ITERATIVE SOLUTION
FOR NONLINEAR INTEGRAL EQUATIONS
OF HAMMERSTEIN TYPE

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Abstract

Let $E$ be a real Banach space with a uniformly convex dual $E^*$. Suppose $N$ is a nonlinear set-valued accretive map of $E$ into itself with open domain $D$; $K$ is a linear single-valued accretive map with domain $D(K)$ in $E$ such that $\text{Im}(N) \subseteq D(K)$; $K^{-1}$ exists and satisfies

$$\langle K^{-1}x - K^{-1}y, j(x - y) \rangle \geq \beta \|x - y\|^2$$

for each $x, y \in \text{Im}(K)$ and some constant $\beta > 0$, where $j$ denotes the single-valued normalized duality map on $E$. Suppose also that for each $h \in \text{Im}(K)$ the equation $h \in z + KNz$ has a solution $x^*$ in $D$. An iteration method is constructed which converges strongly to $x^*$. Explicit error estimates are also computed.
1 Introduction

Let $\Omega$ be a $\sigma$-finite measure space with measure denoted by $dx$. A nonlinear integral equation of Hammerstein type is one of the form:

$$ u(x) + \int_{\Omega} k(x, y)f(y, u(y))dy = h(x), \quad x \in \Omega, $$

where the real kernel $k$ is given in $\Omega \times \Omega$; $f(y, r)$ is a given real-valued function on $\Omega \times \mathbb{R}$; $h$ is a given function on $\Omega$. In operator theoretic form, the problem of determining a solution $u$ of (1) with $u$ lying in a given space of functions defined on $\Omega$ can be put in the form of a nonlinear functional equation:

$$ (I - KF)(u) = h $$

with the linear and nonlinear mappings $K$ and $F$ given by:

$$ (Ku)(x) = \int_{\Omega} k(x, y)v(y)dy; \quad x \in \Omega $$

and

$$ (Fu)(y) = f(y, u(y)) $$

The operator $F$ is called the Niemitskyi operator. We note immediately that several problems that arise in differential equations—for instance, elliptic boundary value problems whose linear parts possess Green's functions—can, as a rule, be transformed into Hammerstein equations (see e.g., [19]). Moreover, operators of Hammerstein-type (i.e., operators of the form $I - AB$) play a crucial role in the study of feedback control systems (see e.g., [12]). Hammerstein equations have been studied extensively by various authors (see e.g., [1-5], [10], [12], [23]).

Many of these authors have used the theory of maximal monotone operators in Banach spaces (defined below) to establish general results on the existence of solutions of equation (2) for reflexive Banach spaces. If $E$ is a real Banach space, $E^*$ its conjugate space, we let $\langle v, w \rangle$ denote the duality pairing between the element $w$ in $E^*$ and the element $v$ in $E$. A subset $M$ of the Cartesian product $E \times E^*$ is said to be monotone if

$$ \langle v_1 - v_2, w_1 - w_2 \rangle \geq 0 $$
for \([v_1, w_1]\) and \([v_2, w_2]\) in \(M\). A monotone set \(M\) is said to be maximal monotone if it is not properly contained in any other monotone set. Let \(T\) be a mapping of \(E\) into \(2^E^*\) the set of all subsets of \(E^*\). The effective domain, \(D(T)\) of \(T\) is the subset of \(E\) given by \(D(T) = \{u : u \in E, T(u) \neq \emptyset\}\). The range, \(R(T)\), of \(T\) is given by \(R(T) = \cup \{T(u) : u \in E\}\). The graph, \(G(T)\), of \(T\) is the subset of the Cartesian product \(E \times E^*\) given by \(G(T) = \{(v, w^* : v \in E, w \in T(v)\}\). \(T\) is said to be monotone if the graph \(G(T)\) is a monotone set. \(T\) is said to be maximal monotone if the graph \(G(T)\) is a maximal monotone set.

Existence results for equation (2) under various monotonicity and compactness conditions on the maps \(K\) and \(F\) abound in the literature (see e.g., [1-5; 12], [23]). A notion which is very closely related to the notion of monotone operators is that of accretive operators. A map \(U\) with domain \(D(U)\) and range \(R(U)\) in an arbitrary Banach space \(E\) is monotone (Kato, [15]) if the inequality

\[
\|x - y\| \leq \|x - y + t(Ux - Uy)\|
\]

holds for all \(x, y \in D(U)\) and some \(t > 0\). If inequality (3) holds for all \(t > 0\) then \(U\) is called accretive (Browder, [6]).

