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ABSTRACT

A scattering theory for the wave equation with compactly supported perturbations was developed by Lax-Phillips in 1967 [L-P]. Using Enss approach [E], Phillips developed a Lax-Philllips scattering theory with short range perturbations of the type: $V(x) = o(\frac{1}{|x|^\beta}), \beta > 2$, see [P1], [P2], and [P3]. In this paper we develop a scattering theory for more general perturbations, i.e. for $V(x) = \frac{\varphi(x)}{|x|^\beta}$, where $\beta = 2 - \frac{n}{s}$; $\varphi \in L^s(\mathbb{R}^n), s > 2$ and $s \geq \frac{n}{2}$.

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INTRODUCTION

A general treatment of scattering theory for the wave equation with short range perturbations of the type: $V(x) = o(\frac{1}{|x|^\beta})$, for some $\beta > 2$, was presented by R. Phillips at [P1], [P2], and [P3]. In this paper we develop a scattering theory for more general perturbations.

We treat the wave equation

$$u_{tt} = Lu \equiv \Delta u - Vu, \tag{0.1}$$

with

$$V(x) = \frac{\varphi(x)}{|x|^\beta}, \text{ where } \beta = 2 - \frac{n}{s}, \ \varphi \in L^s(\mathbb{R}^n), s > 2 \text{ and } s \geq \frac{n}{2}. \tag{0.2}$$

We also require the unique continuation property for the eigenfunctions of L. For simplicity we will assume that L do not have positive eigenvalues, that the null space F of the infinitesimal generator of (0.1) $A_V = \begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix}$ and the null space F_1 of A_V^2 are zero, dimension $n \geq 3$ and odd. We can follow the method in [P1] and [P2] for other cases.

Using Caffarelli-Kohn-Nirenberg inequality [CKN] we prove in Lemma 4.1 that if the perturbation $V(x) = \frac{\varphi(x)}{|x|^\beta}$ is such that $\beta = 2 - \frac{n}{s}$, $\varphi \in L^s(\mathbb{R}^n)$ *and* $s \geq \frac{n}{2}$. Then the norm of the unperturbed energy space H_0 is equivalent to the norm of the perturbed energy space H_V. Furthermore, a necessary and sufficient condition over V that assures equivalent norms is given at Lemma 4.2.

In section 2 we shall prove that the wave operators $W_\pm$ exist on H_0 to H_V. This is, the strong limit $W_\pm f = \lim_{t\to\pm\infty} U_V(-t)U_0(t)f$ exist. The wave operators are isometries intertwining U_0 and U_V i.e. $U_V(t)W_\pm = W_\pm U_0(t)$. The prove of the existence is based on Cook's method and Lemma 4.3 that assures that if u is a solution of the unperturbed wave equation and $V(x) = \frac{\varphi(x)}{|x|^\beta}$ is such that $\beta = 2 - \frac{n}{s}$, $\varphi \in L^s(\mathbb{R}^n)$ and $s > 2$. Then

$$\lim_{r,s\to\infty} \int_r^s \left(\int_{\mathbb{R}^n} |Vu|^2 dx \right)^{\frac{1}{2}} dt = 0.$$

In section 3 we shall prove that the incoming and outgoing subspaces for the perturbed system satisfies the Lax-Phillips axioms relative to U_V on H_V As a consequence of this we obtain:

$$\text{Range}W_+ = \text{Range}W_- = H_V.$$

This is, the wave operators $W_\pm$ are complete. The proof is based on the Enss-Phillips technique [P1] [E] and on section 4.

Then the scattering operator can be defined from H_0 to H_0 as:

$$S = W_+^{-1} W_-.$$

In this paper we can change the Laplacian Δ by $\partial_i a_{ij} \partial_j$ where the matrix $(a_{ij}(x))$ is C^1 and $a_{ij} - \delta_{ij} = o(\frac{1}{r^\alpha}) = \partial a_{ij}$ for some $\alpha > 1$.(see [P1])

1.- DEFINITION OF THE WAVE OPERATORS

For the unperturbed equation we take:

$$(1.1)\qquad \begin{cases} u_{tt} = L_0 u \equiv \Delta u \\ u(x,t) = f_0(x) \\ u_t(x,0) = f_1(x), \end{cases}$$

where $x \in \mathbf{R}^n, n \geq 3, t \in \mathbf{R}$.

Let H_0 denote the Hilbert space of all initial data $f = (f_0, f_1)$ of finite energy, normed by the energy form

$$(1.2)\qquad E_0(f) = \int |\nabla f_0|^2 + |f_1|^2 \, dx.$$

It is convenient to write eq (1.1) in matrix form:

$$A_0 \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} u \\ u_t \end{pmatrix}_t,$$

and

$$\begin{pmatrix} u(x,0) \\ u_t(x,0) \end{pmatrix} = \begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix},$$

where

$$(1.3)\qquad A_0 = \begin{pmatrix} 0 & Id \\ \Delta & 0 \end{pmatrix}.$$

Let $U_0(t)$ denote the operator taking initial data into the solution data at time t. This is, we define $U_0(t) : H_0 \to H_0$ as

$$(1.4)\qquad U_0(t)f = (u, u_t).$$

It is clear that the $\{U_0(t)\}$ form a one-parameter group of unitary (energy-conserving) operators on H_0 and A_0 is its infinitesimal generator, this is :

$$(1.5)\qquad U_0(t)f = e^{tA_0 t}f,$$

for $f \in D(A_0) =$ domain of A_0.

