IC/90/418

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

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H.O. Tejumola

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A.U. Afuwape

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1990 MIRAMARE - TRIESTE

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PERIODIC SOLUTIONS OF CERTAIN THIRD ORDER NONLINEAR DIFFERENTIAL SYSTEMS WITH DELAY *

H.O. Tejumola **
Department of Mathematics, Lincoln University, PA 19352, USA

and

A.U. Afuwape ***
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

This paper investigates the existence of 2π–periodic solutions of systems of third–order nonlinear differential equations, with delay, of the form

$$\dddot{X}(t) + A\ddot{X}(t) + \frac{d}{dt}(Grad\, G(x)) + H(t, X(t-\tau)) + P(t),$$

under varied assumptions on $H(t, X(t-\tau))$ and $G(X)$. The results obtained extend earlier works of Tejumola [14] and generalize to third order systems those of Conti, Iannacci and Nkashama [3] as well as DePascale and Iannacci [5] and Iannacci and Nkashama [7].

MIRAMARE – TRIESTE
December 1990

* Submitted for publication.
** On leave from: Department of Mathematics, University of Ibadan, Ibadan, Nigeria.
*** Permanent address: Department of Mathematics, Obafemi Awolowo University, Ile–Ife, Nigeria.

1 Introduction

Consider the nonlinear system of third order differential equations of the form

$$\dddot{X}(t) + A\ddot{X}(t) + \frac{d}{dt}(Grad\ G(X(t))) + H(t, X(t-\tau)) = P(t), \quad (1.1)$$

together with the periodic boundary conditions

$$X(0) - X(2\pi) = \dot{X}(0) - \dot{X}(2\pi) = \ddot{X}(0) - \ddot{X}(2\pi) = 0, \quad (1.2)$$

where A is an $n \times n$ real symmetric matrix, $G : R^n \to R$ is continuous, $H : [0, 2\pi] \times R^n \to R^n$ is a generalized Caratheodory function, and $P : [0, 2\pi] \to R^n$ is Lebesgue integrable with $\tau \in [0, 2\pi)$ a fixed constant delay. The unknown function $X : [0, 2\pi] \to R^n$ is defined for $0 \leq t < \tau$ by $X(t-\tau) = X(2\pi + \tau - t)$.

Motivated by the recent work of Tejumola [14], extending to third-order nonlinear differential equation with constant delay, the results of E. DePascale and R. Iannacci, [5] and R. Iannacci and M.N. Nkashama [7], on Lienard delay differential equations, we shall in this paper obtain some generalization of the results in [14], and also extend to third-order system the works of G. Conti, R. Iannacci and M.N. Nkashama [3] and P. Omari and F. Zanolin [12]. Our technique is basically an extension of those used in [10 - 12].

We shall consider, to start with, the linear system

$$\dddot{X}(t) + A\ddot{X}(t) + B\dot{X}(t) + C(t)X(t-\tau) = u(t), \quad (1.3)$$

together with the boundary condition

$$X(0) - X(2\pi) = \dot{X}(0) - \dot{X}(2\pi) = \ddot{X}(0) - \ddot{X}(2\pi) = 0, \quad (1.4)$$

where A, B are $n \times n$ matrices, $C(t)$ is continuous $n \times n$ matrix and $u(t + 2\pi) = u(t)$. We shall prove the existence, uniqueness and continuous dependence of solutions on u. To prove the existence of a periodic solution of (1.1), we shall apply Schaefers' fixed point Theorem ([4]) to a suitably defined map associated with (1.1) and use certain extensions of the technique in [10,11] and [14] to obtain the desired a-priori bound.

The setting of the paper is as follows. In section 2, we introduce some notations, and basic results similar to those used in [1,2,3,6,16]. In section 3, we shall consider (1.3) - (1.4), first with $C(t) = C$, a real constant symmetric matrix, and more generally, with $C(t)$ a symmetric continuous $n \times n$, we obtain some basic results which will be used for the nonlinear case (1.1). Section 4 will be devoted to the nonlinear case (1.1) - (1.2). In section 5, we give the uniqueness result.

2 Notations and Basic Results

We shall adapt some of the notations used in [1,2,3,6] and [16]. Let $J = [0, 2\pi]$ and R^n be the real $n-$dimensional Euclidean norm space, equipped with the usual Euclidean norm $\|.\|$. We shall denote by $<, >$ the bilnear pairing in $R^n \times R^n$; that is, for $X, Y \in R^n$, $<X, Y> = \sum_{i=1}^n x_i y_i$. Thus $<X, X> = \|X\|^2$. Next, we introduce the following spaces:

(2.1) $L^p(J, R^n)$ are the usual Lebesgue spaces, with $1 \leq p < +\infty$ (and will be denoted by $L^p_{2\pi}(R^n)$);

(2.2) For $k \geq 0$, an integer,

$$H^k(J, R^n) = \{X : J \to R^n \,|X, \dot{X}, \cdots, X^{(k-1)} \text{ are absolutely continuous, } X^{(k)} \in L^2(J, R^n),$$
$$X^{(i)}(0) - X^{(i)}(2\pi) = 0, (i = 0, 1, 2, \cdots, (k-1))\},$$

with norm

$$\|X\|_{H^k(J, R^n)} = \{\sum_{i=1}^n [(\frac{1}{2\pi}\int_J X_i(t)dt)^2 + (\frac{1}{2\pi})\sum_{j=1}^k \int_J |X_i^{(j)}(t)|^2 dt]\}^{1/2}.$$

We note that $H^0(J, R^n)$ is canonically isomorphic to $L^2(J, R^n)$. We shall sometimes denote $H^k(J, R^n)$ by $H^k_{2\pi}(R^n)$).

(2.3) $\tilde{H}^k(J, R^n) = \{X \in H^k(J, R^n) \mid \bar{X} = \frac{1}{2\pi}\int_J X(t)dt = 0\}$.

