_2\alpha_1}) B^{\alpha_2},
$$
$$
\epsilon_{\alpha\beta}\epsilon_{\alpha\beta} = \tfrac{1}{2}(\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2} + \epsilon_{\alpha_2\beta_1}\epsilon_{\alpha_1\beta_2}).
$$

$$M_{\alpha(2)} = a_\alpha b_\alpha \ , \quad D = \frac{1}{2} a_\gamma b^\gamma ,$$
$$P_{\alpha(2)} = \frac{1}{2} a_\alpha a_\alpha \ , \quad K_{\alpha(2)} = \frac{1}{2} b_\alpha b_\alpha \tag{10}$$

form the conformal algebra so(3,2) under the Poisson bracket (9):

$$[M_{\alpha(2)}, M_{\beta(2)}] = 2\varepsilon_{\alpha\beta} M_{\alpha\beta} \, , \quad [M_{\alpha(2)}, P_{\beta(2)}] = 2\varepsilon_{\alpha\beta} P_{\alpha\beta} ,$$
$$[M_{\alpha(2)}, K_{\beta(2)}] = 2\varepsilon_{\alpha\beta} K_{\alpha\beta} \, , \quad [D, P_{\alpha(2)}] = - P_{\alpha(2)} , \tag{11}$$
$$[D, K_{\alpha(2)}] = K_{\alpha(2)} \, , \quad [P_{\alpha(2)}, K_{\beta(2)}] = \varepsilon_{\alpha\beta} M_{\alpha\beta} + \varepsilon_{\alpha\beta}\varepsilon_{\alpha\beta} D \, .$$

Transition to the tensor notations is performed by means of $\sigma_\mu^{\ \alpha(2)} = (1, \sigma_1, \sigma_3)$ (μ = 0,1,2; I – unit matrix, σ_1 and σ_3 are the Pauli matrices). For example

$$P_\mu = \sigma_\mu^{\ \alpha(2)} P_{\alpha(2)} \, , \quad M_{\mu\nu} = \frac{1}{2} \sigma_\mu^{\ \alpha}{}_\gamma \, \sigma_\nu^{\ \alpha\gamma} M_{\alpha(2)}$$

and so on. Considering the polynomials of all even orders of a_α and b_α, we come to the higher spin generalization of so(3,2) with the generators in the conformal basis [17]

$$T^{(s,c)}_{\alpha(2\ell)} = \frac{1}{2} \sqrt{\frac{(2\ell)!}{(\ell+c)!(\ell-c)!}} \, \underbrace{a_\alpha \dots a_\alpha}_{\ell-c} \, \underbrace{b_\alpha \dots b_\alpha}_{\ell+c} \, (a^\gamma b_\gamma)^{s-\ell} \, . \tag{12}$$

Here s = 1,2,... (2s is a degree of homogeneity of T) defines the signature (s,s) of the so(3,2) spin–s representation (dim D(s,s) = $\frac{(2s+3)(2s+1)(s+1)}{3}$, D(1,1) is the so(3,2) adjoint rep); ℓ = 0,1,...,s is the signature of so(2,1) (2ℓ is a number of spinorial indices) and c = $-\ell, -\ell+1, \dots, \ell$ is a conformal weight $\left([D, T^{(s,c)}_{\alpha(2\ell)}] = c\, T^{(s,c)}_{\alpha(2\ell)}\right)$.

Calculating the Poisson brackets (9) of two basis monomials (12), we obtain an algebra

$$[T^{(s,c)}_{\alpha(2\ell)}, T^{(s',c')}_{\beta(2\ell')}] =$$

$$= \sum_{\ell'',c''} f^{\ell\ell'\ell''}_{cc'c''}(s,s') \, C_{\alpha(2\ell),\beta(2\ell'),}^{\qquad\quad \gamma(2\ell'')} \, T^{(s+s'-1,c'')}_{\gamma(2\ell'')} \, , \tag{13}$$

where C are the so(2,1) spinorial Clebsh–Gordan coefficients [17]