Let \(J\) denote the normalized duality mapping of \(E\) to \(2^{E^*}\) given by

\[
Jx = \{x^* \in E^* : \|x^*\|^2 = \|x\|^2 = \langle x, x^* \rangle\},
\]

where \(\langle ., . \rangle\) denotes the generalised duality pairing. If \(E^*\) is uniformly convex then \(J\) is single-valued and is uniformly continuous on bounded sets. (see e.g., [24]). Kato [15] also proved that \(U\) is monotone (in the sense of inequality (3)) if and only if \(U\) satisfies

\[
\langle Ux - Uy, w \rangle \geq 0
\]

for all \(x, y \in D(U)\) and \(w \in J(x - y)\). Clearly, for Hilbert spaces, monotonicity and accretivity are equivalent. The accretive operators were introduced independently in 1967 by Browder [6] and Kato [15]. If \(E = H\), a Hilbert space, one of the earliest problems in the theory of monotone operators was to solve the equation \(x - Ux = h\) for \(x\) given \(h \in H\) and a monotone operator \(U\). In [6], Browder proved that if \(U\) is locally Lipschitzian and accretive then \(U\) is \(m\)-accretive, i.e., \((I - U)\) is surjective. This result was subsequently generalised by Martin [17] to the continuous accretive operators. Zarantonello [25] also proved that if \(H\) is a Hilbert space and \(U\) is a monotone and Lipschitzian mapping of \(H\) into itself then \(x - Ux = h\) has a
unique solution in $H$. Observe that if $E = H$, a Hilbert space and $K = I$, the identity map of $H$, the Hammerstein equation (2) reduces to

$$x + Fx = h$$

(5)

When a solution to equation (5) is known to exist, iterative methods for approximating such a solution have been studied by various authors. In this connection, Dotson [13] proved that an iteration process of the Mann-type [16], under suitable conditions, converges strongly to the unique solution of (5) when $F$ is a monotone Lipschitz operator with Lipschitz constant 1. This result was extended by one of the authors [8] to mappings with Lipschitz constant $L \geq 1$. In [7], Bruck obtained a solution of equation (5), under certain conditions, as the limit of an iteratively constructed sequence in Hilbert space when $F$ is a set-valued monotone operator. He imposed no continuity assumptions on $F$ but assumed that a solution exists. This result has also been extended by one of the authors [9] to $L_p$ spaces, $p \geq 2$. Further results on the approximation of a solution of equation (5) have been proved in [11] for real Banach spaces $E$ with uniformly convex dual spaces, $E^\ast$.

While several existence theorems have been established, under various monotonicity and continuity conditions on $K$ and $F$ for the general operator equation (2) theorems on methods of approximating a solution (when one exists) have concentrated on the special case of the equation in which $K = I$. If $K$ is compact Brézis and Browder [4] have proved that a suitably defined Galerkin approximations converges strongly to a solution of equation (2). If $K$ is not compact, no strong convergence theorem has been proved. Recently, for $E = H$, a Hilbert space, one of the authors [10] proved that, under suitable conditions on $K$ and $F$, an iteration method of the Mann-type converges weakly to a solution of equation (2).

It is our purpose in this paper to prove, for $K$ not necessarily compact and in real Banach spaces much more general than Hilbert spaces, that the Mann-type fixed point iteration method converges strongly to a solution of equation (2). Our theorems generalize important known results. In particular, the main results of [7], [9], and [10] are special cases of our theorems.
2 Preliminaries

In the sequel, we shall need the following definitions and results: Let $T : E \to 2^E$ be any map. A point $y^* \in E$ will be called a fixed point of $T$ if $y^* \in T(y^*)$. The mapping $T$ will be called bounded if for all $x, y \in E$, $T$ satisfies $\| \xi - \eta \| \leq M$ for some constant $M \geq 0$, $\xi \in T(x)$ and $\eta \in T(y)$.

