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ON THERMAL STABILITY IN INCOMPRESSIBLE SLIP FLOW *

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ABSTRACT

The paper considers the classical problem of the stability of a layer of fluid heated from below, but in the case when the density is low and there is slip flow at the bounding walls. The eigenvalue problem which ensues is tackled by taking cognisance of the orthogonality of Bessel function of the first kind. It is observed that the Rayleigh number for the onset of instability, for the case of marginal stability, is increased by gas rarefication.

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1 INTRODUCTION

The classical problem of the thermal stability of a layer of fliud heated from below has been comprehensively discussed in Chandrasekhar [1], for fluids of high density in incompressible flow. However even in incompressible flows, gas rarefication can occur as a result of low pressure or high temperature. Street [2] has discussed the technological import of such rarefied flows.

This work is therefore addressed to the thermal stability of a layer of fluid heated from below when there is both temperature and velocity slip. The fluid is confined between two rigid walls, since the Benard problem does not make much physical sense in slip flows. Hence in section two the well-known equations of the eigenvalue problem is re-stated together with the now applicable slip boundary conditions. By adopting the orthogonality properties of Bessel function, the determination of the critical Rayleigh number is deduced in section three. Section four is devoted to the discussion of the results.

2 GOVERNING EQUATIONS

We consider a horizontal layer of fluid confined between two plates at non dimensional distances $z = 0$ and $z = 1$ such that gravitation g acts in the reverse direction of z and the physical depth of the fluid layer is d. If all the dependent variables are also non-dimensional, then the equation for the undisturbed mode is

$$\frac{d^2\theta^{(0)}}{dz^2} = 0, \; \theta = \theta_{0,1} + h_1 \frac{d\theta^{(0)}}{dz} \text{ on } z = 0, 1. \qquad (1)$$

Here θ is the fluid temperature, $\theta_{0,1}(\theta_0 > \theta_1)$ are the temperatures of the upper and lower walls respectively and h_1 is the temperature slip coefficient

(h_2 will denote the velocity slip coefficient). The solution to equation (1) is

$$\theta^{(0)} = (1 - h_1)\theta_0 + h_1\theta_1 - \beta z, \; -\beta = T_0 - T_1 \tag{2}$$

The superscript zero in θ indicates the undisturbed state. The perturbed state will be without a superscript.

Hence the boundary conditions on the perturbed state are

$$\theta = h_1 \frac{\partial \theta}{\partial z}, \; (u, v, w) = h_2 \frac{\partial(u, v, w)}{\partial z} \text{ on } z = 0, 1. \tag{3}$$

From the continuity equation, we can show that

$$\frac{\partial(u - h_2 \frac{\partial u}{\partial z})}{\partial x} + \frac{\partial(v - h_2 \frac{\partial v}{\partial y})}{\partial y} + \frac{\partial w}{\partial z} - h_2 \frac{\partial^2 w}{\partial z^2} = 0 \tag{4}$$

Thus in virtue of equation (3) we can deduce form equation (4) the additional boundary conditions

$$\frac{\partial w}{\partial z} - h_2 \frac{\partial^2 w}{\partial z^2} = 0 \,\text{on}\, z = 0, 1 \tag{5}$$

Now the celebrated equations of Chandrasekhar for the stability eigenvalue problem are

$$(D^2 - a^2)(D^2 - a^2 - \sigma)W = (\frac{g\alpha}{\nu} d^2) a^2 \Theta \tag{6}$$

$$(D^2 - a^2 - P_r\sigma)\Theta = -(\frac{\beta}{\kappa} d^2) W$$

where $D^2 = d/dz$ and W(z) and $\Theta(z)$ are the amplitude of the monochromatic representation of w and θ. Further a is the wave number P_r is the Prandtl number, α is the coefficient of volume expansion for a Boussinesq fluid, σ is the time constant, ν is the kinematic coefficient of viscosity and κ is the heat diffusivity. The boundary conditions for equation (6) can now be extracted from equations (3) and (4) as

$$W = h_2 \frac{dW}{dz}, \; \frac{dW}{dz} = h_2 \frac{d^2W}{dz^2}, \; \Theta = h_1 \frac{d\Theta}{dz} \; on z = 0, 1 \tag{7}$$

Equations (6) and (7) now constitute the eigenvalue problem.

3 METHOD OF SOLUTION

We consider the solution of equation (6) and (7) for marginal stability $\sigma = 0$. It is now convenient to write

$$\Theta = \sum_{n=1}^{\infty} J_m(\lambda_n z), \quad m \geq 2 \tag{8}$$

where λ_n are the roots of the equation

$$J_m(x) = h_1 x J'_m(x) \tag{9}$$

and dash on the Bessel function denotes differentiation with respect to the argument. Hence the boundary conditions on Θ in equation (7) are identically satisfied provided $m \geq 2$. Further, putting

$$W = \sum_{n}^{\infty} a_n W_n(z) \tag{10}$$

equation (6a) reduces to

$$(D^2 - a^2)^2 W_n = (\frac{g\alpha}{\nu} d^2) a^2 J_n(\lambda_n z) \tag{11}$$

with solution

$$W_n = \frac{1}{2}(\frac{g\alpha}{\nu} d^2) a^2 \{Acoshaz + Bsinhaz + Czcoshaz + Dzsinhaz$$

$$+ \int_0^z [(z - \varsigma)cosha(z - \varsigma) - \frac{1}{a} sinha(z - \varsigma)] J_m(\lambda_n \varsigma) d\varsigma\} \tag{12}$$

where A,B,C,D are constants.

