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WHY AND HOW TO NORMALIZE THE FACTORIAL MOMENTS OF INTERMITTENCY

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WHY AND HOW TO NORMALIZE THE FACTORIAL MOMENTS OF INTERMITTENCY

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ABSTRACT

The normalization of factorial moments of intermittency, which is often the subject of controverses, is justified and (re-)derived from the general assumption of multi-Poissonian statistical "noise" in the production of particles at high-energy. Correction factors for the "horizontal" vs. "Vertical" analyses are derived in general cases, including the factorial multi-bin correlation moments.

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The present note contains, at least in the essentials, nothing new from the derivation of the factorial moment method as described in Ref.[1]. But, the experience one can get from the now increasing number of papers on the determination of factorial moments and from the discussions at the conference is rather convincing about the usefulness of recalling the reason why and the way how correctly normalized factorial moments are defined. This will also give the opportunity to extend the derivation of normalization to other quantities such as the bin-bin[1] or multi-bin factorial correlations which can be of interest[2] in the discussion of intermittent or other models of multiparticle production.

The definition of the normalized factorial moments does not stem from a personal, even well-educated, guess, but is derived from a precise requirement ; namely, it has to correspond to the deconvolution (suppression) of a multi-Poissonian "statistical noise" from the fluctuations of the event-per-event number of particles around the average density. Of course, the assumption of a multi-Poissonian noise is a strong one and is to be justified from physical considerations but once it is made , the definition of moments and, as we shall see above, of various correlation observables generalizing the moments, does not suffer from any ambiguity problem. In fact, it corresponds to the requirement of eliminating any other source of fluctuation than the unknown dynamical ones. As a matter of fact, the definition of the normalisation is expected to eliminate also the apparent fluctuations due to the non-uniformity in phase space of the average spectrum, by measuring the fluctuations with respect to the actual average in a way which will be recalled later on.

Another exercise is to exhibit extra constraints, such as choosing a-priori a set of collision events with fixed or limited observed multiplicity. Then, the normalization changes but again in a well-defined manner, since one has to express these "external constraints" in complement with the statistical noise distribution which, by definition and in practice, aims at the (simplest) description of the whole part of the process for which we know nothing (except its probable high complexity) and that we want to treat in a statistically independant way.

i) Generating function of moments

Let us recall that the multi-Poissonian noise (one could also work with the multinomial Bernouilli law), initially proposed in Ref.(3), comes from the assumption that if a certain number K_m of particles is observed in a given part of phase-space: the m[th] bin, it corresponds to K_m independant realizations of the probability for finding here a particle with (in fact for each bin), a normalised Poissonian distribution. One can write[1] :

$$Q(K_1,...K_m,...K_M) = \int \mathcal{P}(\rho_1,...\rho_m,...\rho_M) \prod_{m=1}^{M} \frac{(\delta\rho_m)^{K_m}}{K_m!} \exp(-\delta\rho_m)\ d\rho_m \times \times \delta(constraints) \tag{1}$$

where ρ_m is the *asymptotic* density of particles in the m_{th} bin m running from 1 to M, and δ the extension of the bin. In fact *asymptotic* is meant in terms of a limit in the total number of particles per event (or even per bin!) in such a way that $\delta\rho_m$ plays the role of a probability weight ; Q and $\mathcal{P}$ are the probability distributions of the number of particles and densities, which describe respectively, the observational and the dynamical content of the fluctuation patterns. Q corresponds to the statistical distribution for the considered set of events (even only one) , while $\mathcal{P}$ is the distribution which can be discussed in terms of a dynamical mecanism independant of its event-per-event realization in a finite number of particles. The *δ(constraints)* term is placed in formula (1) to specify eventually some extra conditions which are known or choosen to exist the set of events. An exemple is given by the requirement of a fixed multiplicity N, expressed by a factor $\delta(N - K_1 - ... - K_m - ... - K_M)$.