(2.4) $W^{3,2}(J, R^n) = \{X : J \to R^n \mid X, \dot{X}, \ddot{X}$, are absolutely continuous, $\dddot{X} \in H^0(J, R^n)$, and

$$X(0) - X(2\pi) = \dot{X}(0) - \dot{X}(2\pi) = \ddot{X}(0) - \ddot{X}(2\pi) = 0\}$$

with norm

$$\|X\|_{W^{3,2}(J, R^n)} = \{\sum_{i=1}^n [(\frac{1}{2\pi})\sum_{j=0}^2 \int_J |X_i^{(j)}|dt]^2\}^{1/2}.$$

We note that if $X \in H^k_{2\pi}(R^n)$, then the components $X_i \in H^k_{2\pi}(R)$, for $i = 1, 2, \cdots, n$. Moreover, if $X \in H^k_{2\pi}(R^n)$, we may write $X = \bar{X} + \tilde{X}$ where $\tilde{X} \in \tilde{H}^2_{2\pi}(R^n)$ and $\bar{X} = (\frac{1}{2\pi})\int_J X(t)dt$.

(2.5) A solution of (1.1) is a function $X \in W^{3,2}(J, R^n)$ which satisfies (1.1) almost everywhere on R^n.

(2.6) Next, for any $n \times n$ matrix, Q, we shall denote by

$\lambda_i(Q)$, $(i = 1, 2, \cdots, n)$ the eigenvalues of Q;

δ_Q, and Δ_Q, respectively the minimum and maximum eigenvalues of Q.

Lastly, the following standard algebraic results will be used freely.

LEMMA 2.1([1, 6]) Let D and Q be real commuting symmetric $n \times n$ matrices. Then,

(i) for any $X \in R^n$,

$$\delta_D\|X\|^2 \leq <DX, X> \leq \Delta_D\|X\|^2.$$

(ii) $\min_{1\leq j,k\leq n}\{\lambda_j(Q)\lambda_k(D)\} \leq \lambda_i(QD) \leq \max_{1\leq j,k\leq n}\{\lambda_j(Q)\lambda_k(D)\}$.

(iii) $\lambda_i(Q + D)$, for $i = 1, 2, \cdots, n$, is real and

$$\delta_Q + \delta_D \leq \delta_i(Q + D) \leq \Delta_Q + \Delta_D.$$

3 The Linear Delayed System (1.3)

3.1. Consider here the linear delayed system of the form

$$\dddot{X}(t) + A\ddot{X}(t) + B\dot{X}(t) + CX(t - \tau) = 0, \qquad (3.1)$$

$\tau \in [0, 2\pi)$, with boundary conditions

$$X(0) - X(2\pi) = \dot{X}(0) - \dot{X}(2\pi) = \ddot{X}(0) - \ddot{X}(2\pi) = 0, \quad (3.2)$$

where A, B, C are $n \times n$ constant matrices with A and C symmetric and $\delta_A = \min_{1 \leq j \leq n}(\lambda_j(A)) > 0$. The following Lemma is an n-dimensional analogue of Lemma 2.1 of [14].

LEMMA 3.1. *Let the product matrix $A^{-1}C$ be defined and suppose that*

$$(m - 1)^2 < \min_{1 \leq j \leq n} \lambda_j(A^{-1}C)$$
$$\leq \max_{1 \leq j \leq n} \lambda_j(A^{-1}C) < m^2, \qquad (3.3)$$

where $m \geq 1$ is an integer. Then, system (3.1) - (3.2) has no nontrivial 2π-- periodic solution.

Proof of Lemma 3.1 The procedure is as in [14,§2] with the obvious modifications. Let $X(t) = \eta e^{imt}$, for $\eta \in R^n$, m is an integer and $i = \sqrt{-1}$. Substituting in (3.1)-(3.2), we find that (3.1) has no nontrivial 2π-periodic solution if and only if the matrix

$$\chi = -m^2 A + C \cos m\tau$$

is non-singular for all $m \geq 1$ and $\tau \in [0, 2\pi]$. Consider now, the matrix

$$\Phi = A^{-1}\chi = -m^2 I + A^{-1}C \cos m\tau,$$

where I is the unit $n \times n$ matrix. By Lemma 3.1,

$$-m^2 I - A^{-1}C \leq \Phi \leq -m^2 I + A^{-1}C, \ \tau \in [0, 2\pi).$$

From this it is clear that Φ is negative definite if and only if inequalities (3.3) hold. Thus χ is non-singular and the result follows.

Let us now consider the situation in which C is not necessarily a constant matrix. Consider the system

$$\dddot{X}(t) + A\ddot{X}(t) + B\dot{X}(t) + C(t)X(t - \tau) = 0, \qquad (3.4)$$

with boundary conditions

$$X(0) - X(2\pi) = \dot{X}(0) - \dot{X}(2\pi) = \ddot{X}(0) - \ddot{X}(2\pi) = 0. \quad (3.5)$$

Suppose that the matrix $C(t)$ is continuous for all $t \in J$.

The following results hold for system (3.4) - (3.5).

THEOREM 3.2. *Let A, B and $C(t)$ be $n \times n$ matrices with A symmetric and $\delta_A = \min \lambda_j(A) > 0$. Suppose further that*

$$0 < \min_{1 \leq j,k \leq n} \{\lambda_j(A^{-1})\lambda_k(C(t))\}$$
$$\leq \lambda_i(A^{-1}C(t))$$
$$\leq \max_{1 \leq j,k \leq n} \{\lambda_j(a^{-1})\lambda_k(C(t))\}$$
$$< 1, \qquad (3.6)$$

for all $t \in J$. Then, for all real $n \times n$ matrix B, system (3.4) - (3.5) admits in $W^{3,2}(J, R^n)$ only the trivial solution.

THEOREM 3.3. *Suppose that A, B and $C(t)$ are as in Theorem 3.2. Suppose further that instead of (3.6) we have*

$$\begin{aligned}(m-1)^2 < \gamma_m(t) &\le \min_{1\le j,k\le n}\{\lambda_j(A^{-1})\lambda_k(C(t))\}\\ &\le \lambda_i(A^{-1}C(t))\\ &\le \max_{1\le j,k\le n}\{\lambda_j(A^{-1})\lambda_k(C(t))\}\\ &\le \Gamma_m(t)\\ &< m^2 \end{aligned} \tag{3.7}$$

for some integer $m \ge 1$ and for a.e. $t \in J$, with $(m-1)^2 < \gamma_m(t)$ and $\Gamma_m(t) < m^2$ on subsets of J of positive measure. Then, for all real $n \times n$ matrix B, system (3.4) - (3.5) admits in $W^{3,2}(J, R^n)$ only the trivial solution.