**Theorem D. (Dunn, [14] Lemma 1):** Let $\beta_n$ be recursively generated by

$$\beta_{n+1} = (1 - \delta_n) \beta_n + \sigma_n^2$$

with $n \geq 1, \delta_1 \geq 0$, $\{\delta_n\} \subset [0, 1]$ and $\sum \delta_n = \infty; \sum \sigma_n^2 < \infty$. Then, $\beta_n \geq 0$ for all $n \geq 1$ and $\lim \beta_n = 0$.

**Theorem R1 (Reich, [21], p.89):** If $E$ is uniformly convex, then there is a continuous nondecreasing function $b : [0, \infty) \to [0, \infty)$ such that $b(0) = 0, b(\varepsilon t) \leq \varepsilon b(t)$ for $\varepsilon \geq 1$, and

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x) \rangle + \max\{\|x\|^2, 1\} y^* b(\|y\|)$$

for all $x, y \in E$.

**Remark 1:** Nevanlinna and Reich [18] have shown that for any given continuous function $b(\varepsilon)$ with $b(0) = 0$, sequences $\{c_n\}$ always exist satisfying:

(i) $0 < c_n < 1$ for all $n \geq 1$; (ii) $\sum c_n = \infty$; (iii) $\sum c_n b(c_n) < \infty$.

If $E = L_p, 1 < p < \infty$, we can choose any sequence $\{c_n\}$ in $l^p \setminus l^1$ with $s = p$ if $1 < p \leq 2$ and $s = 2$ if $p \geq 2$.

**The Mann Iteration Method (see e.g. [11, 16]):** is defined as follows: For $K$ a convex subset of a Banach space $E$, and $T$ a mapping of $K$ into itself, the sequence $\{x_n\}$ in $K$ is defined by:

$$x_0 \in K,$$

$$x_{n+1} = (1 - c_n)x_n + c_nTx_n, n \geq 0$$

where $\{c_n\}$ is a real sequence satisfying:

$c_0 = 1; 0 \leq c_n < 1$ for all $n \geq 1$ and $\sum c_n = \infty$. The condition $\sum c_n = \infty$ is, in some applications, replaced by $\sum c_n(1 - c_n) = \infty$.

The Mann iteration process has been studied extensively by several authors for approximating solutions of several nonlinear operator equations.
in Banach spaces (see e.g., [7-11], [13-14], [16], [18]). In the sequel the single-valued normalized duality map will be denoted by $j$.

Let $E$ be a Banach space and let $C$ be a nonempty subset of $E$. A mapping $U : C \to E$ is called strongly accretive if for each $x, y \in C$ there exists $w \in J(x - y)$ such that

$$\langle Ux - Uy, w \rangle \geq \beta \|x - y\|^2$$

for some real constant $\beta > 0$.

### 3 Main Results

**Theorem 1:** Let $E$ be a real Banach space with a uniformly convex dual $E'$ and suppose that:

(i) $N$ is a nonlinear set-valued accretive map of $E$ into itself with open domain $D$;

(ii) $K$ is a linear single-valued accretive map of $E$ into itself with domain $D(K)$ such that $Im(N) \subset D(K)$; $K^{-1}$ exists and satisfies

$$\langle K^{-1}x - K^{-1}y, j(x - y) \rangle \geq \beta \|x - y\|^2$$

(4)

for all $x, y \in Im(K)$ and $\beta < 0$. Suppose also that for each $h \in Im(K)$ the equation $h \in x + KNx$ has a solution $x^*$ in $D$. Define the set-valued map $S$ with domain $D$ by $Sx = K^{-1}h - K^{-1}x - Nx + x, x \in D$.

Let $\{c_n\}$ be a real sequence satisfying:

(iii) $0 \leq c_n < 1$ for all $n \geq 1$; (iv) $\sum c_n = \infty$; and (v) $\sum c_n b(c_n) < \infty$.

Then, there exist a neighbourhood $B = B_d(x^*) \subset D$ of $x^*$ and a real number $N_0 \geq 0$ such that for any $n \geq N_0$, and any initial guess $x_1 \in B$, the sequence $\{x_n\}$ generated from $x_1$ by

$$x_{n+1} = (1 - c_n)x_n + c_n \xi_n, \exists \xi_n \in Sx_n,$$

remains in $D$ and converges strongly to $x^*$.