It is now a simple matter to deduce that

$$\frac{dW_n}{dz} = \frac{1}{2}(\frac{g\alpha}{\nu} d^2) a^2 \{(Ba + C)coshaz + (Aa + D)sinhaz$$

$$+ Dazcoshaz + Cazsinhaz +$$

$$a \int_0^z (z - \varsigma) sinha(z - \varsigma) J_m(\lambda_n \varsigma) d\varsigma\} \tag{13}$$

$$\frac{d^2 W_n}{dz^2} = \frac{1}{2}(\frac{g\alpha}{\nu} d^2) a^2 \{a(Aa + 2D)coshaz + a(Ba + 2C)sinhaz$$

$$+ Ca^2 zcoshaz + Da^2 zsinhaz$$

$$+ a \int_0^z [a(z - \varsigma)cosha(z - \varsigma) + sinha(z - \varsigma)] J_m(\lambda_n \varsigma) d\varsigma\} \tag{14}$$

Therefore the boundary conditions on W in equation (7) can, by virtue of equations (12)-(14), be reduced to

$$A = (Ba + C)h_2, \; D = (\frac{1 - a^2h_2^2}{2ah_2})(Ba + C)$$

$$B = \frac{\gamma_4(a)I_{1,2}(\lambda_n) - \gamma_2(a)I_{2,3}(\lambda_n)}{\gamma_1(a)\gamma_4(a) - \gamma_2(a)\gamma_3(a)}, \; C = \frac{\gamma_1(a)I_{2,3}(\lambda_n) - \gamma_3(a)I_{1,2}(\lambda_n)}{\gamma_1(a)\gamma_4(a) - \gamma_2(a)\gamma_3(a)} \quad (15)$$

where

$$\gamma_1 = \frac{1}{2}\{acosha(1 - a^2h_2^2) + sinha(1 + a^2h_2^2) - 2sinha - sinha(\frac{1 - a^2h_2^2}{h_2})\}$$

$$\gamma_2 = \frac{1}{2}\{cosha(1-a^2h_2^2)+2h_2asinha+sinha(\frac{1 + a^2h_2^2}{h_2})-2cosha-sinha(\frac{1 - a^2h_2^2}{ah_2})\}$$

$$\gamma_3 = \frac{1}{2}\{2a^2h_2sinha+a^2sinha(1-a^2h_2^2)-acosha(\frac{1 - a^2h_2^2}{h_2})-sinha(\frac{1 + a^2h_2^2}{h_2})\}$$

$$\gamma_4 = \frac{1}{2}\{2a^2h_2cosha+4ah_2sinha+asinha(1-a^2h_2^2)-cosha(\frac{1 - a^2h_2^2}{h_2})-2sinha$$
$$-sinha(\frac{1 + a^2h_2^2}{ah_2})\}$$

$$I_{1,2} = \int_0^1 [(1-\varsigma)cosha(1-\varsigma)-\frac{1}{a}sinha(1-\varsigma)-ah_2(1-\varsigma)sinha(1-\varsigma)]J_m(\lambda_n\varsigma)d\varsigma\}$$

$$I_{2,3} = a\int_0^1 [(1-\varsigma)sinha(1-\varsigma)-h_2a(1-\varsigma)cosha(1-\varsigma)-h_2sinha(1-\varsigma)]J_m(\lambda_n\varsigma)d\varsigma\}$$

Finally substituting equations (8) and (12) into equation (6b) and invoking the orthogonality property of the Bessel function we can show that

$$\sum_1^\infty a_n[\frac{1}{2}\lambda_n^2\{n/l\} + \frac{1}{2}a^2R < n/l > -(\lambda_n^2 + a^2)\omega_n\delta_{n,l}] = 0$$

which leads to the infinite secular determinant

$$\| \lambda_n^2\{n/l\} + a^2R < n/l > -2(\lambda_n^2 + a^2)\omega_n\delta_{n,l} \| = 0 \quad (16)$$

for the determination of the Rayleigh number R. We have introduced

$$\omega_l = \int_0^1 zJ_m^2(\lambda_n z)dz = \frac{1}{2\lambda_n^2}(\frac{\lambda_n^2}{h_2^2} + \lambda_n^2 - m^2)J_m^2(\lambda_n)$$

$$\{n/l\} = \int_0^1 zJ_m(\lambda_l z)[J_{m-2}(\lambda_n z) + J_{m+2}(\lambda_n z)]dz$$

$$< n/l >= \int_0^1 zJ_m(\lambda_l z)(A_n coshaz + B_n sinhaz + C_n zcoshaz + D_n zsinhaz$$

$$+\int_0^1 zJ_m(\lambda_l z)dz\int_0^z[(z-\varsigma)cosha(z-\varsigma)-\frac{1}{a}sinha(z-\varsigma)]J_m(\lambda_n\varsigma)d\varsigma$$

where R still has its usual definition

$$R=\frac{g\alpha\beta d^4}{\nu\kappa}$$

Generally because of the variational nature of the solution, only a few terms in equation (16) are required to obtain a reasonable accuracy for the determination of R. This is discussed in the next section.

4 DISCUSSION

We solve for R in equation (16) by considering first $n = 1 = l$ and then $n = 2 = l$. When $m = 2$ we find from Table 1 that sufficient convergence

$a = 5.195$	$h_{1,2}$	R
First Approximation	0.05	2858.33
Second Approximation	0.05	2848.00
	0.1	2978.51

is obtained from the second approximation. In the absence of gas rarefication, $h_1 = 0 = h_2$, R is about 1706 as given in Chandrasekhar [1]. Hence we see that rarefication stabilizes the flow.

Indeed when the degree of rarefication increases the critical Rayleigh number for the onset of instability increases accordingly.

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References

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