Remark 3.4 We note that if m = 1, inequalities (3.7) reduce to those in (3.6). Indeed the inequalities in (3.6) correspond to the "first two eigenvalues", while (3.7) relate to "any two arbitrary eigenvalues".

To facilitate the proofs of Theorems 3.2 and 3.3, we state and prove the following n-dimensional analogue of Lemma 1 in [10].

LEMMA 3.5. *Set $M(t) = A^{-1}C(t)$ and let*

$$0 < \Gamma(t) \equiv \max_{1\le j\le n}\lambda_j(M(t)) \le 1, \tag{3.8}$$

with $\Gamma(t) \le 1$, a.e. $t \in J$ and with strict inequality holding on subsets of positive measure. Then, there exists $\delta = \delta(\Gamma) > 0$ such that

$$\begin{aligned}&\tfrac{1}{2\pi}\textstyle\int_J < \dot{\tilde{X}}(t) - \sqrt{\Gamma}(t)\tilde{X}(t),\ \dot{\tilde{X}}(t) + \sqrt{\Gamma}(t)\tilde{X}(t) > dt\\ &\ge \delta < \tilde{X},\ \tilde{X} >_{H^2(J,\ R^n)}\end{aligned} \tag{3.9}$$

for all $\tilde{X} \in \tilde{H}^1(J,\ R^n)$.

Proof of Lemma 3.5. Using the arguments in the proof of Lemma 1 in [10], and Wirtingers inequality, it is easy to verify that

$$\begin{aligned}&\frac{1}{2\pi}\int_J < \dot{\tilde{X}}(t) - \sqrt{\Gamma(t)}\tilde{X}(t),\ \dot{\tilde{X}}(t) + \sqrt{\Gamma(t)}\tilde{X}(t) > dt\\ &\ge \delta < \tilde{X}(t), \tilde{X}(t) >_{H^1(J,R^n)}\end{aligned}$$

for some $\delta = \delta(\Gamma) > 0$, with $\Gamma(t)$ satisfying (3.8).

Proof of Theorem 3.2. Set $M(t) = A^{-1}C(t)$ and consider system (3.4) - (3.5) in the form

$$A^{-1}\dddot{X}(t) + \ddot{X}(t) + A^{-1}B\dot{X} + M(t)X(t-\tau) = 0, \tag{3.10}$$

$$X(0) - X(2\pi) = \dot{X}(0) - \dot{X}(2\pi) = \ddot{X}(0) - \ddot{X}(2\pi) = 0.$$

Let $X(t) = \bar{X} + \tilde{X}$ with $\bar{X}$ and $\tilde{X}$ defined as before. $\tilde{X} \in H^2(J,\ R^n)$, and $\bar{X} = \frac{1}{2\pi}\int_J X(t)dt$.

Scalar multiplying (3.10) with $\bar{X} - \ddot{X}(t)$ and integrating over J with respect to t, we obtain

$$0 = \frac{1}{2\pi}\int_J < \bar{X}(t) - \ddot{X}(t)(t), A^{-1}\dddot{X}(t) + \ddot{X}(t) + A^{-1}B\dot{X} + M(t)X(t-\tau) > dt \tag{3.11}$$

$$\begin{aligned}&= \frac{1}{2\pi}\int_J < \bar{X} - \ddot{X}(t), \ddot{X}(t) + M(t)X(t-\tau) > dt\\ &= \frac{1}{2\pi}\int_J < \dot{\tilde{X}}, \dot{\tilde{X}} > dt + \frac{1}{2\pi}\int_J < \bar{X}, M(t)\bar{X} > dt\\ &\quad - \frac{1}{2\pi}\int_J < \ddot{X}(t), M(t)\tilde{X}(t-\tau) > dt.\end{aligned} \tag{3.12}$$

By adaptation of the arguments in [5,7], it follows that

$$\begin{aligned}0 \ge\ & \tfrac{1}{2\pi}\textstyle\int_J \frac{1}{2}|< \dot{\tilde{X}}(t) - \sqrt{\Gamma(t)}\tilde{X}(t), \dot{\tilde{X}}(t) + \sqrt{\Gamma(t)}\tilde{X}(t) >|dt\\ &+ \tfrac{1}{2\pi}\textstyle\int_J \frac{1}{2}|< \dot{\tilde{X}}(t-\tau) - \sqrt{\Gamma(t)}\tilde{X}(t-\tau), \dot{\tilde{X}}(t-\tau) + \sqrt{\Gamma(t)}\tilde{X}(t-\tau) >|dt,\end{aligned} \tag{3.13}$$

since X is periodic and

$$\int_J < \dot{\tilde{X}}(t), \dot{\tilde{X}}(t) > dt \quad = \quad \int_J < \dot{\tilde{X}}(t-\tau), \dot{\tilde{X}}(t-\tau) > dt.$$

By Lemma 3.5, inequality (3.13) implies the existence of a constant $\delta \;=\; \delta(\Gamma) \;>\; 0$ such that

$$0 \;\geq\; \delta < \tilde{X}, \tilde{X} >_{H^1(J,R^n)}. \tag{3.14}$$

Thus, $\tilde{X} \;=\; 0$ a.e. and hence, $X \;=\; \bar{X}$ a.e.. The conclusion of the Theorem follows, since by (3.8), $\Gamma(t)$ is not the identically zero function.

COROLLARY 3.6. *Suppose all the assumptions of Theorem 3.2 hold. Then, for some fixed* $\tau \in J$ *and every* $u \in L^2(J,\ R^n)$ *the system*

$$\dddot{X}(t) \;+\; A\ddot{X}(t) \;+\; B\dot{X}(t) \;+\; C(t)X(t-\tau) \;=\; u(t), \tag{3.15}$$

with

$$X(0) - X(2\pi) \;=\; \dot{X}(0) - \dot{X}(2\pi) \;=\; \ddot{X}(0) - \ddot{X}(2\pi) \;=\; 0 \tag{3.16}$$

has a unique solution in $W^{3,2}(J,\ R^n)$ *which depends continuously on* $u(t)$.