A solution of (1.1) is called outgoing if it vanishes for $|x| < t,\ 0 < t$. It will be called incoming if it vanishes for $|x| < -t,\ t < 0$.

The set of all initial data for outgoing (incoming) solutions is denoted by D^0_+ (D^0_-).This is:

$$(1.6)\qquad D^0_\pm = \{f \in H_0 \mid U_0(t)f = 0, \text{for} |x| < \pm t, t \gtrless 0\}.$$

It can be shown that these subspaces satisfies the following properties (Lax Phillips axioms).

$$(1.7)\qquad \begin{aligned} &i) U_0(t)D^0_\pm \subset D^0_\pm \text{ for } t \gtrless 0. \\ &ii) \bigcap U_0(t)D^0_+ = \{0\} = \bigcap U_0(t)D^0_-. \\ &iii) \overline{\bigcup U_0(t)D^0_+} = H_0 = \overline{\bigcup U_0(t)D^0_-}. \end{aligned}$$

For the perturbed equation we take:

$$(1.8)\qquad \begin{cases} u_{tt} = Lu \equiv \Delta u - Vu. \\ u(x,t) = f_0(x) \\ u_t(x,0) = f_1(x), \end{cases}$$

where $x \in \mathbf{R}^n, n \geq 3, t \in \mathbf{R}$ and

$$(1.9)\qquad V(x) = \frac{\varphi(x)}{|x|^\beta},\ \beta \geq 2 - \frac{n}{s}\ \varphi \in L^s(\mathbf{R}^n),\quad s > 2 \text{ and } s \geq \frac{n}{s}.$$

Let H_V denote the Hilbert space of all initial data $f = (f_0, f_1)$ of finite energy, normed by the energy form

$$(1.10)\qquad E_V(f) = \int |\nabla f_0|^2 + V|f_0|^2 + |f_1|^2 \, dx.$$

The norm induced by E_0 (1.2) is equivalent to the norm induced by E_V (1.10). See Lemma 4.1. A necessary and sufficient condition over V that assure that the norms are equivalent will be given at Lemma 4.2

It is convenient to write eq (1.8) in matrix form:

$$A_V \begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} u \\ u_t \end{pmatrix}_t,$$

and

$$\begin{pmatrix} u(x,0) \\ u_t(x,0) \end{pmatrix} = \begin{pmatrix} f_0(x) \\ f_1(x) \end{pmatrix},$$

where

$$(1.11)\qquad A_V = \begin{pmatrix} 0 & Id \\ \Delta - V & 0 \end{pmatrix}.$$

Let $U_V(t)$ denote the operator taking initial data into the solution data at time t. This is, we define $U_V(t) : H_V \to H_V$ as

$$(1.12)\qquad U_V(t)f = (u, u_t)$$

(Here u is solution of (1.8)).

It is clear that the $\{U_V(t)\}$ form a one-parameter group of unitary (energy-conserving) operators on H_V and A_V is its infinitesimal generator, this is :

(1.13) $$U_V(t)f = e^{iA_V t}f,$$

for $f \in D(A_V)$ =domain of A_V.

Define the wave operators:

$$W_\pm : H_0 \to H_V,$$

as

(1.14) $$W_\pm f = \lim_{t\to\pm\infty} U_V(-t)U_0(t)f.$$

We define incoming and outgoing subspaces for H_V by

(1.15) $$D_\pm^V = W_\pm D_\pm^0.$$

2.- EXISTENCE OF THE WAVE OPERATORS.

The wave operators were defined on H_0 to H_V as

(2.1) $$W_\pm f = \lim_{t\to\pm\infty} U_V(-t)U_0(t)f.$$

We show in the next Lemma that this operators exists and are isometries. The proof of this Lemma is based on Cook's method and Lemma 4.3 that assures that if u is solution of the unperturbed wave equation then

$$\lim_{r,s\to\infty} \int_s^r \left(\int_{\mathbb{R}^n} |Vu|^2 \, dx \right)^{\frac{1}{2}} dt = 0.$$

Lemma 2.1. *The wave operators $W_\pm$ exists and are isometries on H_0 to H_V.*

Proof. We treat only W_+. Let $f \in (\bigcup U_0(t)D_+^0) \cup D(A_V)$ (a dense set of H_0), call $W(t) = U_V(-t)U_0(t)$. Since H_V is complete, the limit (2.1) exist if and only if $W(t)f$ is Cauchy as $t \to \infty$.

