FR9100253

COMMISSARIAT A L'ENERGIE ATOMIQUE

CENTRE D'ETUDES NUCLEAIRES DE SACLAY
Service de Documentation
F91191 GIF SUR YVETTE CEDEX

CEA-CONF --10252

L1

2D GRAVITY AND RANDOM MATRICES

ZINN-JUSTIN J.
CEA Centre d'Etudes Nucleaires de Saclay, 91 - Gif-sur-Yvette (FR).
Service de Physique Theorique

Communication présentée à : 9. International Conference on Problems in Quantum Field Theory

Dubna (SU)
24-28 Apr 1990

2D Gravity and Random Matrices

J. Zinn-Justin
Service de Physique Théorique de Saclay
F-91191 Gif-sur-Yvette Cedex, FRANCE

Abstract. Recent progress in 2D gravity coupled to $d \leq 1$ matter, based on a representation of discrete gravity in terms of random matrices, is reported. The matrix problem can be solved in many cases by the introduction of suitable orthogonal polynomials. Alternatively in the continuum limit the orthogonal polynomial method can be shown to be equivalent to the construction of representation of the canonical commutation relations in terms of differential operators.

In the case of pure gravity or discrete Ising-like matter the sum over topologies is reduced to the solution of non-linear differential equations.

The $d = 1$ problem can be solved by semiclassical methods.

1. Introduction

It was proposed some time ago [1] that the integral over the internal geometry of a 2D surface can be discretized as a sum over randomly triangulated surfaces. The use of such a lattice regularization allows the partition function of 2D quantum gravity coupled to certain matter systems to be expressed as the free energy of an associated hermitian matrix model. This matrix realization can frequently be solved by means of large N techniques [2] (see also [3] and references therein), and the solutions restricted to fixed topology of

---

(zinn@frsac11.bitnet)

Talk presented at the IX International Conference on "Problems in Quantum Field Theory" (Dubna 1990) April 24-28 SPhT/90/065

the two dimensional spacetime are found to be in agreement with the continuum Liouville results of [4] (for a recent review and more references, see [5]).

More recently [6–8], a continuum limit that includes the sum over topologies of two-dimensional surfaces was defined for certain matter systems coupled to 2D quantum gravity. The continuum limit specific heat for these models was moreover found to satisfy an ordinary differential equation, in principle allowing a full non-perturbative solution. The models in [6–8] were later argued to correspond to 2D gravity coupled to minimal conformal matter[1] of type $(p,q)=(2m-1,2)$, and the differential equations satisfied by their specific heats and correlation functions were found to be closely related to the KdV hierarchy [6,8,10]. In particular, the specific heat for the case of pure gravity (i.e. $(p,q)=(3,2)$, so that $d=0$) was found to satisfy the $2^{\rm nd}$ order Painlevé I equation.

In [11], it was proposed that the more general (p,q) models are similarly related to the generalized KdV hierarchies. General properties of the differential equations of interest were studied in [12]. The equations were shown always to follow from an action principle, the basic action for a (p,q) model taking the form $S_0=\int \mathrm{Res}\, Q^{p/q+1}$. These actions do embody the essential features of the problem in a more compact form than the differential equations themselves. As well, certain properties of the large order behaviour of the solutions of these equations follow from their being derivable from an action principle [13]. Additional results can be found in [14]. Also relevant are the classic articles [15,16] on the generalized KdV hierarchy.

Finally 2D gravity coupled with $d=1$ matter has also been investigated [17].

2. The one matrix model

2.1. Matrix representation of discretized 2D gravity

Following 't Hooft analysis of the large N limit of $SU(N)$ gauge theories[18] it has been recognized that if one considers the integral:

$$Z(\alpha_k,N)=\int \mathrm{d}M \mathrm{e}^{-N\mathrm{tr}\, V(M)},$$

in which M is a $N\times N$ hermitian matrix and:

$$V(\lambda)=\lambda^2+\sum_{k\geq 3}\alpha_k\lambda^k,$$

---

[1] In the nomenclature of [9], "minimal" conformal field theories are specified by two relatively prime integers (p,q), and have dimension (central charge) $d=1-6(p-q)^2/pq$.

the expansion of $F = \ln Z$ for N large takes the form:

$$F = \ln Z = \sum N^{\chi} F_{\chi}(\alpha_k),$$

where F_{χ} is the sum of all Feynman diagrams with Euler-Poincaré characteristics χ, i.e. with $h = \frac{1}{2}(2-\chi)$ handles. It follows that the dual of a Feynman diagram contributing to F_{χ} can be represented as a surface of fixed topology, the powers of $\alpha_3, \alpha_4, \ldots$ counting the number of triangles, squares... of the surface (taking the logarithm $F = \ln Z$, yields the gravity model free energy which represents the sum over connected surfaces only). We have thus a formulation of discrete 2D gravity in terms of random matrices (if by convention the area of each triangle, square... is assumed to be 1, the power of α_k measures the total area of the k-gones).

In what follows it will be convenient to change the normalization in the integrand and consider instead:

$$Z(g, \alpha_k, N) = \int \mathrm{d}M \, \mathrm{e}^{-(N/g) \operatorname{tr} V(M)}. \tag{2.1}$$

Note that to describe pure gravity only one α_k is needed, for instance one can use only triangles. More general models correspond to the addition of some new degrees of freedom on the surface.

The continuum limit. The continuum limit is taken by letting the area of the surfaces tend towards infinity. Surfaces of large area are connected with the large order behaviour of the Taylor series expansion of F_{χ} in powers of g and therefore to the leading singularity of $F_{\chi}(g)$. At g fixed the large N limit selects the surfaces of the topology of the sphere. However it will be shown below that it is possible to take the large N limit and at the same let g approach in some correlated way the location g_c of the leading singularity, in such a way that surfaces of infinite area and arbitrary topology contribute.