Proof of Corollary 3.6. Observe first that the operator

$$T \;:\; W^{3,2}(J,\ R^n) \to L^2(J,\ R^n)$$

$$X \mapsto \dddot{X}$$

is a Fredholm operator of index zero,

and

$$F \;:\; W^{3,2}(J,\ R^n) \to L^2(J,\ R^n)$$

$$X \mapsto A\ddot{X}(t) \;+\; B\dot{X}(t) \;+\; C(t)X(t-\tau)$$

is a completely continuous mapping. Thus, $T \;+\; F$ is Fredholm and of index zero. Hence, (3.15) has a unique solution, since $ker(T \;+\; F) \;=\; \{0\}$.

By the Banach Continuous Inverse Theorem, the continuous dependence of the solution on u(t) follows.

Proof of Theorem 3.3 Set $M(t) \;=\; A^{-1}C(t)$ and $\gamma(t) \;=\; \min_{1\leq j\leq n}\lambda_j(M(t))$. Consider (3.15) in the form

$$A^{-1}\dddot{X}(t) \;+\; \ddot{X}(t) \;+\; A^{-1}B\dot{X}(t) \;+\; M(t)X(t-\tau) \;=\; 0, \tag{3.17}$$

together with

$$X(0) - X(2\pi) \;=\; \dot{X}(0) - \dot{X}(2\pi) \;=\; \ddot{X}(0) - \ddot{X}(2\pi) \;=\; 0. \tag{3.18}$$

Let $X(t) \;=\; \bar{X}(t) \;+\; \tilde{X}(t)$ be a solution of (3.16). Scalar multiply (3.16) by $\bar{X}(t-\tau) \;-\; \tilde{X}(t)$ and integrate over J, with respect to t, to obtain

$$0 \;=\; \tfrac{1}{2\pi}\textstyle\int_J < A^{-1}\dddot{\tilde{X}}(t) \;+ \ddot{\tilde{X}}(t) \;+\; M(t)X(t-\tau), \bar{X}(t-\tau) \;-\; \tilde{X}(t) > dt$$

$$=\; \tfrac{1}{2\pi}\textstyle\int_J < \ddot{\tilde{X}}(t) \;+\; M(t)X(t-\tau), \bar{X}(t-\tau) \;-\; \tilde{X}(t) > dt$$

$$\geq\; \tfrac{1}{2\pi}\textstyle\int_J < \ddot{\tilde{X}}(t) + \dddot{\tilde{X}}(t) + \gamma(t)[\bar{X}(t-\tau) + X(t-\tau)], \bar{X}(t-\tau) - \tilde{X}(t) > dt$$

$$=\; \tfrac{1}{2\pi}\textstyle\int_J < \dot{\tilde{X}}(t), \dot{\tilde{X}}(t) > dt \;-\; \tfrac{1}{2\pi}\textstyle\int_J < \dot{\tilde{X}}(t), \dot{\tilde{X}} > dt$$

$$+\; \tfrac{1}{2\pi}\textstyle\int_J \tfrac{\gamma(t)}{2}[< \bar{X}(t-\tau), \bar{X}(t-\tau) >$$

$$-\; < \tilde{X}(t), \tilde{X}(t) > \;-\; < \tilde{X}(t-\tau), \tilde{X}(t-\tau) >]dt$$

$$+\; \tfrac{1}{2\pi}\textstyle\int_J \tfrac{\gamma(t)}{2}[\bar{X}(t-\tau) + \tilde{X}(t-\tau) - \tilde{X}(t), \bar{X}(t-\tau) + \tilde{X}(t-\tau) - \tilde{X}(t) >]dt, \tag{3.19}$$

where we have used the identity

$$- < u, v > \;=\; \frac{1}{2}[< u, u > \;+\; < v, v > \;-\; < u+v, u+v >], \quad u, v \in R^n.$$

Further, in view of this identity and (3.17), we have that

$$-\int_J < \dot{\tilde{X}}(t), \dot{\tilde{X}}(t-\tau) > dt \;\geq\; -\int_J < \dot{\tilde{X}}(t-\tau), \dot{\tilde{X}}(t-\tau) > dt.$$

Therefore inequality (3.19) becomes

$$
\begin{aligned}
0 \ \geq\ & \tfrac{1}{2\pi}\textstyle\int_J < \dot{\tilde{X}}(t), \dot{\tilde{X}}(t) > dt \ - \ \tfrac{1}{2\pi}\textstyle\int_J < \dot{\tilde{X}}(t-\tau), \dot{\tilde{X}}(t-\tau) > dt \\
& + \tfrac{1}{2\pi}\textstyle\int_J \tfrac{\gamma(t)}{2}[< \bar{X}(t-\tau), \bar{X}(t-\tau) > \\
& - < \tilde{X}(t-\tau), \bar{X}(t-\tau) > - < \tilde{X}(t), \tilde{X}(t) >]dt \\
& + \tfrac{1}{2\pi}\textstyle\int_J \tfrac{\gamma(t)}{2}[< \bar{X}(t-\tau) + \tilde{X}(t-\tau) - \tilde{X}(t), \bar{X}(t-\tau) + < \tilde{X}(t-\tau) - \tilde{X}(t) >]dt \\
& = \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2} < \dot{\tilde{X}}(t) - \sqrt{\Gamma(t)}\tilde{X}(t), \dot{\tilde{X}}(t) + \sqrt{\Gamma(t)}\tilde{X}(t) > dt \\
& + \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2} < \dot{\tilde{X}}(t-\tau) - \sqrt{\Gamma(t)}\tilde{X}(t-\tau), \dot{\tilde{X}}(t-\tau) + \sqrt{\Gamma(t)}\tilde{X}(t-\tau) > dt \\
& - \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2} < \dot{\bar{X}}(t-\tau) - \sqrt{\Gamma(t)}\bar{X}(t-\tau), \dot{\bar{X}}(t-\tau) + \sqrt{\Gamma(t)}\bar{X}(t-\tau) > dt \\
& - \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2} < \dot{\bar{X}}(t-\tau), \dot{\bar{X}}(t-\tau) > dt \\
& + \tfrac{1}{2\pi}\textstyle\int_J \tfrac{\gamma(t)}{2}[\bar{X}(t-\tau) + \tilde{X}(t-\tau) - \tilde{X}(t), \bar{X}(t-\tau) + \tilde{X}(t-\tau) - \tilde{X}(t) >]dt.
\end{aligned}
\tag{3.20}
$$

on using the fact that

$$\int_J < \dot{\tilde{X}}(t), \dot{\tilde{X}}(t) > dt \ \equiv\ \int_J < \dot{\tilde{X}}(t-\tau), \dot{\tilde{X}}(t-\tau) > dt.$$