Using Cook's method we write

$$\begin{aligned} \|W(r)f - W(s)f\|_V &= \| \int_s^r \partial_t W(t)f \, dt\|_V \\ &\le \int_s^r \|\partial_t W(t)f\|_V \, dt \\ &= \int_s^r \|U_V(-t)\begin{pmatrix} 0 & 0 \\ -V & 0 \end{pmatrix} U_0(t)f\|_V \, dt. \end{aligned}$$

Making use of the fact that $U_V(t)$ is unitary with respect to E_V we can rewrite the last integral as

$$\int_s^r \left(\int |V(x)u(x,t)|^2 \, dx \right)^{\frac{1}{2}} dt,$$

where u is solution of (1.1). Then by Lemma 4.3

$$\lim_{r,s\to\infty} \|W(r)f - W(s)f\|_V = 0.$$

This proves that $W(t)f$ is Cauchy for a dense subset of H_0.The existence for all f in H_0 follows from

$$\begin{aligned} \|U_V(t)U_0(t)f\|_V &= \|U_0(t)f\|_V \\ &\le c\|U_0(t)f\|_0 \\ &= c\|f\|_0. \end{aligned}$$

Next we show that W_+ is an isometry on $\bigcup U_0(t)D_+^0$, and by completion we conclude for all H_0. Since U_V and U_0 are unitary operators and the unperturbed norm is equivalent to the perturbed norm we have

$$\begin{aligned} \|U_V(-t)U_0(t)f\|_V^2 &= \|U_0(t)f\|_V^2 \\ &= \|U_0(t)f\|_0^2 + \int V(x)|u(x,t)|^2 dx \\ &= \|f\|_0^2 + \int V(x)|u(x,t)|^2 \, dx, \end{aligned}$$

where u is solution of the unperturbed wave equation (1.1). Lemma 4.5 assures that $\lim_t \to \infty \int V(x)|u(x,t)|^2 \, dx = 0$ and then we can conclude that

(2.2) $$\|W_+(t)f\|_V = \|f\|_0$$

for all $f \in H_0$■

To see that the wave operators are isometries interwining U_0 and U_V, this is:

(2.3) $$U_V(t)W_\pm = W_\pm U_0(t)$$

use (1.13).

3.- COMPLETENESS OF THE WAVE OPERATORS

We shall prove in this section that the wave operators $W_\pm : H_0 \to H_V$ defined as $W_\pm f = \lim_{t\to\pm\infty} U_V(-t)U_0(t)f$ are such that

(3.1) $$\text{Range } W_+ = \text{Range } W_- = H_V.$$

The incoming and outgoing subspaces for the perturbed system are defined as

$$D_\pm^V = W_\pm D_\pm^0. \tag{3.2}$$

We shall prove that $D_\pm^V$ satisfy the Lax-Phillips axioms, a consequence of which is (3.1).
It follows from (1.7) i), ii) and (2.3) and (3.2) that

$$i) U_V(t) D_\pm^V \subset D_\pm^V \text{ for } t \gtrless 0, \tag{3.3}$$

$$ii) \bigcap U_V(t) D_+^V = \{0\} = \bigcap U_0(t) D_-^0.$$

The next Lemma shows that $D_\pm^V$ satisfy the iii) axiom of Lax-Phillips. The proof is based on Enss-Phillips Technique (see (P1) and (E1)) and section 4.

Lemma 3.1. *Let $D_\pm^V$ be as in (3.2), U_V as (1.12) and $\| \, \|_V$ induced by (1.10). Then*

$$iii) \overline{\bigcup U_V(t) D_+^V}^{\| \|_V} = H_V = \overline{\bigcup U_V(t) D_-^V}^{\| \|_V}. \tag{3.4}$$

Proof. Suppose that iii) does not hold for D_+^V. Then there will exist a nonzero $f \in H_V$ which is E_V-orthogonal to $\bigcup U_V(t) D_+^V$. This implies that $U_V(t)f$ is E_V-orthogonal to D_+^V for all t. We can therefore choose a time smoothed version of f such that $A_V f$ is E_V-orthogonal to D_+^V and $A_V f$ belongs to $D(A_0)$.

We shall prove that $A_V f = 0$, and since A_V has no point spectrum in H_V by hypothesis it will follow that $f = 0$ proving that iii) does hold. The prove is due for dimension $n \geq 3$, n odd. For other cases n use [P1] and [P2].

Using Lemma 4.6 (Weak form of local energy decay) we obtain a sequence of t_n, $t_n \to \infty$ such that for all integers n

$$\| A_V U_V(t_n) f \|_0^{|x|<n} < \epsilon_n \tag{3.5}$$

where ϵ_n is chosen so that Lemma 4.7 gives for $|t|$, $|s| < n$

$$\| U_V(t+s) A_V U_V(t_n) f - U_0(t) U_V(s) A_V U_V(t_n) f \|_0 < \frac{k}{n^\epsilon}. \tag{3.6}$$

We will denote $f_n = U_V(t_n) f$.

