IC/90/423

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

A METHOD FOR THE APPROXIMATE SOLUTIONS OF THE UNSTEADY BOUNDARY LAYER PROBLEM

Md. Abdus Sattar

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1990 MIRAMARE - TRIESTE

IC/90/423

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

A METHOD FOR THE APPROXIMATE SOLUTIONS OF THE UNSTEADY BOUNDARY LAYER EQUATIONS *

Md. Abdus Sattar **
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The approximate integral method proposed by Bianchini *et al.* [1] to solve the unsteady boundary layer equations is considered here with a simple modification to the scale function for the similarity variable. This is done by introducing a time dependent length scale. The closed form solutions, thus obtained, give satisfactory results for the velocity profile and the skin friction to a limiting case in comparison with the results of the past investigators.

MIRAMARE – TRIESTE
December 1990

* Submitted for publication.
** Permanent address: Department of Mathematics, Dhaka University, Dhaka–1000, Bangladesh.

I . INTRODUCTION

Bianchini et al.[1] introduced an approximate method to calculate the characteristics of the boundary layer flows, taking the potential flow simply as a function of time. Later, this first approximate method has been developed by Socio & Pozzi[2] with a rigorous mathematical approach. The potential flow in this case is taken to be a function of x and t . The essence of the first approximate method is to assume a similarity solution even in those situations where similarity solutions do not exist, and to find a suitable scale factor for the similarity variable. In comparison with the classical series and numerical solutions and other approximate methods[3] which also heavily rely on numerical computations, the above methods do not require lengthy calculations. Moreover, no linearization is required. A successful application of the first approximate method has recently been done by Palekar & Sarma[4] to the case of a steady boundary layer flow with suction and blowing.

In the present paper, the method of Bianchini et al. is modified by introducing a time dependent length scale along with the scale function for the similarity variable. The first order differential equation that Bianchini et al. obtained for the scale function is obtained here in a much more simpler form which leads to a simple solution. The velocity profiles and the skin friction to a limiting case are thus obtained and compared with the classical results.

II . THE GOVERNING EQUATIONS AND THE METHOD OF SOLUTION

The basic two-dimensional equations governing the unsteady laminar boundary-layer over a semi-infinite flat plate are (Schlichting[5])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{dU}{dt} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

where x and y are the cartesian coordinates along the plate and normal to it, respectively , t is the time , u and v are the velocity components along x and y , and U(t) is the potential flow far away from the plate.

The boundary conditions associated with the above system will be considered as

$$\left.\begin{aligned} u = v = 0 \qquad & y = 0 \\ u = U(t) \qquad & y \to \infty \ . \end{aligned}\right] \tag{3}$$

Now , following Bianchini et al. , the velocity profile is assumed to have the form

$$u = U(t)\, f(\eta) \tag{4}$$

with

$$\eta = Y/h(X) \tag{5}$$

where $Y = y/\sigma$ and $X = x/\sigma$ (6)

and $\sigma\left[= \sigma(t)\right]$ is a length scale but a function of time.

Here eqs. (5) & (6) constitute the point of departure from the assumption of Bianchini et al.

Integrating eq. (2) across the boundary layer from zero to infinity, using the continuity equation (1) and the assumptions (5) & (6) , it is obtained

$$\frac{\sigma^2}{\nu}\frac{1}{U}\frac{dU}{dt}\,\alpha_1 h + \frac{\sigma}{\nu}\frac{d\sigma}{dt}\left[X\frac{dh}{dX} - h\right]\alpha_2 + \frac{U\sigma}{\nu}\,\alpha_3\frac{dh}{dX} = \frac{\alpha_4}{h} \tag{7}$$

where

$$\left.\begin{aligned} \alpha_1 &= \int_0^\infty (1-f)\,d\eta\ , \quad \alpha_2 = \int_0^\infty \eta f'\,d\eta \\ \alpha_3 &= \int_0^\infty f(1-f)\,d\eta \quad \alpha_4 = \left(\frac{\partial f}{\partial \eta}\right)_{\eta=0} . \end{aligned}\right] \tag{8}$$

In order to obtain a simple solution of (7) , one can try a class of solutions of (7) by setting

$$\frac{\sigma^2}{\nu}\frac{1}{U}\frac{dU}{dt} = -\frac{\sigma}{\nu}\frac{d\sigma}{dt} \ . \tag{9}$$

Thus integrating (9) , one obtains

$$U\sigma = \text{constant} = U_0\sigma_0 \tag{10}$$

where U_0 and σ_0 are the reference values of U and σ at $t = t_0$.