Using the arguments in [7,8], inequalities (3.7) and the fact that

$$\int_J < \bar{X}(t-\tau), \tilde{X}(t-\tau) > dt \ = \ 0$$

and

$$\int_J < \bar{X}(t-\tau), \tilde{X}(t-\tau) > dt \ = \ 0.$$

the sum of the last two terms of (3.20) is greater or equal to

$$
\begin{aligned}
& - \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2} < \dot{\bar{X}}(t-\tau), \dot{\bar{X}}(t-\tau) > dt \\
& + \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2}\tfrac{(m-1)^2}{2}[< \bar{X}(t-\tau), \bar{X}(t-\tau) > \\
& + < \tilde{X}(t-\tau) - \tilde{X}(t), \tilde{X}(t-\tau) - \tilde{X}(t) >]dt \\
& \geq \ 0,
\end{aligned}
$$

since $m \geq 1$.

Thus, from (3.20), we have

$$
\begin{aligned}
0 \ \geq\ & \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2} < \dot{\tilde{X}}(t) - \sqrt{\Gamma(t)}\tilde{X}(t), \dot{\tilde{X}}(t) + \sqrt{\Gamma(t)}\tilde{X}(t) > dt \\
& + \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2} < \dot{\tilde{X}}(t-\tau) - \sqrt{\Gamma(t)}\tilde{X}(t-\tau), \dot{\tilde{X}}(t-\tau) + \sqrt{\Gamma(t)}\tilde{X}(t-\tau) > dt \\
& - \tfrac{1}{2\pi}\textstyle\int_J \tfrac{1}{2} < \dot{\bar{X}}(t-\tau) - \sqrt{\Gamma(t)}\bar{X}(t-\tau), \dot{\bar{X}}(t-\tau) + \sqrt{\Gamma(t)}\bar{X}(t-\tau) > dt.
\end{aligned}
$$

Using Lemma 3.5 again, we conclude as in [5,7] and [10,11,14] that $X \equiv 0$, and the result follows.

4 The Nonlinear Case

4.1. The main result of this paper concerns systems of the form

$$\dddot{X}(t) \ + \ A\dot{X}(t) \ + \ \frac{d}{dt}[Grad\ G(X(t))] \ + \ H(t,\ X(t-\tau)) \ = \ P(t) \tag{4.1}$$

with boundary conditions

$$X(0) \ - \ X(2\pi) \ = \ \dot{X}(0) \ - \ \dot{X}(2\pi) \ = \ \ddot{X}(0) \ - \ddot{X}(2\pi) \ = \ 0, \tag{4.2}$$

where $P : J \to R^n$ is Lebesgue integrable, $G : R^n \to R$ is a C^2-function, $H : J \times R^n \to R^n$ is a Caratheodory function in $L^2(J,\ R^n)$; that is $H(\cdot, X)$ is measurable on J for each $X \in R^n$ and $H(t, \cdot)$ is continuous on R^n, a.e. $t \in J$ and for each $r > 0$ there exists a function $\gamma_r \in L^2(J,\ R)$ such that

$$\|H(t,\ X)\| \ \leq \ \gamma_r(t) \quad \text{for a.e. } t \in J \tag{4.3}$$

anf for $\|X\| \in [-r,\ r]$.

Furthermore, we shall assume that A is a constant $n \times n$-real symmetric positive definite matrix and such that the following conditions hold:

Let $\tilde{H}(t,\ X) \ = \ A^{-1}H(t,\ X) \ \equiv \ (\tilde{h}_j(t,X))_{j=1,\cdots,n}$.

Then for any $j = 1, \cdots, n$, assume

$(\tilde{H}_1)$ there exists constants $R_j > 0$ such that

$\tilde{h}_j(t,X)X_j \ \geq \ 0$ for every $X \in R^n$ with $|X_j| \geq R_j$

and

$(\tilde{H}_2)$ $\limsup_{|X_j|\to\infty} X_j^{-1}\tilde{h}_j(t,X) \ \leq \ \Gamma_j(t)$ uniformly for almost all $t \in J$ and uniformly for all $X_l \in R$, $l \neq j$, where $\Gamma_j \in L^2(J, R)$ is such that $\Gamma_j(t) \leq 1$, a.e. on J, with strict inequality on subsets of positive measure.

We remark that condition $(\tilde{H}_1)$ implies that for $j = 1,\cdots,n$, $\Gamma_j(t) \geq 0$ a.e. on J.

Thus, without any loss of generality, we can assume that $\Gamma_j(t) \neq 0$ a.e. on J, for $j = 1,\cdots,n$.

The main result is as follows :-

THEOREM 4.1. *Suppose in the system (4.1)-(4.2), A is a real symmetric positive definite matrix such that* $\delta_A = \min_{1\leq j\leq n}\lambda_j(A) > 0$, *and* G, H *and* P *are as defined above. Suppose further that conditions* $(\tilde{H}_1)$ *and* $(\tilde{H}_2)$ *hold, and that*

$$\bar{P} = \frac{1}{2\pi}\int_0^{2\pi} P(t)dt = 0.$$

Then, for arbitrary C^2*-function* $G : R^n \to R$, *there exists at least one solution* $X \in W^{3,2}(J, R^n)$ *of the boundary value problem (4.1) - (4.2).*

Remark 4.1. Theorem 4.1 is an n-dimensional analogue of Theorem 3.1 in [14]. It also extends some results in [5,7,8] to third-order systems of the form (4.1) - (4.2).