We now decompose $A_V f_n$ into its incoming and outgoing parts. Let $T : H_0 \to L^2(\mathbb{R}, N)$, where N is an auxiliar Hilbert space, be the translation representation (see [L P] pag. 118). Then T is an isometry such that

$$T U_0(t) f = k(s-t) \tag{3.7}$$

$$T A_0 = \partial_s T. \tag{3.8}$$

T is obtained from the Radon transform $Rf(s,w) = \int_{<x,w>=s} f(x) dH_x$ this is

$$T(f_0, f_1) = -D_s^{\frac{n+1}{2}} Rf_0 + D_s^{\frac{n-1}{2}} Rf_1$$

Consider $TA_V f_n$ and choose ϕ, ψ in $C^\infty(\mathbb{R})$ so that $0 \leq \phi \leq 1$ and so that

$$\phi(s) = \begin{pmatrix} 1 & \text{if } s > \frac{1}{2} \\ 0 & \text{if } s < -\frac{1}{2} \end{pmatrix} \tag{3.9}$$

and $\psi = 1 - \phi$.

Denote by g_n and h_n the data whose representers are $\phi T A_V f_n$ and $\psi T A_V f_n$ respectively

$$T g_n = \phi T A_V f_n, \tag{3.10}$$

$$T h_n = \psi T A_V f_n, \tag{3.11}$$

in this case $A_V f_n = g_n + h_n$.

Claim. *Let g_n and h_n be as in (3.10) and (3.11). Then $g_n \in D_+^0$, $A_V g_n \in H_V$ and $h_n \in D_-^0$, $A_V h_n \in H_V$.*

Proof of claim. By (3.10) $\|g_n\|_0 \leq \|A_V f_n\|_0$ and by (3.11) $\|h_n\| \leq \|A_V f_n\|_0$ and since

$$T U_0(t) g_n(x) = k(s-t) = 0 \tag{3.12}$$

if $s - t < -\frac{1}{2}$ we obtain

$$U_0(t) g_n(x) = 0, \tag{3.13}$$

for $|x| < -\frac{1}{2}$, so $g_n \in D_+^0$ and then g_n became the incoming part $A_V f_n$. In the same way we can prove that $h_n \in D_-^0$. This is h_n is the outgoing part of $A_V f_n$.

By (3.7) we have

$$T(A_0 g_n) \leq c T A_V f_n + T A_0 A_V f_n \tag{3.14}$$

and then

$$\begin{aligned} \|A_V g_n\|_V &\leq c_1 \|A_0 g_n\|_0 + c_2 \|g_n\|_0 \\ &\leq c_3 \|A_V f_n\|_0 + c_4 \|A_0 A_V f_n\|_0 + c_2 \|g_n\|_0 \\ &< \infty, \end{aligned}$$

so $A_V g_n \in H_V$ and in the same way $A_V h_n \in H_V$.

Now we shall prove that exists t_n such that

$$(3.15)\qquad \langle A_V f_n, g_n \rangle_0 = \lim_{t_n\to\infty} \langle U_0(t_n)A_V f_n, U_0(t_n)g_n \rangle_0$$
$$\overset{(a)}{=} \lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, U_0(t_n)g \rangle_0$$
$$\overset{(b)}{=} \lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, U_0(t_n)g_n \rangle_V$$
$$\overset{(c)}{=} \lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, W_+U_0(t_n)g_n \rangle_V$$
$$\overset{(d)}{=} 0.$$

In the same way we can prove that $\langle A_V f_n, h_n \rangle_0 = 0$ and since $A_V f_n = g_n + h_n$ one deduce that $A_V f = 0$.

Claim (d).

$$\lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, W_+U_0(t_n)g_n \rangle_V \overset{(d)}{=} 0.$$

Proof of (d). By hypothesis $U_V(t)A_V f_n$ is E_V−orthogonal to D_+^V. By (3.13) $g_n \in D_+^0$ Then by (3.2)

$$W_+U_0(t)g_n \in D_+^V$$

and by hypothesis $U_V(t)A_V f_n$ □

Claim (c).

$$\lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, U_0(t_n)g_n \rangle_V$$

$$\overset{(c)}{=} \lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, W_+U_0(t_n)g_n \rangle_V .$$

Proof of (c). Using the intertwining property (2.3)we can prove that

$$\lim_{t\to\infty} \|W_+U_0(t)g_n - U_0(t)g_n\|_V =$$

$$\lim_{t\to\infty} \|W_+g_n - U_V(-t)U_0(t)g_n\|_V = 0 \quad \square$$

Claim (b).

$$\lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, U_0(t_n)g \rangle_0$$

$$\overset{(b)}{=} \lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, U_0(t_N)g_n \rangle_V .$$

Proof of (b). Using Hölder inequality, Lemma 4.4 and using $()_0$ to denote the first component we obtain:

$$\lim_{t\to\infty} (\langle U_V(t)A_V f_n, U_0(t)g_n \rangle_V - \langle U_V(t)A_V f_n, U_0(t)g_n \rangle_0) \quad \square$$
$$= \lim_{t\to\infty} \int V(x)(U_V(t)A_V f_n)_0 \overline{(U_0(t)g_n)_0}\, dx$$
$$\le \lim_{t\to\infty} \Big(\int V(x)|(U_V(t)A_V f_n)|^2\, dx\Big)^{\frac12}\Big(\int V(x)|(U_0(t)g_n)_0|^2\, dx\Big)$$
$$\le \|A_V f_n\|_V \lim_{t\to\infty} \int V(x)|(U_0(t)g_n)_0|^2 dx$$
$$= 0$$