Thus taking $R = \frac{U\sigma}{\nu} = \frac{U_0\sigma_0}{\nu} = R_0$ as the reference Reynolds number, the equation (7) can be written as

$$-\frac{\sigma}{\nu}\frac{d\sigma}{dt}\alpha_1 h + \frac{\sigma}{\nu}\frac{d\sigma}{dt}\left(X\frac{dh}{dX} - h\right)\alpha_2 + R_0\alpha_3\frac{dh}{dX} = \frac{\alpha_4}{h} \tag{11}$$

The two-dimensional boundary layer equations (1) and (2) are now reduced to a single first order differential equation for the scale function $h(X)$ accept for the term $\frac{\sigma}{\nu}\frac{d\sigma}{dt}$ where time t appears explicitly.

But , since the notion of the present problem is to obtain a similarity solution , the similarity condition requires that $\frac{\sigma}{\nu}\frac{d\sigma}{dt}$ must be a constant. Hence it is supposed that

$$\frac{\sigma}{\nu}\frac{d\sigma}{dt} = K \tag{12}$$

where K is an arbitrary constant.

Now integrating (12) , one obtains

$$\sigma = \sqrt{2K\nu t} \tag{13}$$

where the constant of integration is determined in such a way that $\sigma = 0$ when $t = 0$.

Hence the equation (11) can finally be written as

$$-K\alpha_1 h + K\alpha_2\left(X\frac{dh}{dX} - h\right) + R_0\alpha_3\frac{dh}{dX} = \frac{\alpha_4}{h} . \tag{14}$$

An integration to the above equation gives the scale parameter and hence a solution for the velocity profiles. The constant of integration arising out of this integration will be determined by adding an initial condition to the set of boundary conditions adopted in (4).

III . THE GENERAL SOLUTION

In order to have the general solution to the scale parameter it is now necessary to make an assumption of the function $f(\eta)$ for the velocity distribution proposed in (5) .

A convenient choice for $f(\eta)$ is

$$f(\eta) = \operatorname{erf}(\eta) . \tag{15}$$

Hence from (8) and (13) , one has

$$\alpha_1 = \frac{1}{\sqrt{\pi}} , \alpha_2 = \frac{1}{\sqrt{\pi}} , \alpha_3 = (\sqrt{2}-1)/\sqrt{\pi} , \alpha_4 = \frac{2}{\sqrt{\pi}} . \tag{16}$$

With the above values , a general solution of the equation (14) subject

to the the initial condition

$$h(X) = 0 \qquad X = 0 ,$$

is obtained as

$$\log\left(1 + \frac{\gamma h^2}{\alpha_4}\right) = \beta \log(1 + \delta X) \tag{17}$$

where

$$\delta = \frac{K\alpha_2}{R_0\alpha_3} , \quad \gamma = K(\alpha_1 + \alpha_2) \qquad \beta = \frac{2\gamma}{K\alpha_2} .$$

Now from (17) it is apparent that h is constant for a fixed X . Thus , taking h = constant = c for a fixed X , the velocity profile is obtained from (4) , (5) and (15) as

$$u' = u/U(t) = \operatorname{erf}\{y/(\sigma c)\} \tag{18}$$

But since $\sigma = \sqrt{2K\nu t}$, taking $K = 2$ the above velocity profile can satisfactorily be compared with

$$u' = \operatorname{erf}(y/2\sqrt{\nu t}) ,$$

which is the well known exact solution to the two-dimensional boundary layer problem (see Rosenhead[6]).