4.2. The following n-dimensional analogue of the results by Mawhin and Ward [10,11] will be used in the proof of Theorem 4.1.

LEMMA 4.1.*Suppose that all the conditions on* A, B, $C(t)$ *in Theorem 3.3 hold. Let* δ *be as given in Lemma 3.5. and set* $M(t) \equiv A^{-1}C(t)$.

Suppose further that for any $n \times n$*- continuous symmetric matrix* $V(t)$, *with*

$$0 < V(t) < M(t) + \epsilon I, \quad a.e.\ t \in J, \tag{4.4}$$

for some $\epsilon > 0$, *the inequalities*

$$\begin{aligned} 0 < \delta_V &\leq \min_{1\leq j\leq n}\lambda_j(V(t)) \\ &\leq \max_{1\leq j\leq n}\lambda_j(V(t)), \\ &\leq \min_{1\leq j\leq n}\lambda_j(M(t) + \epsilon I) \end{aligned} \tag{4.5}$$

hold.

Then, for any arbitrary $n \times n$*- matrix* B, *the inequality*

$$\begin{aligned} &\frac{1}{2\pi}\int_J < \bar{X} - \tilde{X}(t),\ A^{-1}\dddot{X}(t) + \ddot{X}(t) + A^{-1}B\dot{X}(t) + V(t)X(t-\tau) > dt \\ &= \frac{1}{2\pi}\int_J < \bar{X} - \tilde{X}(t),\ \ddot{X}(t) + V(t)X(t-\tau) > dt \\ &\geq (\delta - \epsilon) < \tilde{X}(t), \tilde{X}(t) >_{H^1(J,R^n)} \end{aligned} \tag{4.6}$$

holds.

Proof of Lemma 4.1. The proof is essentially the same as that of Lemma 3.5.

4.3.Proof of Theorem 4.1.

Define the operator $S : H^2(J, R^n) \to L^2(J, R^n)$ by

$$u(t) \mapsto AM(t)u(t-\tau) + B\dot{u}(t) + P(t) - \frac{d}{dt}[\frac{\partial}{\partial u}G(u(t))] - H(t, u(t-\tau)).$$

Clearly, S is continuous.

Let $T : L^2(J, R^n) \to W^{3,2}(J, R^n)$, be the operator solution of the problem

$$\dddot{X}(t) + A\ddot{X}(t) + B\dot{X}(t) + C(t)X(t-\tau) = u(t) \tag{4.7}$$

$$X(0) - X(2\pi) = \dot{X}(0) - \dot{X}(2\pi) = \ddot{X}(0) - \ddot{X}(2\pi) = 0. \tag{4.8}$$

By Corollary 3.6, there exists such an operator solution.

Let J_1 be the completely continuous embedding of $W^{3,2}(J,\ R^n)$ into $H^2(J, R^n)$. Then the composition

$$L \equiv J_1 \circ T \circ S \ : \ H^2(J,\ R^n) \to H^2(J,\ R^n)$$

is completely continuous and its fixed point is a solution of (4.1) - (4.2) in $W^{3,2}(J,\ R^n)$.

The existence of at least one fixed point for L will be proved (using Schaefer's Theorem, cf [4]), if we can show that the set of all possible solutions of the parameter dependent system

$$\dddot{X}(t) + A\ddot{X}(t) + (1-\lambda)[B\dot{X}(t) + AM(t)X(t-\tau)]$$

$$+ \lambda\{\frac{d}{dt}(Grad\ G(X(t))) + H(t,\ X(t-\tau)) - P(t)\} = 0 \qquad (4.9)$$

is *a-priori* bounded, with bounds independent of $\lambda \in (0,1)$ and of the solutions.

We note that for $\lambda = 1$, (4.9) reduces to (4.1) and to (3.4) for $\lambda = 0$.

We note that by $(\tilde{H}_1)$ and $(\tilde{H}_2)$, there exist $\bar{r}_j \geq 0$ such that for $j = 1,\cdots,n$,

$$0 \leq X_j^{-1}h_j(t,\ X) \leq \Gamma_j(t) + \delta.$$

Now define

$$\gamma(t,\ X) \equiv (\gamma_1(t,X), \gamma_2(t,X), \cdots, \gamma_n(t,X)) \quad : J \times R^n \to R^n$$

as follows :-

$$\gamma_j(t,X) \equiv \begin{cases} X_j^{-1}h_j(t,X), & \text{if } |X_j| \geq \bar{r}_j; \\ \bar{r}_j^{-1}h_j(t,X_1,\cdots,X_{j-1},-\bar{r}_j,X_{j+1},\cdots,X_n)\frac{X_j}{\bar{r}_j} + (1-\frac{X_j}{\bar{r}_j}), & \\ \qquad \text{if } 0 < X_j < \bar{r}_j;, & \\ \bar{r}_j^{-1}h_j(t,X_1,\cdots,X_{j-1},-\bar{r}_j,X_{j+1},\cdots,X_n)\frac{X_j}{\bar{r}_j} + (1+\frac{X_j}{\bar{r}_j}), & \\ \qquad \text{if } -\bar{r}_j < X_j < 0;, & \\ \Gamma_j(t), & \text{if } X_j = 0 \end{cases}$$

Then, for each $j = 1,\cdots,n$,

$$0 \leq \gamma_j(t,\ X) \leq \Gamma_j(t) + \delta,$$

for every $X \in R^n$ and a.e. $t \in J$.

Next, define

$$\psi = (\psi_1,\psi_2,\cdots,\psi_n) \ : \ J \times R^n \to R^n$$

$$(t,\ X) \mapsto (\gamma_1(t,X)X_1, \gamma_2(t,X)X_2, \cdots, \gamma_n(t,X)X_n).$$

Then ψ is a generalised Caratheodory's map such that $\psi_j(\cdot,X)$ is measurable on J for every $X \in R^n$ and $\psi_j(t,\cdot)$ is continuous on R^n for a.e. $t \in J$.