2.2. The calculation of the integral over matrices

The jacobian. The first step in the calculation of integral (2.1) is the integration over unitary matrices, i.e. the decomposition of the measure $\mathrm{d}M$ in the product of the group measure multiplied by the residual measure over the eigenvalues λ_i of M. The jacobian Δ^2 which arises when passing from the integration over matrices to the integration over eigenvalues can be obtained using the Faddeev–Popov trick. If one writes the hermitian matrix M as $M = U^{\dagger} \Lambda U$, in which Λ is diagonal, and integrates over the unitary matrix U one finds:

$$Z = \int \prod_i \mathrm{d}\lambda_i \, \Delta^2(\Lambda) \exp\left[-\frac{N}{g} \sum_i V(\lambda_i)\right]. \tag{2.2}$$

with:

$$\Delta(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j). \tag{2.3}$$

The method of orthogonal polynomials. The essential remark is that the $\Delta(\lambda)$ is the Vandermonde determinant

$$\Delta(\lambda) = \det \lambda_i^{j-1}.$$

Therefore if one considers a set of polynomials $P_n(\lambda)$ such that:

$$P_n(\lambda) = \lambda^n + O\left(\lambda^{n-1}\right), \tag{2.4}$$

then:

$$\Delta(\lambda) = \det P_{j-1}(\lambda_i). \tag{2.5}$$

One chooses the polynomials to be orthogonal with respect to the measure $d\mu(\lambda) = d\lambda\, e^{-NV(\lambda)/g}$:

$$\int d\mu(\lambda) P_m(\lambda) P_n(\lambda) = s_m \delta_{mn}.$$

If one then expand the determinant Δ, due to the orthogonality conditions only the diagonal terms in the product Δ^2 do not vanish (ε_Π is the signature of the permutation Π):

$$Z = \sum_{\Pi,\Pi'} \int \prod_i d\mu(\lambda_i) \varepsilon_\Pi \varepsilon_{\Pi'} \prod_i P_{\Pi_i}(\lambda_i) \prod_j P_{\Pi'_j}(\lambda_j), \tag{2.6a}$$

$$= N!\, s_0 s_1 \ldots s_{N-1}. \tag{2.6b}$$

The calculation of the initial has been reduced to the determination of the normalization of the polynomials.

Recursion formulae. One sets:

$$P'_n = \sum_{m=0}^{n-1} a_{nm} P_m, \tag{2.7a}$$

$$\lambda P_n = \sum_{m=0}^{n+1} b_{nm} P_m. \tag{2.7b}$$

From the normalization of P_n (equation (2.4)) it follows:

$$a_{n,n-1} = n, \quad b_{n,n+1} = 1.$$

Note that the matrices A, B, which form a representation of the canonical commutation relations, $[A, B] = 1$, can be related to the polynomial $V(\lambda)$ by:

$$\int d\mu(\lambda)\, \lambda P_m P_n = b_{nm} s_m = b_{mn} s_n, \tag{2.8}$$

and:

$$0 = \int d\lambda \frac{d}{d\lambda} \left(P_n P_m e^{-NV(\lambda)/g)} \right) = a_{nm} s_m + a_{mn} s_n - \frac{N}{g} V'_{mn}, \tag{2.9}$$

where the notation

$$V'_{mn} = \int d\mu V' P_m P_n\,, \tag{2.10}$$

has been introduced. Equation (2.8) shows that b_{mn} is different from 0 only if $|m-n| \leq 1$. In what follows it will be convenient to assume that V is an even polynomial. Then the polynomials have a defined parity and (2.7a) reduces to:

$$\lambda P_n = P_{n+1} + b_n P_{n-1}, \tag{2.11}$$

in which b_n is a short-cut notation for $b_{n,n-1}$.

Specializing (2.8) to $m = n - 1$ one obtains the useful relation:

$$\int d\mu\, \lambda P_n P_{n-1} = s_n = b_n s_{n-1} \;\Rightarrow b_n = \frac{s_n}{s_{n-1}},$$

and thus:

$$F = \ln Z = \ln \left(s_0^N N! \right) + \sum_{n=1}^{N-1} (N - n) \ln b_n. \tag{2.12}$$

Specializing also (2.9) to $m = n - 1$ one finds:

$$g \frac{n}{N} = V'_{n,n-1} / s_{n-1}. \tag{2.13}$$

Note that since V'_{mn} can be entirely expressed in terms of b_n, this equation leads to a recursion relation for the coefficient b_n. In the simple example:

$$V(\lambda) = \frac{1}{2}\lambda^2 + \frac{1}{4}\lambda^4,$$

which corresponds to *pure gravity*, one obtains:

$$g \frac{n}{N} = b_n \left(1 + b_{n-1} + b_n + b_{n+1}\right). \tag{2.14}$$

2.3. Large N limit, spherical topology

For N large and n/N finite b_n has a limiting form independent of the boundary conditions, solution of an algebraic equation of the form:

$$g\frac{n}{N} = W(b_n),$$

where W is related to V by:

$$W(b) = \frac{1}{4i\pi}\oint \frac{\mathrm{d}z}{z} V\left[2(z+1/z)\sqrt{b}\right].$$