Claim (a).

$$\lim_{t_n\to\infty} \langle U_0(t_n)A_V f_n, U_0(t_n)g_n \rangle_0$$

$$\overset{(a)}{=} \lim_{t_n\to\infty} \langle U_V(t_n)A_V f_n, U_0(t_n)g \rangle_0 .$$

Proof of (a). It follows from (3.6) that

$$\langle U_V(n)A_V f_n, U_0(n)A_V f_n \rangle_0 - \langle U_0(n)A_V f_n, U_0(n)g_n \rangle_0$$

$$= \langle U_v(n)A_V f_n - U_0(n)A_V f_n, U_0(n)g_n \rangle_0$$

$$\le \|U_V(n)A_V f_n - U_0(n)A_V f_n\|_0 \, \|U_0(n)g_n\|_0$$
$$\le \frac{c}{n^\epsilon} \quad \square$$

With this we finish the proof of Lemma 3.1.■

4.-TECHNICAL INEQUALITIES

First we recall a result of Caffarelli-Kohn-Nirenberg [C-K-N] Let $p, q, r, \alpha, \beta, \sigma$ and δ be fixed real numbers satisfying $p, q \ge 1$, $r > 0, 0 \le \delta \le 1$, and $\frac1p + \frac{\alpha}{n}, \frac1q + \frac{\beta}{n}, \frac1r + \frac{\gamma}{n} > 0$, where $\gamma = \delta\sigma + (1-\delta)\beta$. Then we have the following Theorem see ([C-K-N] pag.259).

Theorem. *There exist a positive constant k such that the following inequality holds for all $u \in C_0^\infty(\mathbb{R}^n)$.*

$$(4.1)\qquad \Big(\int_{\mathbb{R}^n} |u|^r|x|^{r\gamma}\, dx\Big)^{\frac1r} \le k\Big(\int_{\mathbb{R}^n} |\nabla u|^p|x|^{p\alpha}\, dx\Big)^{\frac{\delta}{p}}\Big(\int_{\mathbb{R}^n} |u|^q|x|^{q\beta}\, dx\Big)^{\frac{1-\delta}{q}}.$$

if and only if the following relations hold:

$$\frac1r + \frac{\gamma}{n} = \delta\Big(\frac1p + \frac{\alpha-1}{n}\Big) + (1-\delta)\Big(\frac1q + \frac{\beta}{n}\Big),$$

where $0 \leq \alpha - \sigma$ if $\delta > 0$, and $\alpha - \sigma$ if $\delta > 0$ and $\frac{1}{p} + \frac{\alpha-1}{n} = \frac{1}{r} + \frac{\delta}{n}$.

The next Lemma will show that the unperturbed energy norm

$$\|f_0, f_1\|_0 = (\int_{\mathbb{R}^n} |\nabla f_0|^2 + |f_1|^2 \, dx)^{\frac{1}{2}} \tag{4.2}$$

is equivalent to the perturbed energy norm

$$\|f_0, f_1\|_V = (\int_{\mathbb{R}^n} |\nabla f_0|^2 + V|f_0|^2 + |f_1|^2 \, dx)^{\frac{1}{2}} \tag{4.3}$$

Where

$$V(x) = \frac{\varphi(x)}{|x|^\beta}, \beta = 2 - \frac{n}{s}, \varphi \in L^s(\mathbb{R}^n), \text{ and } s \geq \frac{n}{2}. \tag{4.4}$$

Lemma 4.1. *Let $u(x,t)$ be a solution of the unperturbed wave equation (1.1) and $V(x) = \frac{\varphi(x)}{|x|^\beta}, \beta = 2 - \frac{n}{s}, \varphi \in L^s(\mathbb{R}^n)$, and $\frac{n}{2} \leq s$. Then*

$$\int V(x)|u(x,t)|^2 \, dx \leq k \int |\nabla u(x,t)|^2 dx.$$

Proof. Using Hölder inequality for s and s' and [C-K-N]inequality (4.1) for $r = \frac{1}{2s'}$, $\gamma = -\beta_0$, $\delta = 1$, $p = 2$, $\alpha = 0$, where $\beta_0 = 1 - \frac{n}{2p}$, we obtain:

$$\begin{aligned}
&(\int V(x)|u(x,t)|^2 \, dx)^{\frac{1}{2}} \\
&\leq (\int |\varphi(x)|^s \, dx)^{\frac{1}{2s}} (\int \frac{|u(x,t)|^{2s'}}{|x|^{\beta s'}} \, dx)^{\frac{1}{2s'}} \\
&\leq k_1 (\int |u(x,t)|^{2s'} |x|^{2s' - \beta_0} \, dx)^{\frac{1}{2s'}} \\
&\leq k_2 (\int |\nabla u(x,t)|^2 \, dx)^{\frac{1}{2}} \blacksquare
\end{aligned}$$

Next we give a necessary and sufficient condition over the perturbation that gives equivalence of norms.