$$C_{\alpha(2\ell),\beta(2\ell'),}^{\qquad\quad \gamma(2\ell'')} = A(\ell,\ell',\ell'') \underbrace{\varepsilon_{\alpha\beta} \dots \varepsilon_{\alpha\beta}}_{\ell+\ell'-\ell''} \tag{14}$$
$$\times \underbrace{\delta^\gamma_\alpha \dots \delta^\gamma_\alpha}_{\ell-\ell'+\ell''} \, \underbrace{\delta^\gamma_\beta \dots \delta^\gamma_\beta}_{\ell'-\ell+\ell''} \, , \tag{15}$$
$$A(\ell,\ell',\ell'') = \left[\frac{(2\ell)!(2\ell')!(2\ell''+1)!}{(\ell+\ell'-\ell'')!(\ell'-\ell+\ell'')!(\ell''-\ell'+\ell)!(\ell+\ell'+\ell''+1)!} \right]^{\frac{1}{2}}$$

and f are reduced structure constants of the form

$$f^{\ell\ell'\ell''}_{cc'c''}(s,s') = \delta(c+c'-c'')$$
$$\times \Big\{ \frac{1}{2} \sqrt{(\ell-c)(\ell'+c')(\ell+\ell'-\ell'')(\ell+\ell'+\ell''+1)} \; C^{\ell-\frac{1}{2},\ell'-\frac{1}{2},\ell''}_{c+\frac{1}{2},c'-\frac{1}{2},c''} +$$
$$+ \frac{1}{2} \sqrt{(\ell+c)(\ell'-c')(\ell+\ell'-\ell'')(\ell+\ell'+\ell''+1)} \; C^{\ell-\frac{1}{2},\ell'-\frac{1}{2},\ell''}_{c-\frac{1}{2},c'+\frac{1}{2},c''} + \tag{16}$$
$$+ [(s'-\ell')c - (s-\ell)c'] \; C^{\ell,\ell',\ell''}_{c,c',c''} \Big\} ,$$
$$\delta(n) = 1 (0) \text{ at } n = 0 \ (n \neq 0),$$

where C are the usual Clebsh–Gordan coefficients (our conventions are such as in Ref.[23]).

Note that the generators with s=1 form the maximal fin–dim subalgebra isomorphic to so(3,2) (11): $P_{\alpha(2)} \sim T^{(1,-1)}_{\alpha(2)}$, $K_{\alpha(2)} \sim T^{(1,1)}_{\alpha(2)}$, $M_{\alpha(2)} \sim T^{(1,0)}_{\alpha(2)}$, $T^{(1,0)} \sim D$.

The algebra (13–16) is a classical (Poisson–bracket) version hs*(4) of the higher spin algebra hs(4). It was obtained originally in Ref.[14] as a higher spin generalization of the anti–De Sitter algebra so(3,2) in D=4 and applied in Ref.[20] to built up the cubic invariant interaction of the higher spin massless gauge fields in ADS_4, and in Ref.[21] to construct consistent equations of motion for the interacting higher spins. The operator realization of hs(4) was proposed in Ref.[16]. In Ref.[17] it was used to construct the conformal higher spin Chern–Simons theory in D = 2+1 generalizing the conformal supergravity (in [17] hs(4) was denoted also hsc(3), higher spin conformal algebra in D=3). The conformal basis (12) was introduced in [17] and all our consideration here concerning hsc*(3) ≃ hs*(4) (* means the Poisson–bracket version) follows this work. The similar construction for 4D conformal superalgebra su(2,2|N) was developed in Ref.[18] and applied in Ref.[22] to construct the cubic interaction in 4D conformal higher spin theory (generalization of the conformal supergravity).