In the example above:

$$W(b) = b + 3b^2.$$

The singularities of the free energy are related to the singularities of the function $b(g)$. The function b is singular when $W'(b)$ vanishes. Let us call b_c the corresponding value of b. If $W''(b)$ does not vanish, b has a square root singularity. This is the case for pure gravity where:

$$b_c = -1/6, \Rightarrow b - b_c \sim \left(\frac{(gn/N)+1/12}{3}\right)^{1/2},$$

or:

$$\frac{b_c - b(g)}{b_c} \sim \left(\frac{g_c - g}{g_c}\right)^{1/2}. \tag{2.15}$$

(the solution has to be continuously connected with solution $b = 0$ at $g = 0$). If V is an even polynomial of degree $2m$, $W(b)$ is a polynomial of degree m and its coefficients can be adjusted in such a way that $m-1$ successive derivatives of W vanish. Then:

$$W(b) = [b_c^m - (b_c - b)^m] / (m b_c^{m-1}),$$

and b behaves like (setting $g_c = b_c/m$):

$$\frac{b_c - b(g)}{b_c} \sim \left(\frac{g_c - g}{g_c}\right)^{1/m}. \tag{2.16}$$

In large N limit the generating function F is given by:

$$F = N^2 \int_0^1 \mathrm{d}x(1-x)\ln b(gx) + F_0 \equiv F_{\mathrm{sing.}} + F_0,$$

where F_0 is less singular. Setting $gx = x'$ and differentiating twice with respect to g one finds:

$$\left(g^2 F_{\mathrm{sing.}}(g)\right)'' \sim N^2 \ln b(g), \tag{2.17}$$

and thus

$$F''_{\text{sing.}}(g) \sim \frac{N^2}{g_c^2} \ln b(g) \sim \frac{N^2}{g_c^2 b_c} \left[b(g) - b_c \right] \text{ for } g \to g_c, \tag{2.18}$$

equation which relates directly the singular part of the generating function to the function $b(g)$. If b has the behaviour (2.16) then:

$$F_{\text{sing.}}(g) \sim -\frac{N^2 m^2}{(2m+1)(m+1)} \left(\frac{g_c - g}{g_c} \right)^{2+1/m}. \tag{2.19}$$

This suggests the existence of a scaling region in which N becomes large at $N^2 (g_c - g)^{2+1/m}$ fixed.

Interpretation of multicritical models. If the case $m = 2$ has a simple interpretation in terms of pure gravity, the interpretation of the m-model $m > 2$ is less clear in general. The $m = 3$ model has been discovered to describe the Yang–Lee edge singularity on a random lattice. More generally it is conjectured that the m model can associated with the $(2, 2m-1)$ minimal model of the Conformal Field Theory (CFT) classification.

The minimal critical model. If one considers the minimal critical model, i.e. the model in which the polynomial V has the smallest degree possible at a fixed value of m, one finds, from the explicit form of W given above, that the term of highest degree in integral (2.1) is positive for m odd and negative for m even. This shows that in the latter case the minimal integrals can be defined only by analytic continuation. This is the source of serious difficulties.

2.4. Beyond the spherical approximation: pure gravity

In this section only the simplest case, pure gravity, which can be generated by a quartic function V, will be considered. To evaluate the corrections to the large N limit one notes that, for N large and n/N of order 1, b_n becomes a function of the product gn/N only. Therefore:

$$b_{n+1} = b(gn/N) + \frac{g}{N} b' + \frac{g^2}{2N^2} b'' + \cdots \tag{2.20}$$

Note that if one believes in the scaling postulated above, $N^{-2}(g_c - g)^2$ goes to zero. Since b has an algebraic singularity the successive terms in the expansion are less and less singular. Introducing expansion (2.20) in equation (2.14) one sees that the first derivative cancels. It follows at leading order:

$$\frac{g_c - g}{g_c} = \left(\frac{b_c - b(g)}{b_c} \right)^2 + \frac{g_c^2}{3N^2} \frac{b''(g)}{b_c}. \tag{2.21}$$

Setting:

$$b(g)/b_c = 1 - N^{-2/5} f\left[N^{4/5}(g_c - g)/g_c\right], \tag{2.22}$$

choice consistent with the scaling form anticipated above, one transforms equation (2.21) into:

$$z = f^2 - f''/3. \tag{2.23}$$

In terms of the scaling function f, $F_{\text{sing.}}$ is then given by:

$$F''_{\text{sing.}} = -\frac{N^{8/5}}{g_c^2} f\left[N^{4/5}(g_c - g)/g_c\right]. \tag{2.24}$$

This equation can be more conveniently rewritten:

$$\frac{\mathrm{d}^2 F_{\text{sing.}}}{(\mathrm{d}z)^2} = -f(z), \quad \text{with } z = N^{4/5}(g_c - g)/g_c.$$

The Painlevé equation. Equation (2.23) is a Painlevé equation of first kind. It has been extensively studied in the literature. Its only moveable singularities are double poles of residue 2. Equation (2.24) shows that these poles correspond to double zeros of Z. The solution depends on two parameters. We are interested only in solutions which have an asymptotic expansion for z large (the topological expansion) which begins with the spherical result (2.15), i.e. $f(z) \sim \sqrt{z}$. This expansion takes the form:

$$f(z) = \sum_{k=0} f_k z^{1/2-5k/2}, \quad f_0 = 1.$$

It is easy to verify that all terms in the expansion of F are positive except the first one. But the first term goes to zero when g goes to g_c (and thus z goes to zero) and is thus dominated by the regular part.