Lemma 4.2. *The unperturbed energy norm (4.2) is equivalent to the perturbed energy norm (4.3) if and only if V is such that*

$$sup_{0,r<\infty} r^{2-n} \int_0^r t^{n-1} V(t) \, dt < \infty. \tag{4.5}$$

Proof. (see [C-D-G]).

The behaviour of the L^2-norm of $V(x)u(x,t)$ where u is solution of the unperturbed wave equation is crucial to prove existence of the wave operators i.e. of the scattering operator. The next Lemma will show us that under condition (4.4) the L^2-norm of Vu behaves good enough. (Condition (4.5) is not enough to have existence of the wave operators.)

Lemma 4.3. *Let $u(x,t)$ be a solution of the unperturbed wave equation such that $u(x,t) \in D^0_+$ (see (1.6)) and $V(x) = \frac{\varphi(x)}{|x|^\beta}, \beta = 2 - \frac{n}{s}, \varphi \in L^s(\mathbb{R}^n)$, and $s \in (2, \infty]$. Then*

$$\lim_{r,s\to\infty} \int_r^s (\int_{\mathbb{R}^n} |V(x)u(x,t)|^2 \, dx)^{\frac{1}{2}} \, dt = 0.$$

Proof. It is known (see [R-S, III], pag. 46) that if $u(x,t)$ is a solution of (1.1) with initial data in $H_0(\mathbb{R}^n)$. Then:

(a) For any N and $\epsilon > 0$, there is a $C_{N,\epsilon}$ so that

$$|u(x,t)| \leq C_{N,\epsilon}(1 + |x| + |t|)^{-N}, \tag{4.6}$$

where $|x| \geq (1+\epsilon)|t|$.

(b) There is a d so that

$$|u(x,t)| \leq (\frac{d}{(1+|t|)})^{\frac{n-1}{2}} \tag{4.7}$$

for all x and t.

Now let assume that $u(x,t) \in D^0_+$ (a dense set on H_0). This is, there exist T such that $u(x,t) = 0$ if $|x| < t + T$.

First we assume that $s < \infty$. Then using Hölder inequality for $\frac{2}{s}$ and $\frac{s}{s-2}$ we obtain:

$$\begin{aligned}
&\lim_{r,s\to\infty} \int_r^s (\int_{\mathbb{R}^n} |V(x)u(x,t)|^2 \, dx)^{\frac{1}{2}} \, dt \\
&\leq \lim_{r,s\to\infty} \int_r^s (\int_{\mathbb{R}^n} |\varphi(x)|^s \, dx)^{\frac{1}{s}} (\int_{\mathbb{R}^n} (\frac{|u(x,t)|^2}{(1+|x|)^{2\beta}})^{\frac{s}{s-2}} \, dx)^{\frac{s-2}{2s}} \, dt
\end{aligned}$$

$$\leq \lim_{r,s\to\infty} \int_r^s k (\int_{|x|>t+T} (\frac{|u(x,t)|^2}{|x|^{2\beta}})^{\frac{s}{s-2}} \, dx)^{\frac{s-2}{2s}} \, dt, \tag{4.8}$$

where $k = \int_{\mathbb{R}^n} |\varphi(x)|^s \, ds)^{\frac{1}{2s}}$.

Consider the following decomposition

$$\begin{aligned}
A(t) &= \{x \in \mathbb{R}^n \mid (1+\epsilon)|t| \leq |x|, \epsilon > 0\} \\
B(t) &= \{x \in \mathbb{R}^n \mid |t| + T \leq |x| \leq (1+\epsilon)|t|\}
\end{aligned} \tag{4.9}$$

Now we do two steps to calculate (4.8) first we restict to A(t) and second we restrict to B(t).

Step 1.- Choose N at 4.6 big enough such that:

$$\lim_{r,s\to\infty} \int_r^s k (\int_{A(t)} (\frac{|u(x,t)|^2}{(1+|x|)^{2\beta}})^{\frac{s}{s-2}} \, dx)^{\frac{s-2}{2s}} \, dt$$

$$\leq \lim_{r,s\to\infty} \int_r^s k\Big(\int_{A(t)} \Big(\frac{C_{N,\epsilon}}{|x|^{2\beta}(1+|x|+|t|)^{-N}}\Big)^{\frac{s}{s-2}}\, dx\Big)^{\frac{s-2}{2s}}$$

$$\leq \lim_{r,s\to\infty} \int_r^s \frac{k_1}{t^{1+\epsilon}}\, dt$$

$$= 0. \tag{4.10}$$

Step 2.- Now we calculate (4.8) over B(t). If $T > 0$ then $u(x,t)$ is zero on $B(t)$ so we assume $T < 0$. Since $\beta = 2 - \frac{n}{s}$ and $s > 2$ we have that $n + \frac{2s}{s-1} < (2\beta + n - 1)\frac{s}{s-1}$. Now using 4.7 we obtain:

$$\lim_{r,s\to\infty} \int_r^s k\Big(\int_{B(t)} \Big(\frac{|u(x,t)|^2}{(1+|x|^{2\beta}}\Big)^{\frac{s}{s-2}}\, dx\Big)^{\frac{s-2}{2s}}\, dt$$

$$\leq \lim_{r,s\to\infty} \int_r^s k\Big(\int_{B(t)} \Big(\frac{d}{|x|^{2\beta}(1+|t|)^{n-1}}\Big)^{\frac{s}{s-2}}\, dx\Big)^{\frac{s-2}{2s}}\, dt$$

$$\leq k_2 \int_r^s \frac{vol(B(t))^{\frac{s-2}{2s}}}{t^{\beta+\frac{n-1}{2}}}\, dt$$

$$\leq k_3 \int_r^s \frac{t^{\frac{n}{2}-\frac{n}{s}}}{t^{\beta+\frac{n-1}{2}}}\, dt = 0. \tag{4.11}$$

Second we consider the case when $s = \infty$. Then $\beta = 2$ and φ is bounded. Arguing as in (4.11) we have

$$\lim_{r,s\to\infty} \int_r^s \Big(\int_{B(t)} \frac{\varphi(x)|u(x,t)|^2}{|x|^4}\, dx\Big)^{\frac{1}{2}}\, dt$$

$$\leq \lim_{r,s\to\infty} k \int_r^s \Big(\int_{B(t)} \frac{1}{|x|^4(1+t)^{n-1}}\, dx\Big)^{\frac{1}{2}}\, dt$$

$$\leq \lim_{r,s\to\infty} k_2 \int_r^s t^{\frac{n}{2}-\frac{3}{2}-\frac{n}{2}}\, dt = 0 \blacksquare$$

Lemma 4.4. *Let V be such that $V(x) = \frac{\varphi(x)}{|x|^\beta}$ where $\beta = 2 - \frac{n}{s}$, $\varphi \in L^s(\mathbb{R}^n)$, $s > 2$ and $s \geq \frac{n}{2}$.*

Then :

(i) The functional

$$V(f_0, f_1) = \int_{\mathbb{R}^n} V|f_0|^2\, dx$$

is a compact continuous mapping from H_0 into $\mathbb{R}$.

(ii) Given $\epsilon > 0$, there is a constant C_ϵ such that for all f in $D(A_1)$

$$\int_{\mathbb{R}^n} |Vf_0|^2\, dx \leq \epsilon \int_{\mathbb{R}^n} |\Delta f_0|^2\, dx + C_\epsilon \int_{\mathbb{R}^n} |\nabla f_0|^2\, dx.$$

Proof. (i) By Lemma 4.1, V as a functional is well-define and continuos on H_0. Let $u_m = (u_{m_1}, u_{m_2})$ be a sequence in H_0 such that

$$sup_m \|u_m\|_0^2 = M < \infty \tag{4.12}$$

Let $\epsilon > 0$. Choose $R > 0$ such that

$$R^{2-\frac{n}{s}-\beta} < \epsilon(1+M)^{-1} \tag{4.13}$$

. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \phi \leq 1$, $\phi(x) = 1$ when $\|x\| \leq R$, $\phi(x) = 0$ when $\|x\| \geq 2R$, $sup_{x\in\mathbb{R}^n}|\nabla\phi(x)| \leq 1$. Put $v_m = \phi u_m$. By (4.13), Lemma 4.1 and (4.12)

$$\int_{\mathbb{R}^n} |u_{m_1} - v_{m_1}|^2 V(x) dx$$

$$\leq R^{2-\frac{n}{s}-\beta} \int_{R^n} |u_{m_1}|^2 \frac{\varphi(x)}{|x|^{2-\frac{n}{s}}} dx$$

$$\leq \frac{\epsilon}{M+1} \int_{\mathbb{R}^n} |\nabla u_{m_1}|^2 dx \leq \epsilon \tag{4.14}$$

It is clear that v_m is bounded in H_0. Since $s > \frac{n}{2}$, $\frac{2s}{s-1} < \frac{2n}{n-2}$. Therefore, by the Rellich-Kondrachov Theorem ([A],pag. 144), there is a subsequence v_{m_k} of v_m and $v \in H_0$ such that

$$\lim_{k\to\infty} \int_{B(0,R)} |v_{m_k 1} - v_1|^{\frac{2s}{s-1}} dx = 0.$$

Thus by Hölder inequality and Lemma 4.1, we get

$$\lim_{k\to\infty} \int_{B(0,R)} |v_{m_k 1} - v_1|^2 |V(x)| dx$$

$$\leq \lim_{k\to\infty} \{int_{B(0,R)}|V|^s dx\}^{\frac{1}{s}} \Big\{\int_{B(0,R)} |v_{m_k 1} - v_1|^{\frac{2s}{s-1}} dx\Big\}^{1-\frac{1}{s}} = 0 \tag{4.15}$$

From (4.13) and (4.14) we get (i).

