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THE EFFECT OF ULTRASOUND ON ARTERIAL BLOOD FLOW: I - STEADY FULLY DEVELOPED FLOW *

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ABSTRACT

The paper models the effects of ultrasound heating of the tissues and the resultant perturbation on blood flow in the arteries and veins. It is assumed that the blood vessel is rigid and the undisturbed flow is fully developed. Acoustical perturbation on this Poiseuille flow, for the general three–dimensional flow with heat transfer in an infinitely long pipe is considered. Closed form analytical solutions are obtained to the problem. It is discovered that the effects of the ultrasound heating are concentrated at the walls of the blood vessels.

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1. INTRODUCTION

It is now well–known that, if the thermal equilibrium of the human body and its environment is disturbed, heat flows through the epidermis, the dermis, the fat, the muscles and so on. A detailed investigation of this heat transfer process has been carried out by Wilson and Spence [1] and Hodson *et al.* [2]. Hence the application of ultrasound heats up the tissue, thereby causing free convection heat transfer to flow in the blood vessels embedded in the tissues. The extent to which the ultrasound disturbs the flow in the blood vessels, forms the central theme of this paper.

The problem is a very difficult one, since blood vessels distend under a pulsatile flow situation. Here we make this simplifying assumption that the blood vessel is rigid, and the flow before the incident of the ultrasonic beam is steady and fully developed. The acoustical approximations are then applied to the general three–dimensional flow with heat transfer in an infinitely long cylindrical tube.

In Section 2 the governing equations are developed, and the solutions of these equations are obtained in a closed form in Section 3. Section 4 is devoted to the discussion of these solutions. In Part II, the analysis is extended to the case of an initially oscillatory flow, and additional deductions are made about the overall flow structure.

2. GOVERNING EQUATIONS

We consider the flow in an infinitely long rigid pipe with free convection heat transfer. If p is the pressure, θ the temperature, (u, v, w) the velocity components in the cylindrical polar coordinates (r, φ, z) such that t is time, then following Bestman [3, 4] the governing equations could be expressed in the form

$$\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{r\partial \varphi} = 0 \\
&Re\left\{\frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{u\partial v}{\partial r} + \frac{v\partial v}{r\partial \varphi} + \frac{uv}{r}\right)\right] - \frac{1}{r}\frac{\partial}{\partial \varphi}\left(\frac{u\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \varphi} - \frac{v^2}{r}\right)\right\} = \\
&\quad = \left(\nabla^2 - \sigma\frac{\partial}{\partial t}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{1}{r}\frac{\partial u}{\partial \varphi}\right] + G_r\left(\frac{\partial \theta}{\partial r}\sin\varphi + \frac{1}{r}\frac{\partial \theta}{\partial \varphi}\cos\varphi\right) \\
&Re\left(\frac{u\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \varphi}\right) = -\frac{\partial p}{\partial z} + \left(\nabla^2 - \sigma\frac{\partial}{\partial t}\right)w \\
&P_r\sigma\frac{\partial \theta}{\partial t} + Re\,P_r\left(\frac{u\partial \theta}{\partial r} + \frac{v}{r}\frac{\partial \theta}{\partial \varphi}\right) = \nabla^2\theta
\end{aligned} \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial \varphi^2}\,.$$

The problem depends on the frequency parameter σ, the Reynolds number Re, the free convection parameter G_r and the Prandtl number P_r.

We assume that initially a fully developed flow, $(0\,,0\,,w^{(0)}(r))$, exists in the tube such that

$$-\frac{dp^{(0)}}{dz} \equiv K^{(0)} = \frac{d^2 w^{(0)}}{dr^2} + \frac{1}{r}\,\frac{dw^{(0)}}{dr}\,, \qquad (2)$$

in which $K^{(0)}$ is the driving pressure. The elementary solution of Eq.(2) subject to the no–slip condition at the wall $r = 1$, is

$$w^{(0)} = \frac{1}{4}\,K^{(0)}(1 - r^2)\,. \qquad (3)$$

An ultrasound beam of frequency parameter σ, is subsequently applied on the skin which then heats up the tissue above the body ambient temperature θ_c (say). The heating of the tissue then causes free convection current to flow in the blood vessel. In step with the acoustical approximation, we write

$$u = u'(r,\theta,t)\,,\ v = v',\, w = w^{(0)}(r) + w',\ p = K^{(0)}z + p'(r,\varphi,t)\,, \theta = \theta_c + \theta'\,, \qquad (4)$$