Stability of Painlevé equation and large order behaviour. To analyze the stability of equation (2.23) near a solution f_0 one sets:

$$f(z) = f_0(z) + \epsilon(z).$$

ϵ satisfies the linearized equation:

$$\epsilon'' - 6 f_0 \epsilon = 0.$$

It follows:

$$\epsilon \underset{z \to +\infty}{\propto} z^{-1/8} e^{\pm(4\sqrt{6}/5) z^{5/4}}.$$

The existence of the asymptotic expansion fixes one parameter, to suppress the growing exponential. A second parameter remains free. Also this result implies that the topological expansion is a divergent series whose terms grow like $\Gamma(2k - 1/2)$ and are all negative at large orders (they are actually negative for $k > 0$). Therefore the series is non-Borel summable. The sum of series is ambiguous and depends on one parameter, result consistent with the stability analysis. The meaning of this new parameter is obscure.

3. A general method: the canonical commutation relations

It is possible to study the recursion formulae (2.13) for general polynomials $V(\lambda)$. In the continuous limit one finds a non-linear differential equation of more general type for a scaling function f. However a simpler algebraic method has been found, which easily generalizes to the several matrix problem. It is convenient to first introduce normalized orthogonal polynomials Π_n:

$$P_n = \sqrt{s_n}\Pi_n,$$

and define now the matrices A and B in terms of Π_n:

$$\Pi'_n = \sum_{m=0}^{n-1} a_{nm}\Pi_m, \quad \lambda\Pi_n = \sum_{m=0}^{n+1} b_{nm}\Pi_m . \tag{3.1}$$

With this new definition the matrix B is symmetric. In terms of the coefficients b_n, now directly *defined by equation* (2.11), the recursion formula for the orthogonal polynomials becomes:

$$\lambda\Pi_n = \sqrt{b_{n+1}}\Pi_{n+1} + \sqrt{b_n}\Pi_{n-1}.$$

The equation for A now reads:

$$A + {}^tA = \frac{N}{g}V'(B),$$

and one still has:

$$[A,B] = 1.$$

It is convenient to shift A, introducing a matrix C:

$$C = A - \frac{N}{2g}V'(B).$$

Then C is antisymmetric and still satisfies:

$$[C,B] = 1. \tag{3.2}$$

A remarkable property can be verified: From the commutation relation (3.2) and the condition $c_{mn} = 0$ for $|m-n| > 2l-1$, where $2l$ is the degree of V, it is possible to calculate c_{mn} in terms of b_n. In addition an equation for b_n is obtained which is the finite difference derivative of equation (2.13). The coefficient c_{mn} are obtained by solving recursion relations whose boundary conditions replace the coefficients of V. Since the coefficients in V are determined by the criticality conditions, the original problem can thus be entirely reformulated in terms of the matrix B satisfying the commutation relation

(3.2), from which the singular part of the free energy F can be calculated. It remains to take the large N, scaling limits directly in these expressions.

The large N limit. One can expand the matrix B, considered as a function of the continuous variable $x = n/N$:

$$
\begin{aligned}
(B\Pi)_n &= \sqrt{b_{n+1}}\Pi_{n+1} + \sqrt{b_n}\Pi_{n-1}, \\
&= \sqrt{b_c}\left[2\Pi_n + \frac{b_n - b_c}{b_c}P_n + \frac{1}{N^2}\Pi_n''\right] + \cdots
\end{aligned}
$$

The leading term, proportional to the identity, does not contribute to the commutation relation. The next term is a second order differential operator. It is convenient to parametrize B in the form:

$$(B\Pi)_n = \frac{\sqrt{b_c}}{N^2}\left(2N^2 + Q\right)\Pi_n,$$

with:

$$Q = \mathrm{d}^2 - u(x), \quad u = N^2\frac{b_c - b_n}{b_c}, \tag{3.3}$$

where d is a notation for $\mathrm{d}/\mathrm{d}x$. The formal hermiticity of the operator Q follows directly from the symmetry of B. It is then not too surprising that in the same limit the matrix C becomes also a differential operator P which is moreover formally antihermitian because C is antisymmetric. The degree of the P is seen to be at most $2l-1$ when the conditions of muticriticality are met.

Finally, as in the pure gravity case, one verifies that the singular part is still given by:

$$\frac{\mathrm{d}^2 F_{\text{sing.}}}{(\mathrm{d}x)^2} = -u(x).$$

The problem has thus been reduced to finding solutions of the canonical commutation relations (3.2).

Construction of the differential equations. The differential equations following from (3.2), $[P, Q] = 1$, may be derived as follows. The differential operator that can satisfy (3.2) is constructed as a fractional power of the oper[illegible] (3.3). Formally, $Q^{1/2}$ may be represented within an algebra of formal pseudo-[illegible] operators as

$$Q^{1/2} = \mathrm{d} + \sum_{i=1}^{\infty}\left\{e_i, \mathrm{d}^{-i}\right\}. \tag{3.4}$$

The differential equations describing the $(2l-1, 2)$ minimal model are given by

$$[Q_+^{l-1/2},\, Q] = 1, \tag{3.5}$$

where $Q_+^{l-1/2}$ indicates the part of $Q^{l-1/2}$ with only non-negative powers of d. The formal expansion of $Q^{l-1/2}$ (an anti-hermitian operator) in powers of d is given by

$$Q^{l-1/2} = \mathrm{d}^{2l-1} - \frac{2l-1}{4}\left\{u, \mathrm{d}^{2l-3}\right\} + \cdots \tag{3.6}$$