Now, the calculation is rather straightforward. Let us introduce the generating function of moments Φ; By definition one writes:

$$\Phi(\lambda_1,...,\lambda_m,...\lambda_M) = \left\langle \lambda_1^{K_1}...\lambda_m^{K_m}...\lambda_M^{K_M} \right\rangle_Q / \langle 1 \rangle_Q \tag{2}$$

where $\langle...\rangle_Q$ represents the averaging of a function of the K'_Ms with respect to the distribution Q. It is easy to realize that the next relation follows from the very definition (2):

$$\begin{aligned} \langle K_{m_0}(K_{m_0}-1)...(K_{m_0}-p+1)\rangle &= \left.\frac{\partial^p \Phi}{(\partial\lambda_{m_0})^p}\right|_{all\lambda's=1} \\ \langle K_{m_0}...(K_{m_0}-p_0+1)K_{m_1}...(K_{m_1-p_1+1})K_{m_2}...\rangle &= \left.\frac{\partial^{p_0}}{(\partial\lambda_{m_0})^{p_0}}\frac{\partial^{p_1}}{(\partial\lambda_{m_1})^{p_1}}...\Phi\right|_{\lambda'_s=1} \end{aligned} \tag{3}$$

where m_0, m_1, m_2... are different (fixed) bins and p_0, p_1, ... arbitrary integers and the expressions in brackets are (unormalized) factorial moments and higher multi-bin factorial correlations of interest in the problems of fluctuations and intermittency[1,2,3]

ii) Vertical analysis

Let us first consider the case of the so-called "vertical analysis" when one is interested in the study of fluctuations in a fixed bin, say in the rapidity of particles. In this kind of inclusive anlysis, and if one studies a reasonably but not too[4]small interval in the central region, one can ignore

the δ-function of constraints, those due in particular to energy-momentum conservation. Inserting expression (1) without the δ-function in the definition (2) and inverting summation and integration, one finds :

$$\Phi = \int \mathcal{P}(\rho_1, ..., \rho_M) \prod_m exp(\delta\rho_m(\lambda_m - 1))\, d\rho_m \tag{4}$$

and thus by repeated use of the derivative relations (3), one obtains the *vertical* factorial moments F_p^V and correlators $C^V_{p_0,p_1...}$

$$\begin{aligned} \langle F_p^V \rangle_Q &= \frac{\langle K_{m0} ... (K_{m_0} - p + 1) \rangle_Q}{[\langle K_{m_0} \rangle_Q]^p} = \frac{\langle \rho^p_{m_0} \rangle_{\mathcal{P}}}{[\langle \rho_{m_0} \rangle_{\mathcal{P}}]^p} \\ \langle C^V_{p_0,p_1...} \rangle &= \frac{\langle K_{m_0} ... (K_{m_0} - p_0 + 1) K_{m_1} ... (K_{m_1} - p_1 + 1) ... \rangle_Q}{\langle K_{m_0} \rangle_Q^{p_0} \langle K_{m_1} \rangle_Q^{p_1} ...} = \frac{\langle \rho^{p_0}_{m_0} \rho^{p_1}_{m_1} ... \rangle_{\mathcal{P}}}{\langle \rho_{m_0} \rangle_{\mathcal{P}}^{p_0} \langle r_{m_1} \rangle_{\mathcal{P}}^{p_1}} \end{aligned} \tag{5}$$

where the averaging $\langle ... \rangle_{\mathcal{P}}$ is over the distribution of densities.

Note that the combination $\langle C^V_{p_0,p_1...} \rangle / \langle F^V_{p_0} \rangle \langle F^V_{p_1} \rangle$ is shown to have particularly attractive properties in the framework of random cascading models of intermittency; they are independant of the bin *size*[1] and related to the dynamical mechanisms responsible for the intermittency *patterns*[2].