where the primes quantities are small, so that we can neglect their squares and products. Substituting Eqs.(4) in Eqs.(1), we get

$$\begin{aligned}
&\frac{1}{r}\,\frac{\partial}{\partial r}\,(ru') + \frac{1}{r}\,\frac{\partial v'}{\partial \varphi} = 0\\
&0 = \left(\nabla^2 - \sigma\,\frac{\partial}{\partial r}\right)\left[\frac{1}{r}\,\frac{\partial}{\partial r}\,(rv') - \frac{1}{r}\,\frac{\partial u'}{\partial \varphi}\right] + G_r\left(\frac{\partial \theta'}{\partial r}\sin\varphi + \frac{1}{r}\,\frac{\partial \theta'}{\partial \varphi}\cos\varphi\right)\\
&Re\,\frac{dw^{(0)}}{dr}\,u' = \left(\nabla^2 - \sigma\,\frac{\partial}{\partial t}\right)w'\\
&P_r\,\sigma\,\frac{\partial \theta'}{\partial t} = \nabla^2\theta'\,.
\end{aligned} \qquad (5)$$

The velocity vector satisfies the no–slip boundary conditions on $r = 1$. We then take the temperature at the tube wall to be of the form

$$\theta_w(\varphi,t) = (\theta_0 + \theta_1\cos\varphi + \ldots + \theta_n\cos n\varphi + \ldots)\,e^{it} \qquad (6a)$$

in virtue of the Fourier decomposition, since generally the heating of the tube wall will be asymmetric.

The form of Eq.(6), dictates the solution to Eqs.(5). First, for the temperature we seek solution in the form

$$\theta' = \left[\Theta^{(0)}(r) + \Theta^{(1)}(r)\cos\varphi + \ldots + \Theta^{(n)}(r)\cos n\varphi + \ldots\right]\,e^{it}\,, \qquad (6b)$$

Hence we have

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\,\frac{d}{dr} - \frac{n^2}{r^2} - i\sigma P_r\right)\Theta^{(n)} = 0\,, \quad \Theta^{(n)}(r) = \theta_n\,; \quad n = 0,1,2,\ldots \qquad (7)$$

Then for the velocity components, we now write

$$
\begin{aligned}
u' &= U^{(1)}(r)\cos\varphi + U^{(2)}(r)\cos 2\varphi + \left(U^{(3)}(r)\cos 3\varphi + U^{(3-1)}\sin\varphi\right) + \ldots + \\
&\quad + \left(U^{(n+1)}\cos(n+1)\varphi + U^{(n-1)}\sin(n-1)\varphi\right) + \ldots \\
v' &= V^{(1)}(r)\sin\varphi + V^{(2)}\sin 2\varphi + \left(V^{(3)}\sin 2\varphi + V^{(3-1)}\cos\varphi\right) + \ldots + \\
&\quad + \left(V^{(n+1)}\sin(n+1)\varphi + V^{(n-1)}\cos(n-1)\varphi\right) + \ldots \\
w^{(1)} &= W^{(1)}(r)\cos\varphi + W^{(2)}(r)\cos 2\varphi + \ldots + \\
&\quad + \left(W^{(n+1)}\cos(n+1)\varphi + W^{(n-1)}\sin(n-1)\varphi\right) + \ldots ,
\end{aligned}
\tag{8}
$$

such that

$$
\frac{1}{r}\frac{d}{dr}\left(rU^{(1)}\right) + \frac{1}{r}V^{(1)} = 0 \tag{9a}
$$

$$
\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - i\sigma\right)\left[\frac{1}{r}\frac{d}{dr}\left(rV^{(1)}\right) + \frac{1}{r}U^{(1)}\right] + G_r\frac{d\Theta^{(0)}}{dr} = 0\ ; \tag{9b}
$$