(where only symmetrized odd powers of d appear in this case). It is easy to show that the commutator takes the form

$$[Q_+^{l-1/2}, Q] = (l+1/2)R_l' \ . \tag{3.7}$$

After integration, the equation $[Q_+^{l-1/2}, Q] = 1$ thus becomes

$$(l+\tfrac{1}{2})R_l[u] = x \ . \tag{3.8}$$

The quantities R_l in (3.7) are easily seen to satisfy the recursion relation:

$$R_{l+1}' = \frac{1}{4}R_l''' - uR_l' - \frac{1}{2}u'R_l \ . \tag{3.9}$$

While this recursion formula only determines R_l', by demanding that the R_l ($l \neq 0$) vanish at $u = 0$, one for example obtains

$$\begin{aligned} R_0 &= \frac{1}{2}\,, \qquad R_1 = -\frac{1}{4}u\,, \qquad R_2 = \frac{3}{16}u^2 - \frac{1}{16}u''\,, \\ R_3 &= -\frac{5}{32}u^3 + \frac{5}{32}\left(uu'' + \tfrac{1}{2}u'^2\right) - \frac{1}{64}u^{(4)} \ . \end{aligned} \tag{3.10}$$

The corresponding first $Q_+^{l-1/2}$ are,

$$\begin{aligned} Q_+^{1/2} &= \mathrm{d}\,, \qquad Q_+^{3/2} = \mathrm{d}^3 - \frac{3}{4}\{u, \mathrm{d}\}\,, \\ Q_+^{5/2} &= \mathrm{d}^5 - \frac{5}{4}\{u, \mathrm{d}^3\} + \frac{5}{16}\left\{(3u^2 + u''), \mathrm{d}\right\} \ . \end{aligned} \tag{3.11}$$

The R_l's satisfy as well a functional relation that allows to write eq. (3.8) as the variation of an action. To derive this, and for later purposes, it is useful to introduce the "residue" and "trace" of a formal differential operator $A = \sum_{i=-\infty}^{k} a_i(x)\,\mathrm{d}^i$ as

$$\begin{aligned} \mathrm{Res}\, A &\equiv a_{-1}\ , \\ \mathrm{tr}\, A &\equiv \int \mathrm{d}x\ \mathrm{Res}\, A = \int \mathrm{d}x\, a_{-1}\ . \end{aligned} \tag{3.12}$$

This trace can be interpreted as the "logarithmic divergence" of $\mathrm{tr}\, A = \int \mathrm{d}x\, \langle x|A|x\rangle$, from which follows the cyclicity property $\mathrm{tr}\, AB = \mathrm{tr}\, BA$ for any two differential operators A, B. Since $R_{l+1} = \frac{1}{2}\,\mathrm{Res}\, K^{l+1/2}$:

$$\begin{aligned} \frac{\delta}{\delta u}\int \mathrm{d}x\, R_{l+1}[u] &= \frac{\delta}{\delta u}\frac{1}{2}\mathrm{tr}\, Q^{l+1/2} = -(l+\tfrac{1}{2})\frac{1}{2}\,\mathrm{Res}\, Q^{l-1/2} \\ &= -(l+\tfrac{1}{2})R_l[u] \ . \end{aligned} \tag{3.13}$$

The differential equation (3.8) therefore results as the variational derivative with respect to u of the action

$$S = \int \mathrm{d}x\, \left(R_{l+1} + xu\right) \ . \tag{3.14}$$

4. $(q-1)$-matrix models

It is also possible to solve, by a method of orthogonal polynomials, models involving integration over several matrices $M^{(\alpha)}$. The basic identity is:

$$\int \mathrm{d}M^{(2)} \exp\left(-\operatorname{tr} V_2\left(M^{(2)}\right) + c \operatorname{tr} M^{(1)} M^{(2)}\right) \\ \propto \int \mathrm{d}\Lambda^{(2)} \exp\left[\sum_i -V_2\left(\lambda_i^{(2)}\right) + c\lambda_i^{(1)}\lambda_i^{(2)}\right] \frac{\Delta\left(\Lambda^{(2)}\right)}{\Delta\left(\Lambda^{(1)}\right)},$$

where the $M^{(\alpha)}$ are $N \times N$ hermitian matrices, the $\lambda_i^{(\alpha)}$ their eigenvalues, and $\Delta(\Lambda)$ is the Vandermonde determinant $\det(\lambda_i^{j-1})$. It follows:

$$Z = \int \prod_{\alpha=1}^{q-1} \mathrm{d}M^{(\alpha)}\, \mathrm{e}^{-\operatorname{tr}\left(\sum_{\alpha=1}^{q-1} V_\alpha(M^{(\alpha)}) - \sum_{\alpha=1}^{q-2} c_\alpha M^{(\alpha)} M^{(\alpha+1)}\right)} \\ = \int \prod_{\substack{\alpha=1,q-1\\ i=1,N}} \mathrm{d}\lambda_i^{(\alpha)}\, \Delta\left(\Lambda^{(1)}\right) \mathrm{e}^{-\sum_{i,\alpha} V_\alpha(\lambda_i^{(\alpha)}) + \sum_{i,\alpha} c_\alpha \lambda_i^{(\alpha)} \lambda_i^{(\alpha+1)}}\, \Delta\left(\Lambda^{(q-1)}\right). \tag{4.1}$$

The result in the second line of (4.1) depends on having c_α's that couple matrices along a line (with no closed loops so that the integrations over the relative angular variables in the $M^{(\alpha)}$'s can be performed.) Via a diagrammatic expansion, the matrix integrals in (4.1) can be interpreted to generate a sum over discretized surfaces, where the different matrices $M^{(\alpha)}$ represent $q-1$ different matter states that can exist at the vertices. The quantity Z in (4.1) thereby admits an interpretation as the partition function of 2D gravity coupled to matter. Note that matter has only a finite number of states. Furthermore the only possible symmetry which can be implemented corresponds to reversing the line. Therefore these matrix models can only represent Ising-like matter on a random lattice.