The rather simple expressions one obtains relating *normalized* factorial moments to the density distributions are no more valid if the Poissonian distribution is truncated for high number of produced particles by, e.g. energy momentum conservation. Then, it is preferable to come back to the full definition (1) and to express the corresponding truncation constraint in a simulation[4]. As a result, there is a range in which the relations (5) are realized. Outside, the discussion has to be done through the simulation. The major point in all this discussion is that the normalizations in (5) correspond to the averaging over differents events in the same fixed region of phase space. The requirement of a phase-space-dependant normalization factor is essential to avoid correction factors[1,5] for the shape of the averaged distribution when one changes the location of the bin m_0. On a more dynamical point of view, the distribution $\mathcal{P}$ could depend on the location m_0 and in facts seems to do[6], but this is independent from the normalization problem.

iii) "Horizontal analysis"

In an horizontal analysis, the focus is put a-priori on the analysis of a unique event, when one considers the distribution of fluctuations among different bins, as in the initial study on intermittency of Ref.[3]. However, one needs to know the average distribution around which the fluctuations are to be measured. We will thus consider the extension of the original horizontal analysis to a set of events of the same, fixed, multiplicity N. By definition, the averaged distribution will be defined by this set of events. As discussed previously, the definition (1) has to be used with a constraint $\delta(N - K_1 - ... - K_M)$. In the absence of other constraints, one gets :

$$\Phi_N = \int \prod_m \mathrm{d}\rho_m \mathcal{P}(\rho_1, ..., \rho_M) \left\{ \frac{\delta\rho_1\lambda_1 + ... + \delta\rho_M\lambda_M}{N} \right\}^N \delta\left(N - \sum_m \delta\rho_m\right) \tag{6}$$

where one expresses the multiplicity constraint, namely: $\sum_m K_m = \sum_m \delta\rho_m = N$.

It is worth noting the compatibility of expressions (4) and (6) for the generating functionales Φ and Φ_N. It comes from the general definitions (1) and (2), where one computes Φ either directly (see eqn.(4)) or as a sum over the different event multiplicities N. One gets:

$$\begin{aligned} \Phi \equiv \sum_N \int \prod_m \mathrm{d}\rho_m \mathcal{P}(\rho_1 ..., \rho_M) \left(\sum_m \frac{\delta\rho_m\lambda_m}{N}\right)^N \frac{e^{-N}}{N!} \delta\left(N - \sum_m \delta\rho_m\right) \\ \equiv \sum_N \Phi_N \, e^{-N}/N! \end{aligned} \tag{7}$$

Note that, for this comparison, one sums over a limited range of multiplicities in order to avoid the known *dynamical* dependance of the distribution $\mathcal{P}$ over the multiplicity.

Using (6), let now compute the horizontal factorial moments F_Q^H, as they were defined in the original paper, Ref.[3].

$$\langle F_p^H \rangle \equiv \frac{\frac{1}{M}\sum_m \langle K_m ... (K_m - p + 1)\rangle_Q}{\frac{N}{M} ... \frac{(N - p + 1)}{M}} = \frac{\frac{1}{M}\sum_m \langle \rho_m^p \rangle_{\mathcal{P}}}{\left(\frac{1}{M}\sum \langle \rho_m \rangle_{\mathcal{P}}\right)^p} \tag{8}$$

where one made use of the relation $\frac{1}{M}\sum_m \langle \rho_m \rangle_{\mathcal{P}} = N/M\delta$. If one would like to make the connection of horizontal with vertical factorial moments of Eq.(5), that is with the normalized normal moments of densities, the expression (8) is to be compared to the following expression, derived from (5) :

$$\frac{1}{M} \sum_m \langle F_p^V \rangle_Q \equiv \frac{1}{M} \sum_m \frac{\langle \rho_m^p \rangle_{\mathcal{P}}}{[\langle \rho_m \rangle_{\mathcal{P}}]^p} \tag{9}$$