The algebra hsc*(3) defined by eqs.(13–16) with the generators (12), where

$$s = 1, 2, \dots; \qquad \ell = 0, 1, \dots; \qquad \ell \le s; \qquad c = -\ell, -\ell+1, \dots, \ell, \tag{17}$$

is a straightforward three–dimensional analog of the little higher spin conformal algebra hs*(2) in D=1

$$[L^s_m, L^{s'}_{m'}] = (sm' - s'm) L^{s+s'-1}_{m+m'}, \tag{18}$$

where

$$|m| \le s, \qquad |m'| \le s'. \tag{19}$$

(L^1_m form so(2,1)).

To pass to the Virasoro algebra and its higher spin generalization hsc*(1) (higher spin conformal algebra in D=1) all we have to do is to abolish the conditions (19), and after this the generators become L_m^s with $s = 1,2,\dots$ and $m \in \mathbf{Z}$. The Poisson–bracket realization similar (3) is

$$L^s_m = \frac{1}{2} (p)^{s+m} (q)^{s-m} \qquad (L^1_m = L_m \text{ in eq.(3)}). \tag{20}$$

Now we are approaching to the culmination point of our consideration. What do we have to do to obtain the quasiconformal (Virasoro–like) algebra and its higher spin generalization like (18) in D=3? It turns out we only have to abolish the restriction $\ell \le s$ in (17)! That is we permit now to appear both positive and negative powers of $(a^\gamma b_\gamma)$ in (12) because $s-\ell$ now is not necessary supposed to be non–negative ($\ell = 0,1,2,\dots,\infty$ for fixed s). The commutation relations of the resulting algebra PAC(so(3,2)) (when s=1) and its higher spin generalization PAC(hsc*(3)) (when s = 1,2,...) are the same as in so(3,2) and in hsc*(3) resp. and given by the eqs.(13–16), but only the regions of the definition of parameters are different. This is a procedure of the analytic continuation (here we have revoked the restrictions $\ell \le s$ likely the revocation of $|m| \le s$ in the Virasoro case).

The algebra PAC(so(3,2)) is given by the commutation relations*)

$$(L^c_{\alpha(2\ell)} = T^{(1,c)}_{\alpha(2\ell)}, \; \ell = 0,1,2,\dots, \; |c| \le \ell)$$

$$[L^c_{\alpha(2\ell)}, L^{c'}_{\beta(2\ell')}] = \sum_{\ell'', c''} f^{\ell\ell'\ell''}_{cc'c''} C_{\alpha(2\ell),\beta(2\ell'),}{}^{\gamma(2\ell'')} L^{c''}_{\gamma(2\ell'')}, \tag{21}$$

where the structure constants $f^{\ell\ell'\ell''}_{cc'c''} = f^{\ell\ell'\ell''}_{cc'c''}(1,1)$ are given by the eq.(16).

The Minkowski space reality conditions are

$$(T^{(s,c)}_{\alpha(2\ell)})^\dagger = T^{(s,c)}_{\alpha(2\ell)}, \quad (L^c_{\alpha(2\ell)})^\dagger = L^c_{\alpha(2\ell)}. \tag{22}$$

---

*) The structure of AC(so(4,1)) and PAC(so(3,2)) is schematically illustrated on the Fig.2 in Appendix. On the Fig.1 the Virasoro algebra is represented for comparison.

The Euclidean quasiconformal algebra AC(so(4,1)) is extracted by the following reality conditions:

$$\left(T^{(s,c)}_{\alpha(2l)}\right)^{\dagger} = \underbrace{\varepsilon^{\alpha\beta}\cdots\varepsilon^{\alpha\beta}}_{2l}\, T^{(s,c)}_{\beta(2l)},$$
$$\left(L^{c}_{\alpha(2l)}\right)^{\dagger} = \varepsilon^{\alpha\beta}\cdots\varepsilon^{\alpha\beta}\, L^{c}_{\beta(2l)}. \tag{22}$$