However , with the solution of $h(X)$ from (17) , velocity profiles for various values of X are obtained and shown in figure 1 . It is apparent from figure 1 that the velocity profiles become fuller as one approaches the leading edge, or otherwise as expected , the boundary layer thickness increases with the increase of the downstream position.

IV . APPROXIMATE SOLUTIONS

From equation (17) it follows that $h(X) \to 0$ as $X \to 0$. Thus for small X , one can consider $\frac{\gamma h^2}{\alpha_4} < 1$.

Hence, expanding both sides of (17) , and subsequently neglecting the terms of higher powers of $\frac{\gamma h^2}{\alpha_4}$ and δX , one obtains

$$\frac{\gamma h^2}{\alpha_4} = \beta\delta X$$

or

$$h(X) = \sqrt{\frac{\alpha_4 \beta \delta X}{\gamma}} \; .$$

Therefore, from (15) , the velocity profiles for small X are obtained as

$$u' = \mathrm{erf}\left(\bar{z} \Big/ \sqrt{\frac{\alpha_4 \beta \delta}{\gamma}} \right) \qquad (19)$$

where $\bar{z} = y/\sqrt{X}$.

A comparison of this result is made in figure 2 with those of Blasius' exact solution and the experiments of Hill & Stennig[7]. The comparison shows quite a good agreement.

Again , inorder to provide a good approximate solution of the steady boundary layer flow over a flat plate, it is assumed that $\sigma = L$ and hence $K = 0$ in equation (14). Then one obtains the solution of $h(X)$ as

$$h = \sqrt{\frac{2\alpha_4}{R_0 \alpha_3} X}$$

Thus

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{U_0}{L\,h(X)} \times \frac{2}{\sqrt{\pi}} \; .$$

Therefore, the skin friction denoted by C_f , is obtained as

$$\frac{1}{2} C_f = \frac{\tau_w}{\rho U_0^2} = \sqrt{\frac{2\alpha_3}{\pi \alpha_4}} \left(\frac{\nu}{U_0 x}\right)^{\frac{1}{2}}$$

$$= \cdot 363109 \left(\frac{\nu}{U_0 x}\right)^{1/2} .$$

In the limit situation of steady flow , the above result is found to be in good comparison with Blasius' solution (see Rosenhead[6])

$$\frac{\tau_w}{\rho U_0^2} = \cdot 33206 \left(\frac{\nu}{U_0 x}\right)^{1/2} .$$

V . CONCLUSIONS

Following Bianchini et al. a method has been proposed for the solution of the unsteady, incompressible boundary layer problem which reduces the original two dimensional partial differential equations to a

simple ordinary first order differential equation. The method does not require rigorous mathematical approach, rather a much more simple form of solution has been obtained in comparison with those of Bianchini et al. and Socio & Pozzi. With this simple form of solutions, one can attain the well known exact solution of the problem under consideration for a fixed downstream position by taking the weight factor of the length scale σ equal to 2.Due to this reason all the other calculations have been done with K=2. One can, certainly, calculate the velcity profiles with other values of K to assertain the variation of the boundary layer thickness with K. From the calculation it is also observed that the usual Blasius' similar velocity profile is valid for small X , but breaks down otherwise. Nevertheless, the calculation of the skin friction in the limit situation of steady flow corresponds well with Blasius' exact formula. Therefore, as far as the simplicity is concerned, the present method is advantageous over the classical and other time consuming numerical methods. To test further capability of the method works are under progress.

ACKNOWLEDGMENTS

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

REFERENCES

[1] Bianchini, A., Socio, L. de., Pozzi, A., ASME J. Appl. Mech., 43, (1976) , 396 .

[2] Socio, L. de., Pozzi, A., ASME J. Appl. Mech., 46 , (1979), 269 .

[3] Miller, R.W., Han, L.S., ASME Paper72-WA/FE-2 .

[4] Palekar, M.G., Sarma, D.P., Indian J. Pure Appl. Maths., 15, 6 , (1984), 671 .

[5] Schlichting, H., Boundary Layer Theory. McGraw-Hill , NY ,(1968).