**Proof:** Observe first that $x^*$ is a fixed point of $S$ and that, without loss of generality, we may assume $\beta \in (0,1)$. Using inequality (4) and the monotonicity of $N$ we obtain that

$$\langle \xi - \eta, j(x - y) \rangle \leq (1 - \beta)\|x - y\|^2$$

(*)

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for all $\xi \in S(x), \eta \in S(y)$ and $x, y \in D$. This implies that $(I - S)$ is monotone and so is \textit{locally bounded} at each point of its effective domain (Rockafellar, [22]). In particular, $(I - S)$ is locally bounded at $x^*$. Thus, we can choose $B = B_d(x^*)$, the closed ball of radius $d > 0$ centred at $x^*$ such that $B \subset D$ and $(I - S)(B)$ is bounded. Let $\|\xi - x\| \leq C$ for each $\xi \in S(x)$ and $x \in B$. Then, $\|x - x^*\| \leq d + C$. Starting with an initial guess $x_1 \in D$ define $\{x_n\}$ by

$$x_{n+1} = (1 - c_n)x_n + c_n\xi_n, \exists \xi_n \in S(x_n), n \geq 1. \quad (6)$$

Observe that (v) implies $\lim c_n = 0$. This condition and the continuity of $b$ imply there exists an integer $N_0 > 0$ such that for $n \geq N_0$,

$$\max\{(1 - c_n)d, 1\}(C + d) \max\{C + d, 1\}b(c_n) < d^2(1 - \beta)$$

\textbf{CLAIM:} $x_n$ is well defined and $x_n \in B_d(x^*)$ for all $n \geq N_0$. We establish this claim by induction. For $n = 1, x_1 \in B_d(x^*)$ by our choice. Assume $x_n$ is well defined and $x_n \in B_d(x^*)$ for some $n$. Then, $\|x_n - x^*\| \leq d$ and $\|\xi_n - x^*\| \leq C + d$. Moreover, for all $n \geq N_0$ and using the induction hypothesis we obtain, using inequality (*) and the nondecreasing nature of $b$:

$$\|x_{n+1} - x^*\|^2 = \|(1 - c_n)(x_n - x^*) + c_n(\xi_n - x^*)\|^2$$

$$\leq (1 - c_n)^2\|x_n - x^*\|^2 + 2c_n(1 - c_n)\langle \xi_n - x^*, b(x_n - x^*) \rangle$$

$$+ \max\{(1 - c_n)\|x_n - x^*\|, 1\}c_n\|\xi_n - x^*\|b(c_n\|\xi_n - x^*\|)$$

$$\leq ((1 - c_n)^2 + 2\beta c_n(1 - c_n))\|x_n - x^*\|^2$$

$$+ \max\{(1 - c_n)\|x_n - x^*\|, 1\}c_n\|\xi_n - x^*\|\max\{\|\xi_n - x^*\|, 1\}b(c_n)$$

$$\leq [1 - (1 - \beta)c_n]d^2 + c_n\beta^2C_d(1 - \beta) = d^2,$$

so that $x_{n+1} \in B_d(x^*)$, completing proof of the claim.

Set $M = \sup\{(1 - c_n)\|x_n - x^*\|, 1\}\sup\|\xi_n - x^*\|\sup\{\|\xi_n - x^*\|, 1\}$,

$$\rho_n = \|x_n - x^*\|^2, (1 - \gamma_n) = [1 - (1 - \beta)c_n]^2 \geq 0.$$ Then, as above,

$$\|x_{n+1} - x^*\|^2 \leq [1 - (1 - \beta)c_n]\|x_n - x^*\|^2 + M c_n b(c_n)$$

so that

$$\rho_{n+1} \leq (1 - \gamma_n)\rho_n + M c_n b(c_n), n \geq 1 \quad (6)$$

Inequality (7) and a simple induction (see e.g., [11]) now give, for $n \geq 1$,

$$0 \leq \rho_n \leq \lambda^2 a_n, \quad (7)$$
where $\alpha_n \geq 0$ is recursively generated by

$$\alpha_{n+1} = (1 - c_n)\alpha_n + c_n b(\epsilon_n), \alpha_1 = 1,$$

and $\lambda^2 = \max\{\rho_1, M^*\}$ with $M^* = \sup\{||\xi - x^*||^2, \xi \in S(x), x \in D\}$. Conditions (iv) and (v) imply, by Theorem D, that $\alpha_n \to 0$ as $n \to \infty$ and so (by inequality (8)), $\{x_n\}$ converges strongly to $x^*$. This completes the proof of the Theorem.