In the Cartan–Weyl basis (with respect to the fin–dim conformal subalgebra) our algebras take the form

$$[\, T^{(s,c)}_{l,m},\ T^{(s',c')}_{l',m'} \,] = \sum_{l'',c'',m''} f^{l l' l''}_{c c' c''}(s,s')$$
$$\times\, C^{l\, l'\, l''}_{m\, m'\, m''}\, T^{(s+s'-1,c'')}_{l'',m''}, \tag{23}$$

and $\left(L^{l}_{m,c} = T^{(1,c)}_{l,m} \right)$

$$[\, L^{l}_{m,c},\ L^{l'}_{m',c'} \,] = \sum_{l'',c'',m''} \Big\{ \tfrac{1}{2}\sqrt{(l-c)(l'+c')(l+l'-l'')(l+l'+l''+1)}$$
$$\times\, C^{l-\frac{1}{2},\, l'-\frac{1}{2},\, l''}_{c+\frac{1}{2},\, c'-\frac{1}{2},\, c''} + \tfrac{1}{2}\sqrt{(l+c)(l'-c')(l+l'-l'')(l+l'+l''+1)}\, C^{l-\frac{1}{2},\, l'-\frac{1}{2},\, l''}_{c-\frac{1}{2},\, c'+\frac{1}{2},\, c''} \tag{24}$$
$$+ [\,(1-l')c-(1-l)c'\,]\, C^{l\, l'\, l''}_{c\, c'\, c''} \Big\}\, C^{l\, l'\, l''}_{m\, m'\, m''}\, L^{l''}_{m'',c''}.$$

Let us list the important subalgebras in AC(so(4,1)) (and PAC(so(3,2))):

1) the maximal finite-dimensional subalgebra is so(4,1) (so(3,2)). The corresponding generators are $T^{c}_{\alpha(2l)}\ (L^{l}_{m,c})$ with $l\leq 1$;

2) a subalgebra formed by the zero conformal weight generators in AC(so(4,1)) turns out to be isomorphic to the algebra su(∞) of area–preserving (symplectic) diffeomorphisms on the sphere S^2 considered in Ref.[24]

$$[\, L^{l}_{m,0},\ L^{l'}_{m',0} \,] = \tfrac{1}{2}\sum_{l'',m''} \left(1-(-1)^{l+l'-l''}\right)$$
$$\times \sqrt{l l'(l+l'-l'')(l+l'+l''+1)}\, C^{l-\frac{1}{2},\, l'-\frac{1}{2},\, l''}_{-\frac{1}{2},\ \frac{1}{2},\ 0}$$
$$\times\, C^{l\, l'\, l''}_{m\, m'\, m''}\, L^{l''}_{m''\, 0}. \tag{25}$$

In this way there are two methods to obtain Eq.(25). One is a limit $N \to \infty$ in SU(N) as in Ref.[24] and the other is to calculate the Poisson brackets [12],[28]. The corresponding zero-conformal weight subalgebra in PAC(so(3,2)) is a non-compact version su(∞,∞), the algebra of are-preserving diffeomorphisms on the hyperboloid $S^{1,1}$ = so(2,1)/so(1,1) (see Refs.[24-29],[12]). The zero-conformal weight subalgebra in the higher spin algebra AC(hsc*(3)) turns out to be isomorphic to the non-negative frequency subalgebra in the Kac-Moody algebra for su(∞,∞).

3) The following generators

$$M_{\alpha(2)} = \tfrac{1}{\sqrt{2}} L^{0}_{\alpha(2)} \left(= \tfrac{1}{\sqrt{2}} T^{(1,0)}_{\alpha(2)}\right),\ P_{\alpha(2)} = L^{-1}_{\alpha(2)} \left(= T^{(1,-1)}_{\alpha(2)}\right),$$
$$P_{\alpha(2l)} = L^{-l}_{\alpha(2l)} \left(= T^{(1,-l)}_{\alpha(2l)}\right),\ l = 2,3,\ldots \tag{26}$$

form a subalgebra which can be viewed as an extension of the Poincare algebra:

$$[\, P_{\alpha(2l)},\ P_{\beta(2l')} \,] = (l-l')\, P_{\alpha(2l)\beta(2l')},$$
$$[\, M_{\alpha(2)},\ P_{\beta(2l)} \,] = 2l\, \varepsilon_{\alpha\beta}\, P_{\alpha\beta(2l-1)},\ l = 1,2,\ldots, \tag{27}$$

or in the tensor notations

$$[\, P_{\mu(l)},\ P_{\nu(l')} \,] = (l-l')\, P_{\mu(l)\nu(l')},$$
$$[\, M_{\mu\nu},\ P_{\rho(l)} \,] = l\left(\eta_{\mu\rho} P_{\rho(l-1)\nu} - \eta_{\nu\rho} P_{\rho(l-1)\mu}\right), \tag{28}$$
$$(\mu,\nu,\rho,\sigma = 0,1,2\,)$$

We also want to point out the algebras AC(so(4,1)) and AC(so(3,2)) as well as their higher spin extensions have an important involutive automorphism

$$\mathcal{R}\left(T^{(s,c)}_{\alpha(2\ell)}\right) = (-1)^{s+\ell}\, T^{(s,-c)}_{\alpha(2\ell)}, \qquad \mathcal{R}^2 = 1,$$
$$\mathcal{R}\left(L^{c}_{\alpha(2\ell)}\right) = (-1)^{\ell+1}\, L^{-c}_{\alpha(2\ell)}, \tag{29}$$

(for the generating elements we have $\mathcal{R}(a_a)=b_a$, $\mathcal{R}(b_a)=a_a$). As a matter of fact $\mathcal{R}$ is an automorphism from the Weyl group of the conformal algebra. Similar the conformal algebra can be obtained from the Poincaré algebra by means of $\mathcal{R}(K_\mu = \mathcal{R}P_\mu\mathcal{R}$, and D appears in [P,K]), the analytic continuation of that may be obtained from the extension (27,28) of the Poincaré algebra $\mathcal{R}(P_{\mu(\ell)}) = (-1)^{1+\ell} K_{\mu(\ell)} = L^{\ell}_{\mu(\ell)}$ and all the other generators $L^{\ell}_{\mu(\ell)}$ with $c \neq \pm\ell$ appear in the RHS of the commutators $[P_{\mu(\ell)}, K_{\nu(\ell')}]$.

5. Extensions with the Abelian Kernels

Until recently we have considered the higher–dimensional generalizations of the centreless Virasoro algebra. However in the quantum theory the Virasoro algebra acquires the central term. It is an anomalous term appearing in the operator product expansion $\Theta_{\mu\nu}(x)\Theta_{\rho\sigma}(x+\varepsilon)$ of the two energ–momentum tensors. The problem arising in the quantum theory is: **what is an analog of the central charge in D>2?**

As a matter of fact the physical answer is contained in Refs.[3–5]. The important result obtained therein consists in the following [5]: nontrivial solutions of the D–dimensional models are possible only if three appear a scalar field P(x) with the scale dimension d_P=D–2 in the operator product expansion $\Theta_{\mu\nu}(x)\Theta_{\rho\sigma}(x+\varepsilon)$. This is an operatorial Schwinger term. For D=2 the field P(x) has a zero scale dimension and turns out to be a constant [5]

$$P(x)\Big|_{D=2} = -\frac{c}{144\pi} \tag{30}$$

where c coincides with the central charge of 2D theories.

Now, having in our hands the inf–dim algebras, we might speculate about a cohomological interpretation of this phenomenon.