Note that for $j = 1,\cdots,n$, $\psi_j(t,X) = h_j(t,X)$ for all $X \in R^n$ with $|X_j| \geq \bar{r}_j$.

Finally, define

$$\Psi(t,X) = (\Psi_1(t,X),\cdots,\Psi_n(t,X)) \ : \ J \times R^n \to R^n$$

with

$$\Psi_j(t,X) = h_j(t,X) - \psi_j(t,X) \qquad = h_j(t,X) - \gamma_j(t,X)X_j$$

for $j = 1,\cdots,n$.

Again, $\Psi(t,X)$ is a generalized Caratheodory map. In particular, for some $\alpha(t) = (\alpha_1(t),\cdots,\alpha_n(t)) \in L^2(J,\ R^n)$,

$$|\Psi_j(t,\ X)| \leq \alpha_j(t), a.e.\ t \in J,\ for\ \ all\ \ X \in R^n, \qquad (4.10)$$

with $\alpha_j(t)$ depending on $\Gamma(t)$ and $\gamma_r(t)$, where γ_r is as defined in (4.3) with

$$r = \min\{\bar{r}_j |\ j = 1,\cdots,n\}$$

and

$$\Gamma(t) = \min\{\Gamma_j(t)|j = 1,\cdots,n,\ \ a.e.\ \ t \in J\}.$$

Thus, we can now replace each component $h_j(t,X)$ in (4.9) of $H(t,X)$ with

$$\Psi_j(t,X) + \gamma_j(t,X)X_j, \text{for } j = 1,\cdots,n.$$

Setting

$$U(t) = (1-\lambda)AM(t) + \lambda\gamma(t,X(t-\tau))$$

we can rewrite (4.9) in the form

$$\begin{aligned}&\ddot{X}(t) + A\tilde{X}(t) + (1-\lambda)B\dot{X}(t) + \lambda\tfrac{d}{dt}(Grad\ G(X(t)))\\ &+ U(t)X(t-\tau) + \lambda\ \Psi(t,\ X(t-\tau)) - \lambda P(t) = 0.\end{aligned} \tag{4.11}$$

or, in a more convenient form :

$$\begin{aligned}&A^{-1}[\ddot{X}(t) + (1-\lambda)[B\dot{X}(t)] + [\tilde{X}(t) + V(t)X(t-\tau)]\\ &+ \lambda A^{-1}\{\tfrac{d}{dt}[Grad\ G(X(t))] \quad + \Psi(t,\ X(t-\tau)) - P(t)\} = 0\end{aligned} \tag{4.12}$$

where $V(t) = A^{-1}U(t)$.

Taking the scalar product of (4.12) with $\bar{X} - \tilde{X}(t)$, and integrating over J with respect to t, we obtain

$$\begin{aligned}&\tfrac{1}{2\pi}\textstyle\int_J < \bar{X} - \tilde{X}(t), A^{-1}\ddot{X}(t) + (1-\lambda)A^{-1}B\dot{X}(t) > dt\\ &+ \tfrac{1}{2\pi}\textstyle\int_J < \bar{X} - \tilde{X}(t), \ddot{X}(t) + V(t)X(t-\tau) > dt +\\ &+ \tfrac{\lambda}{2\pi}\textstyle\int_J < \bar{X} - \tilde{X}(t), A^{-1}(\tfrac{d}{dt}(Grad\ G(X(t)))) > dt\\ &+ \tfrac{\lambda}{2\pi}\textstyle\int_J < \bar{X} - \tilde{X}(t), A^{-1}[\Psi(t,X(t-\tau)) - P(t)] > dt\\ &= 0.\end{aligned} \tag{4.13}$$

By (4.9), the first integral in (4.13) is zero. Observe also that the third integral, when expressed in terms of its components, reduces to terms majorized by

$$\frac{\delta_A^{-1}}{2\pi}\sum_{j=1}^{n}\int_J(\bar{X}_j - \tilde{X}_j(t))\frac{d}{dt}(\frac{\partial}{\partial X_j}G(X(t)))dt$$

which is zero, where $\delta_A = \min_{1\le j\le n}\lambda_j(A) > 0$. Finally, by (4.11), and our earlier definitions, the fourth integral in (4.13) is greater than or equal to

$$-\frac{1}{\delta_A}(|\bar{X}|_C + \|\tilde{X}(t)\|_C)(|\alpha(t)| + \|P(t)\|_{L^2(J,R^n)}).$$

Thus, from (4.13), we have

$$\begin{aligned}0 &\ge \tfrac{1}{2\pi}\textstyle\int_J < \bar{X} - \tilde{X}(t), \ddot{X}(t) + V(t)X(t-\tau) > dt\\ &- \tfrac{1}{\delta_A}(|\bar{X}|_C\ \|\tilde{X}(t)\|_C)(|\alpha(t)| + \|P(t)\|_{L^2(J,R^n)})\\ &\ge (\delta-\epsilon) < \tilde{X}, \tilde{X} >_{H^1(J,R^n)}\\ &- \tfrac{1}{\delta_A}(|\bar{X}|_C + \|\tilde{X}(t)\|_C)(|\alpha(t)| + \|P(t)\|_{L^2(J,R^n)}),\end{aligned} \tag{4.14}$$

on using Lemma 4.1.

Setting $\epsilon = \delta/2$, we have

$$\begin{aligned}0 &\ge \tfrac{\delta}{2}\|\tilde{X}\|^2_{H^1(J,R^n)}\\ &- \tfrac{1}{\delta_A}(|\bar{X}| + \|\tilde{X}\|_{L^2(J,R^n)})\cdot(|\alpha(t)|_{L^2(J,R^n)} + \|P(t)\|_{L^2(J,R^n)}).\end{aligned} \tag{4.15}$$

Thus for some constant $d_1 > 0$, independent of X, we obtain,

$$\|\tilde{X}\|^2_{H^1(J,R^n)} \le d_1(\|\bar{X}\| + \|\tilde{X}\|_{L^2(J,R^n)}). \tag{4.16}$$

A repetition of the arguments in [14], will now readily show that

$$\|\bar{X}\| \le k + 2\pi\|\tilde{X}\|_{H^1(J,R^n)} \tag{4.17}$$

for some constant $k > 0$.