Generalized orthogonal polynomials. To solve the matrix models one defines generalized orthogonal polynomials $\Pi_n(\lambda)$ satisfying:

$$\int \mathrm{d}\mu\left(\lambda^{(1)}, \ldots, \lambda^{(q-1)}\right) \Pi_m\left(\lambda^{(1)}\right) \Pi_n\left(\lambda^{(q-1)}\right) = \delta_{mn}, \tag{4.2}$$

where the measure $\mathrm{d}\mu$ is:

$$\mathrm{d}\mu\left(\lambda^{(1)}, \ldots, \lambda^{(q-1)}\right) = \prod_{\alpha=1,q-1} \mathrm{d}\lambda^{(\alpha)}\, \mathrm{e}^{-\sum_\alpha V_\alpha(\lambda^{(\alpha)}) + \sum_\alpha c_\alpha \lambda^{(\alpha)} \lambda^{(\alpha+1)}}.$$

To derive recursion formulae one then inserts $\lambda^{(\alpha)}$ and $\mathrm{d}/\mathrm{d}\lambda^{(\alpha)}$ respectively in the integral (4.2). It is convenient to introduce a matrix B_α associated with $\lambda^{(\alpha)}$ and defined by:

$$\lambda^{(\alpha)} \int \mathrm{d}\mu\left(\lambda^{(1)} \ldots \mathrm{d}\lambda^{(\alpha-1)}\right) \Pi_m\left(\lambda^{(1)}\right) = [B_\alpha]_{mn} \int \mathrm{d}\mu\left(\lambda^{(1)} \ldots \mathrm{d}\lambda^{(\alpha-1)}\right) \Pi_n\left(\lambda^{(1)}\right),$$

where $\mathrm{d}\mu\left(\lambda^{(1)} \ldots \mathrm{d}\lambda^{(\alpha-1)}\right)$ is the same integrand as in (4.2) but integrated over the $\alpha - 1$ first variables. Note that one can also define $\widetilde{B}_\alpha$, using a similar expression but in which the roles of $\Lambda^{(1)}$ and $\Lambda^{(q-1)}$ are exchanged, i.e. by integrating over $\lambda^{(\alpha+1)}, \ldots, \lambda^{(q-1)}$. Then, integrating over all λ's, one obtains the relation:

$$\widetilde{B}_\alpha = {}^t B_\alpha .$$

One also defines the matrix A_1:

$$\Pi'_m = [A_1]_{mn} \Pi_n .$$

With these definitions, after inserting $\mathrm{d}/\mathrm{d}\lambda^{(\alpha)}$ in (4.2), one finds:

$$\begin{aligned} A_1 + c_1 B_2 &= V_1'(B_1), \\ c_{\alpha-1} B_{\alpha-1} + c_\alpha B_{\alpha+1} &= V_\alpha'(B_\alpha) \quad \text{for } 1 < \alpha < q-1, \\ \widetilde{A}_{q-1} + c_{q-2} B_{q-2} &= V_{q-1}'(B_{q-1}), \end{aligned} \tag{4.3}$$

with $\widetilde{A}_{q-1} = {}^t A_1$, $\widetilde{B}_{q-1} = B_1$, and $[A_1, B_1] = 1$.

Equations (4.3) imply:

$$c_\alpha [B_\alpha, B_{\alpha+1}] = c_{\alpha-1} [B_{\alpha-1}, B_\alpha] = 1, \tag{4.4}$$

and thus:

$$\left[\widetilde{A}_{q-1}, B_{q-1}\right] = -1. \tag{4.5}$$

Inspired by the one-matrix case one then introduces matrices C_α defined by:

$$\begin{aligned} C_1 &= A_1 - \tfrac{1}{2} V_1'(B_1), \\ C_\alpha &= c_{\alpha-1} B_{\alpha-1} - \tfrac{1}{2} V_\alpha'(B_\alpha) \quad \text{for } \alpha > 1. \end{aligned}$$

It follows from these definitions and equations (4.4), (4.5):

$$\begin{aligned} [C_\alpha, B_\alpha] &= 1, \\ C_\alpha &= \tfrac{1}{2} V_\alpha'(B_\alpha) - c_\alpha B_{\alpha+1} \quad \text{for } \alpha < q-1, \\ C_{q-1} &= \tfrac{1}{2} V_{q-1}'(B_{q-1}) - \widetilde{A}_{q-1}. \end{aligned} \tag{4.6}$$

$\mathbb{Z}_2$ *symmetry.* When $V_\alpha = V_{q-\alpha}$ and $c_\alpha = c_{q-1-\alpha}$, the matrix problem has a $\mathbb{Z}_2$ symmetry corresponding to the mapping of matrices $M^{(\alpha)} \mapsto M^{(q-\alpha)}$. This symmetry yields the relations:

$$\widetilde{B}_\alpha = {}^t B_\alpha = B_{q-\alpha}.$$

It follows:

$${}^t C_\alpha = -C_{q-\alpha}.$$

The free energy. Finally it is necessary to calculate the coefficient of highest degree $1/\sqrt{s_n}$ of the polynomials Π_n. One finds the relation:

$$[B_1]_{n+1,n} \equiv \sqrt{b_n} = (s_{n+1}/s_n)^{1/2},$$

and thus:

$$\begin{aligned} F = \ln Z &= \ln(N!\, s_0 s_1 \ldots s_{N-1}), \\ &= \ln(N!\, s_0^N) + 2\sum_{n=1}^{N-1}(N-n)\ln b_n. \end{aligned} \tag{4.7}$$

Note that with the help of the recursion relations it is easy to show that the matrices B_1 and C_1 have non-vanishing matrix elements only for a strip along the diagonal, the width of the strip being related to q and the degrees of the polynomials V_α.