Above all let us remind some facts conserning cohomology of the Lie algebras (see e.g. Refs.[31,32]). Let g be a Lie algebra and π be some module over g. Then one can define the cohomologies $H^q(g;\pi)$ with the coefficients in the module π. In particular we are interested in the cohomologies $H^2(g;\pi)$ of the bilinear antisymmetric map g×g ⟶ π. When π is a one–dimensional module, i.e. π=**C** or **R** (for complex or real g), the elements of $H^2(g)$ are non–trivial cocycles g×g ⟶ **C**(**R**) defining non–trivial central extensions of g. The more general cohomologies $H^2(g;\pi)$ for nontrivial π with dim π>1 have a similar meaning. Their elements define non–splitted extensions of g with the Abelian kernel π (see Ref.[32]). The extension of g with the Abelian kernel π is an algebra

$$[T_A, T_B] = f_{AB}{}^{C}\, T_C + C_{AB}{}^{a}\, P_a, \tag{31a}$$
$$[T_A, P_a] = g_{Aa}{}^{b}\, P_b, \quad [P_a, P_b] = 0, \tag{31b}$$

where T_A form a basis in g, P_a form a basis in π, f^{C}_{AB} are the structure constants of g, g^{b}_{Aa} are the matrix elements of the representation g in π and C^{a}_{AB} is a cocylce g×g ⟶ π. When the cocycle C^{a}_{AB} is non–trivial (belongs to $H^2(g;\pi)$) the extension of g with the Abelian kernel π turns out to be nonsplitted, i.e. it cannot be splitted into a semidirect sum of g and the Abelian ideal π (the additional term with P in (31a) cannot be taken away by means of any changing of the basis).

Now let us return to our quasiconformal algebras AC(so(D+1,1)). The above–described result of Refs.[3–5] might be possibly formulated in the following form

$$\dim H^2(g;\pi) \neq 0, \tag{32}$$

where g=AC(so(D+1,1)) (or its Minkowski version) and π is an elementary representation of so(D+1,1)χ=(0,d_P) (scalar field P(x) with the scale dimension d_P=D–2). We denote the corresponding extension with this Abelian kernel (field P(x)) as AC(so(D+1,1)). In principal it is not impossible to conjecture that the result of [5] indicates more strong statement: $\dim H^2(g;\pi)=1$, for $\pi=(0,d_P)$, and $H^2(g;\pi)=0$ (g=AC(so(D+1,1)) for some other elementary representations of so(D+1,1).

It should be mentioned that in Ref.[2] we have conjectured that the algebra AC(g) has a unique central extension $\widetilde{AC}(g)$ (Conjecture 1). However more correctly $\widetilde{AC}(g)$ is not a usual $\left(\text{for } g\neq sl_2\right)$ central extension but might be an extension by means of the non–trivial representation of g.

To conclude this section, we want to point out that the 3D algebra AC(so(4,1)) indeed does not admit any non–trivial central extension, i.e. $\dim H^2(AC(so(4,1));\mathbb{R})=0$ in agreement with our conjecture and the fact established in [5] that there are no non–trivial usual central extensions in D=3.

The most general possible expression for cocycle of AC(so(4,1)) is (see (21))

$$C^{\ell}_{m,c}\Big|^{\ell'}_{m',c'} = (-1)^m \delta(m+m')\delta(c+c')\,\phi^{\ell}_{c},\quad \phi^{\ell}_{c} = -\phi^{\ell}_{-c}\,. \tag{33}$$

The Jacobi identities require in particular

$$(-1)^{\ell''}\phi^{\ell''}_{c''} f^{\ell,\ell',\ell''}_{c,c',-c''}\sqrt{2\ell''+1} + (-1)^{\ell'}\phi^{\ell'}_{c'} f^{\ell'',\ell,\ell'}_{c'',c,-c'}\sqrt{2\ell'+1} + (-1)^{\ell}\phi^{\ell}_{c} f^{\ell',\ell'',\ell}_{c',c'',-c}\sqrt{2\ell+1} = 0\,. \tag{34}$$

Taking into account $\phi^{\ell}_{0}=0$ followed from antisymmetry of the commutator and setting $\ell=c=1$ in (34) and $\phi^{1}_{1}=0$ (it means that there are no central terms in the conformal algebra SO(4,1)), we obtain the general solution is only $\phi^{\ell}_{c}=0$ for all ℓ and c.