Convergence rates

For the rate of convergence of the sequence constructed in Theorem 1 we need the following definition and result:

The modulus of convexity of a real Banach space $E$ is the function $\delta_E : [0, 2] \to [0, 1]$ defined by the following formula:

$$\delta_E(\epsilon) = \inf\{1 - 1/2||x + y|| : x, y \in E, ||x|| = ||y|| = 1, ||x - y|| \geq \epsilon\}.$$

Theorem R2 (Reich[20]) If $\delta_E(\epsilon) \geq k\epsilon^r$ for some $k > 0$ and $r \geq 2$, then for $t \leq M, b(t) \leq ct^{r-1}$, where $s = r(r - 1)^{-1}$. In particular, if $E = L_p, 1 < p < \infty$ then $s = p$ if $1 < p \leq 2$; and $s = 2$ if $2 \leq p < \infty$.

By setting $c_n = s(1 - \beta)^{-1}(n + 1)^{-1}$ in our theorem and using Theorem R2 (see also Reich[20], Chidume[11]) it follows that the sequence $\{x_n\}$ generated in our theorem satisfies:

$$||x_n - x^*|| = O(n^{-(r-1)/2})$$

In particular, if $E = L_p, 1 < p < \infty$, we have:

$$||x_n - x^*|| = O(n^{-(p-1)/2}), \text{ if } 1 < p \leq 2$$

and

$$||x_n - x^*|| = O(n^{-1/2}), \text{ if } 2 \leq p < \infty.$$
Proof Since $K$ is strongly accretive bounded and linear it follows that $K^{-1}$ exists and satisfies $(K^{-1}x,x) \geq \beta^*\|x\|^2$ for all $x \in \text{Im}(K)$ where $\beta^* = \beta\|K\|^{-2}$. The Corollary then follows from Theorem 1.

Remark 2 Brézis and Browder ([4], Theorem 1, p. 126) proved that under suitable conditions on $K$ and $F$ the equation $h \in x + KFx$ has a unique solution in $L^p(\Omega;\mathbb{R}^n)$ where $\Omega$ is a $\sigma$–finite measure space, and $1 < p < +\infty$.

If the conditions on $F$ in [4] are suitably modified and $K$ is a strongly accretive bounded linear map of $L^p(\Omega;\mathbb{R}^n)$ into itself, then $K$ satisfies the conditions (6)-(8) of [4] so that the existence of a unique solution to $h \in x + KFx$ in $L^p(\Omega;\mathbb{R}^n)$ follows from Theorem 1 of [4], and Theorem 1 can then be applied to approximate the solution when $1 < p < \infty$.

Remark 3. If in Theorem 1, we set $E = L_p, p \geq 2$ and $K = I$, the identity operator of $E$ we obtain the main result of [9] (which itself is a generalization of the main result of Bruck[7]).

Furthermore, if the $D$ in Theorem 1 is bounded and the range of $N, R(N)$, is also bounded, then following the method of proof of Theorem 1, the following theorem is easily proved:

Theorem 2: Let $E, N, K, D, c_n$ and $x^*$ be as in Theorem 1. Suppose $D$ (not necessarily open) is bounded and the range of $N$ is bounded. Then, for any initial guess $x_1 \in D$, the sequence $\{x_n\}$ defined in (6) converges strongly to $x^*$.

Remark 4. If the continuity requirement on the map $T$ in Theorem 1 of [11] is replaced by the requirement that the equation $x + Tx = f$ has a solution in the domain of $T$ then Theorem 2 extends the conclusion of that Theorem (and Corollaries 1 and 2 of [11]) to the more general operator equation (2) for set-valued monotone maps $T$. (The continuity condition imposed in Theorem 1 of [11] is needed only to ensure the existence of a solution. (See Remark 3 of [11]).)

Remark 5. Finally, we observe that the error estimates obtained in Theorems 1 and 2 agree with those obtained in [7],[8],[9],[11] and [13].
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