4.1. The continuum limit

In the $N \to \infty$ limit, it can be argued, with arguments similar to those presented in the one-matrix case, that the matrices B_α and C_α again become differential operators of finite order, say p, q respectively (where $p > q$ is assumed), Q_α, P_α. These still satisfy

$$[P, Q] = 1. \tag{4.8}$$

In the continuum limit of the matrix problem (i.e. the "double" scaling limit, which here means couplings in (4.1) tuned to certain critical values), Q becomes a differential operator of the form

$$Q = \mathrm{d}^q + \{v_{q-2}(x), \mathrm{d}^{q-2}\} + \cdots + 2v_0(x). \tag{4.9}$$

(By a change of basis of the form $Q \to f^{-1}(x)Qf(x)$, the coefficient of d^{q-1} may always be set to zero.) The continuum scaling limit of the multi-matrix models is thus abstracted to the mathematics problem of finding solutions of (4.8).

The "grade" of d is defined to be 1, so that the grade[2] of $v_{q-\alpha}$ is α for an operator Q of overall grade q. The $(q-1)$-matrix model provides the *minimum* number of matrices required to realize a $q^{\rm th}$ order differential operator Q. (We can of course realize lower order differential operators by suitable (de-)tuning of couplings in (4.1) since k-matrix models with $k < q-1$ are always embedded as submodels.)

The function v_{q-2} can be identified (up to normalization) in the continuum scaling limit with the second derivative of the free energy with respect to the cosmological constant (here proportional to x). Equivalently v_{q-2} can be written in terms of the 2-point function of a puncture operator $\mathcal{P}$, $v_{q-2} \propto \langle \mathcal{P}\mathcal{P} \rangle$.

The differential equations (4.8) may be constructed as follows. For p, q relatively prime, a $p^{\rm th}$ order differential operator that can satisfy (4.8) is constructed as a fractional power of the operator Q of (4.9). Formally, a $q^{\rm th}$ root may again be represented within an algebra of formal pseudo-differential operators as

$$Q^{1/q} = {\rm d} + \sum_{i=1}^{\infty} \{e_i, {\rm d}^{-i}\} \ . \tag{4.10}$$

The differential equations describing the (p,q) minimal model are given by

$$[Q_+^{p/q},\ Q] = 1 \ , \tag{4.11}$$

where $Q_+^{p/q}$ indicates the part of $Q^{p/q}$ with only non-negative powers of d.

The parametrization in (4.9) has been chosen so that if $v_{q-2l-1} = 0$ (for l integer), then Q is formally hermitian for q even, and anti-hermitian for q odd. This is the situation with no $\mathbb{Z}_2$ symmetry breaking (where the $\mathbb{Z}_2$ refers to the symmetry under reflecting the line of matrices about its center). In the $\mathbb{Z}_2$ symmetric case, all of the e_{2l}'s in (4.10) vanish — the parameters v_{q-2l-1} are $\mathbb{Z}_2$ odd, and enter into the e_{2l}'s of (4.10) only in odd powers, and into the e_{2l+1}'s only in even powers.

4.2. Example: the critical Ising model (4,3)

The Ising model has a natural realization as a two-matrix model [20] in which the two matrices represent the $+/-$ states of an Ising spin. The two-matrix model has been first

[2] This notion of grade is related to the conventional scaling weights of operators. It can also be used to determine the terms that may appear in many equations, since these will only relate terms of overall equal grade.

solved directly using recursion relations [21]. A simpler derivation follows from considering the commutation relation (4.10) [11,12] with

$$\begin{aligned} Q &= \mathrm{d}^3 - \frac{3}{4}\{u, \mathrm{d}\} + \frac{3}{2}w = (\mathrm{d}^2 - u)_+^{3/2} + \frac{3}{2}w \ , \\ P &= Q_+^{4/3} = (\mathrm{d}^2 - u)^2 + \{w, \mathrm{d}\} + v \ , \end{aligned} \tag{4.12}$$

where w is a $\mathbb{Z}_2$ breaking field, ultimately resulting in coupling to a magnetic field (and where v can be set equal to $-8R_2/3$, but it is convenient to leave it free, later to be determined equivalently by equations of motion). For generality, a term $-t(\mathrm{d}^2 - u)$, which represents a deviation from criticality of the Ising model, i.e. in the direction of pure gravity (as described by the Painlevé equation $R_2[u] \sim x$ — recall that $[Q_+^{3/2}, Q] = 4R_2')$ will also be added to P. From (4.11) follow two equations given by setting the d^2 and d coefficients to zero. After integration one finds

$$4R_2 + \frac{3}{2}v = 0 \ , \tag{4.13a}$$

$$w'' - 3uw - 3tw = -h \tag{4.13b}$$

(where the constant of integration can be set to zero in (4.13a) by shifting v). The last equation, coming from the non-differential term, can be combined with (4.13a,b) and integrated to yield

$$x = -8R_3 + \frac{3}{2}uv - \frac{1}{4}v'' + \frac{3}{2}w^2 + 4t\,R_2 \ . \tag{4.13c}$$

It can be verified that (4.13a,b,c) can be derived as derivatives respectively with respect to v, w, u of the action

$$\begin{aligned} S_{\text{Ising}}(u,v,w) = \int \mathrm{d}x \Big[&\frac{16}{7}R_4 + 4R_2 v + \frac{3}{2}R_0 v^2 - \frac{8}{5}tR_3 \\ &+ \frac{1}{2}w'^2 + \frac{3}{2}(u+t)w^2 - hw - xu \Big] \ . \end{aligned} \tag{4.14}$$

Correlation functions of the spin field and energy operator (which have dressed gravitational weights (KPZ) equal to $\frac{5}{6}$ and $\frac{1}{3}$) can be calculated by taking derivatives of the solution u with respect to h and t.