(ii) Fix a $R > 0$. By Lemma 4.1 and Lemma 4.2 we have

$$\int_{|x|>R} |Vf_0|^2 dx \leq c \int_{\mathbb{R}^n} |\nabla f_0|^2 dx. \tag{4.16}$$

If $n = 3$, by the Sobolev Imbedding Theorem we get

$$\int_{|x|<R} |Vf_0|^2 dx \leq (\int_{|x|<R} |V|^2 dx)(sup_{|x|<R}|f_0(x)|)^2$$

$$\leq M(\int_{|x|<R} |V|^2 dx) \sum_{0\leq|\alpha|\leq 2} \int_{|x|<R} |D^\alpha f_0|^2 dx. \tag{4.17}$$

If $n = 4$, put $p = \frac{s}{s-2}$. Applying the Sobolev Theorem we get

$$\int_{|x|<R} |Vf_0|^2 dx$$

$$\leq \{\int_{|x|<R} |V|^s dx\}^{\frac{2}{s}} \{\int_{|x|<R} |f_0|^{2p} dx.\}^{\frac{1}{p}}$$

$$\leq M_2 \sum_{0\leq|\alpha|\leq 2} \int_{|x|<R} |D^\alpha f_0|^2 dx. \tag{4.18}$$

Similarly for $n > 4$.
Using Theorem 8.12 in [G-T] pag.186 we have

$$\int_{|x|<R} \sum_{0\leq|\alpha|\leq 2} |D^\alpha f_0|^2 dx$$

$$\leq M_3 \int_{|x|<R} |\triangle f_0|^2 + |f_0|^2 dx. \tag{4.19}$$

Now choosing a sufficiently small R, from (4.16), (4.17), (4.18) and (4.19) we get

$$\int_{\mathbb{R}^n} |Vf_0|^2 dx$$

$$\leq \epsilon \int_{\mathbb{R}^n} |\triangle f_0|^2 dx + M_4\{\int_{R^n} |\nabla f_0|^2 dx + \int_{|x|<R} |f_0|^2 dx\}. \tag{4.20}$$

Applying Lemma 4.1 (for the caracteristic function on $|x| < R$) to the last integral in (4.20) we get (ii).■

Lemma 4.5. *Let $u(x,t)$ and $V(x)$ be as in Lemma 4.3. Then*

$$\lim_{t\to\infty} \int_{\mathbb{R}^n} V(x)|u(x,t)|^2 dx = 0.$$

The proof of this lemma follows from the proof of Lemma 4.3.

Lemma 4.6. *(WEAK FORM OF LOCAL ENERGY DECAY) Suppose $f \in H_V \bigcap D(A_V)$. Then there exists a sequence of t_{n_s}, $t_n \to \infty$, such that for all $R_0 > 0$*

$$\lim_{n\to\infty} \|A_V U_V(t_n) f\|^{|x|<R_0} = 0.$$

Proof Since A_V has no point spectrum in H_V it can be shown (see [L-P] pag.145) that there exists a sequence t_n tending to infinity such that

$$\lim_{t_n\to\infty} < U_V(t_n)A_V f, k >_V = 0$$

for all $k \in H_V$.
Since the E_V and E_0 forms are equivalent on H_0 then

$$\lim_{t_n\to\infty} < U_V(t_n)A_V f, k >_0 = 0$$

for all $k \in H_0$. By the Rellich compactness criterion the set of $f \in D(A_V)$ is compact in $H_0(B_R)$. This implies that if the weak limit is zero then the strong limit is zero. This means that exists a sequence t_n such that

$$\lim_{t_n\to\infty} \|A_V U_v(t_{n_s'}) f\|_0^{|x|<R} = 0■$$

Lemma 4.7. *Suppose $f \in H_V \bigcap D(A_V)$ and $\|A_V U_V(t_n) f\|_{0,|x|<R}$. Then for $|t|, |s| < R$*

$$\|U_V(t+s)A_V f_V - U_0(t)U_0(s)A_V f_n\|_0^2 \leq \frac{c}{R^\epsilon} + C,$$

with $\epsilon > 0$.

Proof The proof of this lemma is based on Duhamel's principle, Lemma4.4, Lemma 4.8 and Lemma 4.9.For details see [P1].

Lemma 4.8. *If f belongs to $H_V \bigcap D(A_V)$. Then*

$$\|QA_V U_V(t) f\|_{0,|x|<R}^2 \leq \frac{k\|A_V U_V(t) f\|_V}{R^\beta},$$

where $\beta = 2 - \frac{n}{s}$ and $Q = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}$.

Lemma 4.9. *For any f in H_V such that*

$$\|f\|_0^{2R} < \epsilon$$

exists C_R such that

$$\|U_V(t) f\|_0^R \leq C_R \epsilon$$

for $|t| < \frac{3R}{4c}$.

The proof of this Lemmas are given at [P1], pag.628 630.

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