7. Conclusion

We have demonstrated that in D>2 there exist inf–dim extensions of the conformal algebras similar the Virasoro extension of the little conformal algebras in D=1,2. In the authors' opinion such algebras might become a basis of the algebraic theory of exactly solvable conformal quantum field models in D>2 like the theory on the Viraso algebra in D=2. Corresponding quasiconformal symmetry might be a hidden underlying symmetry of solvable models. Remarkably that the main features of the Virasoro algebra have natural analogs in D>2. Practically all kinds of extensions of Vir take place also in the general case in question. We have already considered supersymmetric and classical higher–spin extensions. Extensions similar the W_N–algebras seemingly also exist in higher dimensions (see also Ref.[28]).

Anyway, we believe that the new class of algebras conserned may be of interest both in physical and mathematical problematics.

Appendix

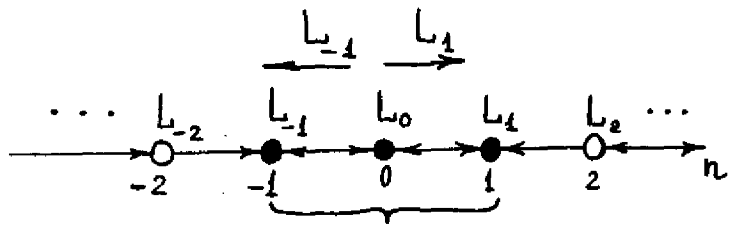

Fig.1. The Virasoro algebra. ● – the sl_2 generators; ○ – the additional generators L_n with $|n|>1$. The raising and lowering generators L_1 and L_{-1} act along the rows → and ← resp.

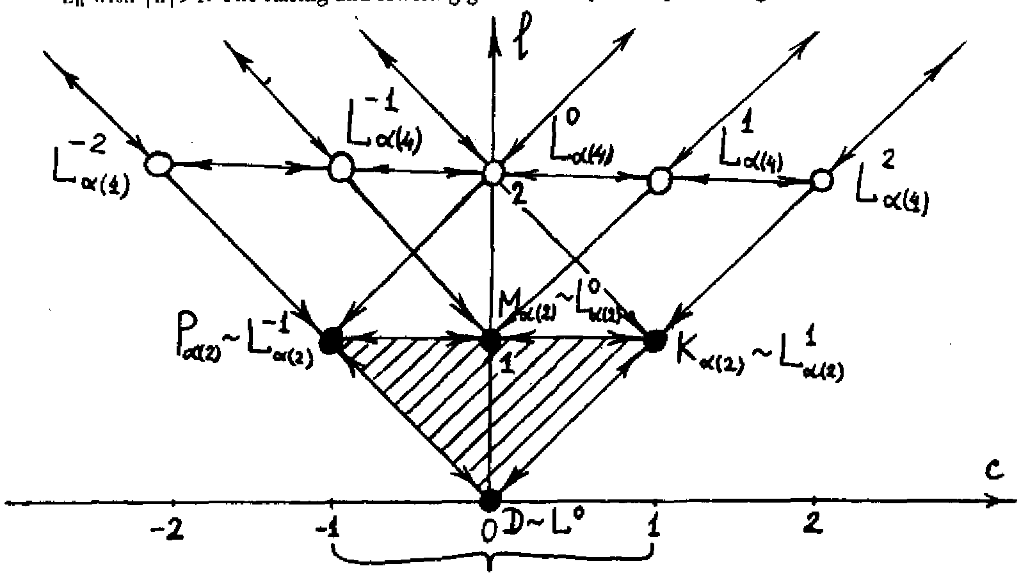

Fig.2. The Infinite–Dimensional Quasiconformal Extension (Analytic Continuation) AC(so(4,1)) (or PAC(so(3,2))) of the 3D Conformal Algebra so(4,1) (or so(3,2)). ● – the usual conformal generators; ○ – the additional generators $L^{c}_{\alpha(2\ell)}$ with $\ell>1$. The raising and lowering generators $K_{\alpha(2)}$ and $P_{\alpha(2)}$ act along the rows → and ← resp.

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