4.3. Generalizations, open problems

General actions. It can be verified that action (4.14) can be written in a more general form:

$$S_{\text{Ising}} = \mathrm{tr}\left(Q^{7/4} + t_2\,Q^{3/4} + t_1\,Q^{1/2} + t_0\,Q^{1/4}\right) , \tag{4.15}$$

where we take $Q = (\mathrm{d}^2 - u)^2 + \{w, \mathrm{d}\} + v$. More generally the action for $(q-1)$-matrix models which generates all equations derived from the commutation relation $[P, Q] = 1$ can be written:

$$S = \mathrm{tr}\left(Q^{p/q+1} + \sum_{k=1} \sum_{\alpha=0}^{q-2} t_{(k),\alpha}\, Q^{k+(\alpha+1)/q} + \sum_{\alpha=0}^{q-2} t_\alpha\, Q^{(\alpha+1)/q}\right) , \qquad (4.16)$$

with

$$Q = \mathrm{d}^q + \sum_{\alpha=0}^{q-2} \{v_\alpha, \mathrm{d}^\alpha\} , \qquad (4.17)$$

Open problems. Many questions require further investigation. For example it is non-trivial to solve many of the equations which have been derived. The perturbation expansion can be shown to be non-Borel summable in many cases of interest, and the solutions of the differential equations depend on at least one parameter which cannot be fixed even in principle by requiring the solution to have the right perturbative expansion.

It is possible to integrate over unitary matrices provided the connection between matrices involves no loop. However the method of orthogonal polynomials is only applicable in the case explained above.

Finally in the most interesting problems (in particular to extend the analysis beyond $d = 1$ matter) it is not possible even to integrate over the unitary group.

References

[1] F. David, Nucl. Phys. B257[FS14] (1985) 45, 543;
J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. B257[FS14] (1985) 433; J. Fröhlich, in: Lecture Notes in Physics, Vol. 216, ed. L. Garrido (Springer, Berlin, 1985);
V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. 157B (1985) 295; D. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. B174 (1986) 87; Nucl. Phys. B275[FS17] (1986) 641.

[2] E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, Comm. Math. Phys. 59 (1978) 35.

[3] D. Bessis, C. Itzykson, and J.-B. Zuber, "Quantum field theory techniques in graphical enumeration," Adv. Appl. Math. 1 (1980) 109.

[4] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819;
F. David, Mod. Phys. Lett. A3 (1988) 1651;
J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.

[5] V. A. Kazakov and A. A. Migdal, Nucl. Phys. B311 (1988) 171.

[6] M. Douglas and S. Shenker, "Strings in less than one dimension," Rutgers preprint RU-89-34.

[7] E. Brézin and V. Kazakov, Phys. Lett. B236 (1990) 144.

[8] D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 127; "A non-perturbative treatment of two-dimensional quantum gravity," Princeton preprint PUPT-1159 (1989).

[9] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.

[10] T. Banks, M. Douglas, N. Seiberg, and S. Shenker, "Microscopic and macroscopic loops in non-perturbative two dimensional gravity," Rutgers preprint RU-89-50.

[11] M. R. Douglas, "Strings in less than one dimension and the generalized KdV hierarchies," Rutgers preprint RU-89-51.

[12] P. Ginsparg, M. Goulian, M. R. Plesser, and J. Zinn-Justin "(p,q) String Actions", Harvard preprint HUTP-90/A015, Saclay SPhT/90-049

[13] P. Ginsparg and J. Zinn-Justin, in preparation

[14] R. Dijkgraaf and E. Witten, "Mean field theory, topological field theory, and multi-matrix models," IAS preprint IASSNS-HEP-90/18 (1990), ;
F. David, "Loop equations and non-perturbative effects in two-dimensional quantum gravity" Saclay preprint SPhT/90-043 ;
O. Alvarez and P. Windey, "Universality in two-dimensional quantum gravity", Paris VI preprint PAR-LPTHE 90-12.

[15] I. M. Gel'fand and L. A. Dikii, Russian Math. Surveys 30:5 (1975) 77.

[16] I. M. Gel'fand and L. A. Dikii, Funct. Anal. Appl. 10 (1976) 259.

[17] P. Ginsparg and J. Zinn-Justin, "2D gravity + 1d matter", Harvard preprint HUTP-90/A004, ;

E. Brézin, V. A. Kazakov, and Al. B. Zamolodchikov, "Scaling violation in a field theory of closed strings in one physical dimension," LPS-ENS 89-182, ;

D.J. Gross and N. Miljkovic, Princeton preprint, PUPT-1160, ;

D.J. Gross and I. Klebanov, "One dimensional string theory on a circle" Princeton preprint

[18] G. 't Hooft, Nucl. Phys. B72 (1974) 461.

[19] E. Witten, "On the topological phase of two dimensional gravity," IAS preprint IASSNS-HEP-89/66 (1989).

[20] V. Kazakov, Phys. Lett. 119A (1986) 140;

D. Boulatov and V. Kazakov, Phys. Lett. 186B (1987) 379.

[21] E. Brézin, M. Douglas, V. Kazakov, and S. Shenker, "The Ising model coupled to 2D gravity: a non-perturbative analysis," Rutgers preprint RU-89-47;

D. Gross and A. Migdal, Phys. Rev. Lett. 64 (1990) 717;

C. Crnković, P. Ginsparg, and G. Moore, "The Ising model, the Yang-Lee edge singularity, and 2D quantum gravity," Harvard preprint HUTP-89/A058, to appear in Phys. Lett. B.