[6] Rosenhead, L. (editor), Laminar Boundary Layers, Oxford at the Clarendon Press, (1963).

[7] Hill, P.G., Stenning, A.H., ASME J. Basic Engg., 82 ,(1960), 593 .

FIGURE CAPTIONS

Fig.1 - Velocity profiles for various values of downstream positions

Fig.2 - Similarity profiles (a) present method, '
(b) Blasius' exact method
(c) experiments of Hill and Stenning [7]

11

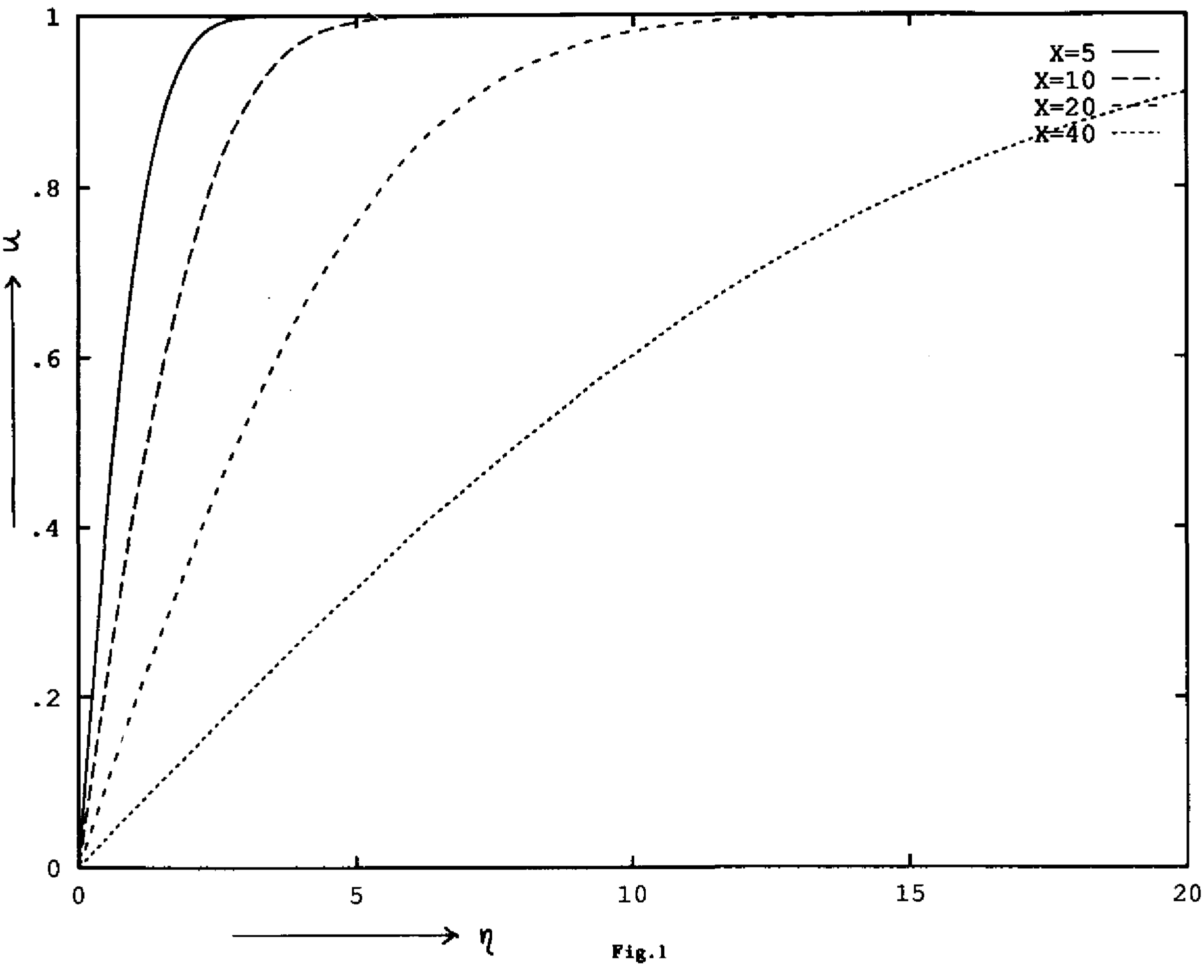

Fig.1

12

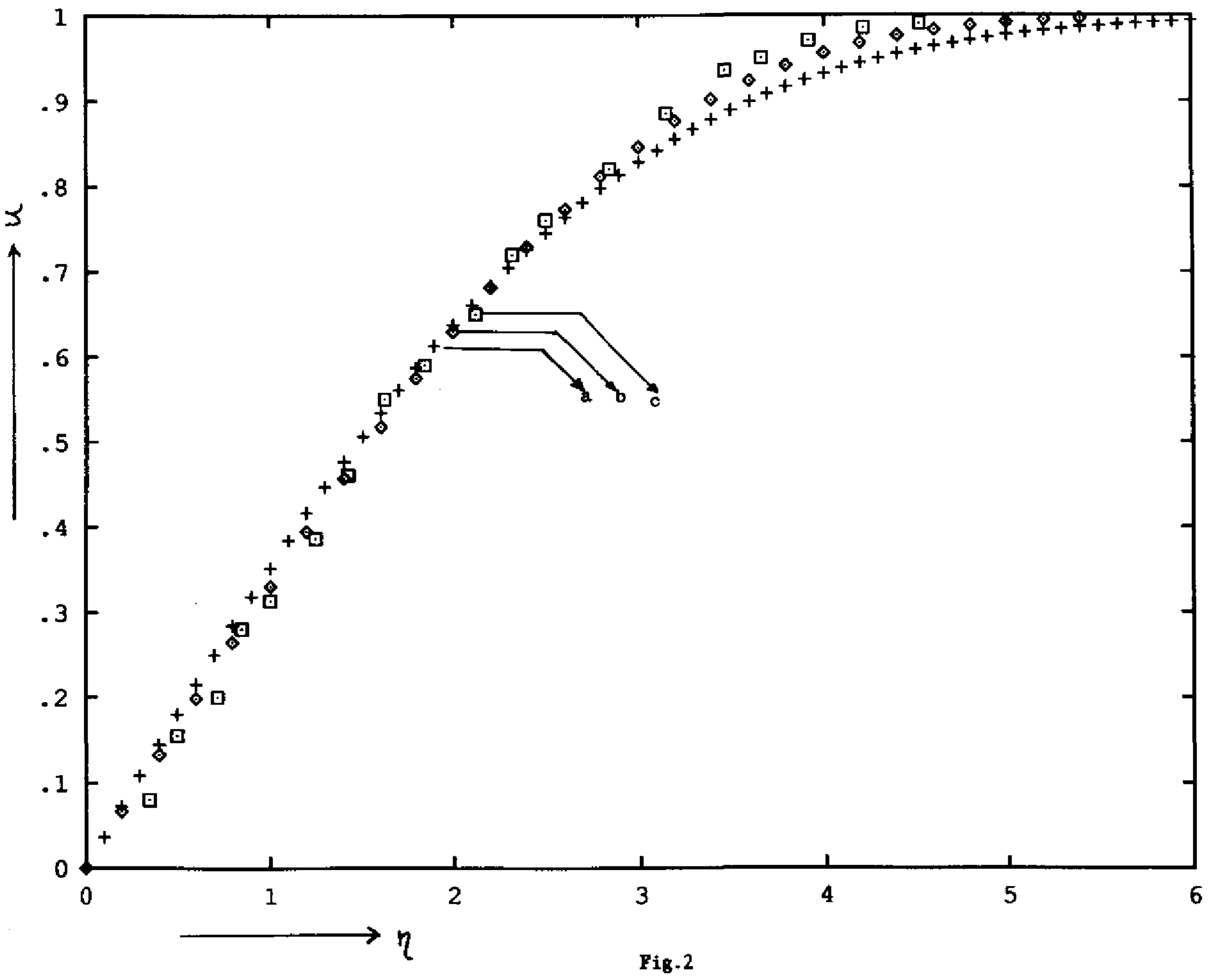

Fig.2