Assuming that $\langle F_q^V \rangle_Q$ is bin-independant one obtains :

$$\langle F_p^H \rangle = \langle F_p^V \rangle_Q \cdot \frac{\frac{1}{M}\sum_{m_0} \langle \rho_{m_0} \rangle_{\mathcal{P}}^p}{\left(\frac{1}{M}\sum_{m_0} \langle \rho_{m_0} \rangle_{\mathcal{P}}\right)^p} = \langle F_p^V \rangle_Q \cdot \mathrm{R}_p \tag{10}$$

where one finds the correction factor, R_p, already advocated in Ref.[1] and derived in Ref.[5]. However, this factor is correct only if the vertical factorial moment, thus the density fluctuations, are bin-independant in the considered range , as also noted at this conference[6]. Beyond these technical considerations, it is a highly non-trivial conjecture that the dynamical distribution $\mathcal{P}$ remains the same, e.g. that the vertical and horizontal analyses, once the corrections made, lead to similar patterns of intermittency. Indeed, there could have been no link between the distribution of fluctuations among different events in the same phase-space location and the fluctuations observed in a unique event for different phase-space bins. Such a link would correspond to a kind of "ergodic property" of the multiparticle dynamics which seems to be experimentally verified. It is also implemented for the class of intermittency models based on random cascading[1,3]. However certain observed[6] small variations with the bin locations deserves more systematic study.

The same comparison can be made for factorial correlators. Taking as a (generalizable) example the horizontal analysis of bin-bin correlations, one would define :

$$\langle C_{p_0,p_1}^H \rangle = \frac{1}{M-D} \sum_{\substack{m_0 \\ m_1 - m_0 = D}} \frac{\langle K_{m_0},...,(K_{m_0} - p_0 + 1)\, K_{m_1},...,(K_{m_1} - p_1 + 1) \rangle_Q}{\frac{N}{M}\frac{(N-1)}{M}\cdots\frac{(N - p_0 - p_1 + 1)}{M}} \tag{11}$$

where the two bins are separated by a fixed distance D (see Ref.[1] for details). Following the same analysis as the one leading to equations (8-10), one easily gets:

$$\langle C_{p_0,p_1}^H \rangle = \frac{\frac{1}{M-D}\sum_{m_0} \langle \rho_{m_0}^{p_0} \rho_{m_1}^{p_1} \rangle_{\mathcal{P}}}{\left(\frac{1}{M}\sum_{m} \langle \rho_m \rangle_{\mathcal{P}}\right)^{p_0+p_1}} \tag{12}$$

provided the assumption of bin-independance of $\langle C_{p_0}^V \rangle$ is made. One gets a generalized correction factor for the correlation, namely:

$$R_{p_0,p_1} = \frac{\langle C_{p_0,p_1}^H \rangle}{\langle C_{p_0,p_1}^V \rangle} = \frac{\frac{1}{M-D}\sum_{m_0, m_1 - m_0 = D} \langle \rho_{m_0}^{p_0} \rho_{m_1}^{p_1} \rangle_{\mathcal{P}}}{\left(\frac{1}{M}\sum_{m} \langle \rho_m \rangle_{\mathcal{P}}\right)^{p_0+p_1}} \tag{13}$$

which corrects for the extra contributions to the correlations due to the non uniform distribution of the average density spectrum of particles. This correction factors can be easily generalized to multi-bin correlations. Note also that the particular combination which leads to a binsize-independant determination of correlations[1] requires its own correction factor, namely :

$$\frac{\langle C^V_{p_0,p_1}\rangle}{\langle F^V_{p_0}\rangle\langle F^V_{p_1}\rangle} = \frac{\langle C^H_{p_0,p_1}\rangle}{\langle F^H_{p_0}\rangle\cdot\langle F^H_{p_1}\rangle} / \frac{R_{p_0,p_1}}{R_{p_0}.R_{p_1}}. \tag{14}$$

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