Lastly, using the arguments in [5,10,14], we have

$$\|\tilde{X}\|_{L^2(J,R^n)} \le d_2\|\tilde{X}\|_{H^1(J,R^n)} \tag{4.18}$$

for some constant $d_2 > 0$. Combining the estimates (4.17) and (4.18) with (4.16), we have readily that

$$\|\tilde{X}\|^2_{H^1(J,R^n)} \le d_3(\|\tilde{X}\|_{H^1(J,R^n)}). \tag{4.19}$$

for some constant $d_3 > 0$.

Thus, there exists a constant $d_4 > 0$, such that

$$\|\ddot{X}\|_{H^1(J,R^n)} \leq d_4. \tag{4.20}$$

Since (4.17) holds, we finally have that

$$\|X\|_{H^1(J,R^n)} \leq d_5, \tag{4.21}$$

for some constant $d_5 > 0$.

The remaining part of the proof now follows as in [14]. That is, from (4.21), we obtain

$$\|X\|_C \leq d_6 \tag{4.22}$$

for some constant $d_6 > 0$, and using the notations in section 2, we have

$$\|\dot{X}\|_{L^2(J,R^n)} \leq d_7 \equiv 2\pi d_6. \tag{4.23}$$

Now multiplying (4.11) by A^{-1}, then scalar multiply the resulting equation by $\dot{X}(t)$ and lastly integrate over J with respect to t, on using the boundedness of $M(t)$, $\gamma(t,X)$ (hence of $V(t)$), and the fact that $\bar{P} = 0$, we obtain

$$0 \geq d_8 \int_J < \ddot{X}(t), \ddot{X}(t) > dt - d_9 \left(\frac{1}{2\pi}\int_J < \ddot{X}(t), \ddot{X}(t) > dt\right)^{\frac{1}{2}} \tag{4.24}$$

for some constants $d_8 > 0$ and $d_9 > 0$.

Thus, as in [5,14],

$$\|\ddot{X}\|_{L^2(J,R^n)} = \int_J < \ddot{X}(t), \ddot{X}(t) > dt \leq d_{10} \tag{4.25}$$

for some constant $d_{10} > 0$.

Therefore, from (4.21), (4.23) and (4.25), we obtain

$$\|X\|_{W^{3,2}(J,R^n)} \leq d_{11} \tag{4.26}$$

with the constant $d_{11} > 0$ independent of $\lambda \in (0,1)$ and the solutions of (4.8) in $W^{3,2}(J,R^n)$. This completes the proof of Theorem 4.1.

5 Uniqueness Result

5.1. Let us finally consider the uniqueness conditions for systems of the form

$$\dddot{X}(t) + A\ddot{X}(t) + B\dot{X}(t) + H(t, X(t-\tau)) = P(t) \tag{5.1}$$

with

$$X(0) - X(2\pi) = \dot{X}(0) - \dot{X}(2\pi) = \ddot{X}(0) - \ddot{X}(2\pi) = 0, \tag{5.2}$$

where A is an $n \times n$ real symmetric matrix.

THEOREM 5.1. *Let* $\delta_A = \min_{1\leq j\leq n}\lambda_j(A) > 0$ *and suppose that* H *satisfies conditions* $(\tilde{H}_1)$ *and* $(\tilde{H}_2)$, *and that* $\bar{P} = 0$. *Suppose further that for all* $X, Y \in R^n$, $X \neq Y$,

$$0 < \frac{\|H(t,X) - H(t,Y)\|}{\delta_A^2\|X - Y\|} \leq \Gamma(t) < 1, \tag{5.3}$$

for a.e. $t \in J$.

Then, for all arbitrary constant $n \times n$ *real matrix* B, *system (5.1) - (5.2) has a unique solution in* $W^{3,2}(J,R^n)$.

Proof of Theorem 5.1 Since all the conditions $(\tilde{H}_1)$ and $(\tilde{H}_2)$, of Theoren 4.1 hold, there exists a solution $X \in W^{3,2}(J,R^n)$ of (5.1) - (5.2). Let $Y \in W^{3,2}(J,R^n)$ be another such solution and set $Z = X - Y$. Then, Z is a solution of the system

$$\dddot{Z}(t) + A\ddot{Z}(t) + B\dot{Z}(t) + H(t, Z+Y) - H(t,Y) = 0 \tag{5.4}$$

with

$$Z(0) - Z(2\pi) = \dot{Z}(0) - \dot{Z}(2\pi) = \ddot{Z}(0) - \ddot{Z}(2\pi) = 0. \tag{5.5}$$

Define, for $i = 1,\cdots,n$, the matrix $M(t,Z) = diag(M_i(t,Z))$, by

$$M_i(t,Z) = \begin{cases} \frac{H_i(t,Z+Y) - H_i(t,Y)}{Z_i}, & \text{if } Z_i \neq 0, \\ 0, & \text{if } Z_i = 0 \end{cases} \tag{5.6}$$

Then, (5.4) - (5.5) becomes

$$\dddot{Z}(t) + A\ddot{Z}(t) + B\dot{Z}(t) + M(t,Z)Z(t-\tau) = 0 \quad (5.7)$$

$$Z(0) - Z(2\pi) = \dot{Z}(0) - \dot{Z}(2\pi) = \ddot{Z}(0) - \ddot{Z}(2\pi) = 0. \quad (5.8)$$

and the condition

$$0 < \lambda_j(M(t,Z)) < 1$$

holds for all $j = 1,\cdots,n$, for a.e. $t \in J$ and for all $z \in R^n$ By Theorem 3.3, we clearly have $Z \equiv 0$, that is $X = Y$, which proves the uniqueness of solutions in $W^{3,2}(J,R^n)$.

Acknowledgements

One of the authors (A.U.A.), would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality in the International Centre For Theoretical Physics, Trieste,Italy. He would also like to thank the Swedish Agency for Research Co-operation with Developing Countries (SAREC), for financial support during his visit to ICTP under the Associateship Scheme.

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