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dr}\left(rU^{(2)}\right) + \frac{2}{r}V^{(2)} = 0 \\
&\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2} - i\sigma\right)\left[\frac{1}{r}\frac{d}{dr}\left(rV^{(2)}\right) + \frac{2}{r}U^{(2)}\right] + \frac{1}{2}G_r\left(\frac{d\Theta^{(1)}}{dr} - \frac{1}{r}\Theta^{(1)}\right) = 0;
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\frac{1}{r}\frac{d}{dr}\left(rU^{(n+1)}\right) + \frac{n+1}{r}V^{(n+1)} = 0 = \frac{1}{r}\frac{d}{dr}\left(rU^{(n-1)}\right) + \frac{n-1}{r}V^{(n-1)} \\
&\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(n+1)^2}{r^2} - i\sigma\right]\left[\frac{1}{r}\frac{d}{dr}\left(rV^{(n+1)}\right) + \frac{(n+1)}{r}U^{(n+1)}\right] + \\
&\quad + \frac{1}{2}G_r\left(\frac{d\Theta^{(n)}}{dr} - \frac{n}{r}\Theta^{(n)}\right) = 0 \\
&\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(n-1)^2}{r^2} - i\sigma\right]\left[\frac{1}{r}\frac{d}{dr}\left(rV^{(n-1)}\right) + \frac{(n-1)}{r}U^{(n-1)}\right] + \\
&\quad + \frac{1}{2}G_r\left(\frac{d\Theta^{(n)}}{dr} + \frac{n}{r}\Theta^{(n)}\right) = 0\ ; \quad n = 3, 4, \ldots,
\end{aligned}
\tag{11}
$$

$$
Re\,\frac{dw^{(0)}}{dr}U^{(1)} = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} - i\sigma\right)W^{(1)}\ , \tag{12}
$$

$$
Re\,\frac{dw^{(0)}}{dr}U^{(2)} = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2} - i\sigma\right)W^{(2)}\ , \tag{13}
$$

$$
\left.
\begin{aligned}
Re\,\frac{dw^{(0)}}{dr}U^{(n+1)} &= \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(n+1)^2}{r^2} - i\sigma\right]W^{(n+2)} \\
Re\,\frac{dw^{(0)}}{dr}U^{(n-1)} &= \left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(n-1)^2}{r^2} - i\sigma\right]W^{(n-2)}
\end{aligned}
\right\} n = 3, 4, \ldots , \tag{14}
$$

with homogeneous boundary conditions. The mathematical statement of the problem now reduces to the solution of Eqs.(7), (9)–(14).

3. SOLUTION OF EQUATIONS

Eq.(7) is standard, and its solution is given by

$$\Theta^{(n)} = \theta_n \frac{I_n\left[(1+i)\left(\frac{1}{2}\sigma P_r\right)^{1/2} r\right]}{I_n\left[(1+i)\left(\frac{1}{2}\sigma P_r\right)^{1/2}\right]}, \tag{15}$$

in which $I_n(z)$ is the modified Bessel's function of order n and argument z. With these values of $\Theta^{(n)}$, Eqs.(9)–(11) may now be integrated. First of all, we set

$$\zeta^{1/2} = (1+i)\left(\frac{1}{2}\sigma\right)^{1/2},$$

then Eq.(9b) may be integrated once to give

$$\frac{1}{2}\frac{d}{dr}(rV^{(1)}) + \frac{1}{r}U^{(1)} = \sigma \cdot const_1^{(1)}\, I_1(\zeta^{1/2} r) - G_r\,\theta_0 \cdot \frac{P_r^{1/2}}{(P_r-1)} \cdot \frac{I_1(\zeta^{1/2}P_r^{1/2} r)}{I_0(\zeta^{1/2}P_r^{1/2})}.$$

Solving this last equation simultaneously with Eq.(9a), we can show that

$$rU^{(1)} = \frac{G_r\,\theta_0}{i\sigma\, P_r^{1/2}(P_r-1)}\left[const_2^{(1)} \cdot r - const_1^{(1)} \cdot I_1(\zeta^{1/2} r) + \frac{I_1(\zeta^{1/2}P_r^{1/2} r)}{I_0(\zeta^{1/2}P_r^{1/2})}\right]$$

$$rV^{(1)} = -\frac{G_r\,\theta_0}{i\sigma\, P_r^{1/2}(P_r-1)}\left[const_2^{(1)} \cdot r - const_1^{(1)} \cdot \zeta^{1/2} r\, I_1'(\zeta^{1/2} r) + \zeta^{1/2}P_r^{1/2} r \cdot \frac{I_1'(\zeta^{1/2}P_r^{1/2} r)}{I_0(\zeta^{1/2}P_r^{1/2})}\right] \tag{16}$$

such that the constants can be expressed as

$$const_1^{(1)} = \frac{I_1(\zeta^{1/2}P_r^{1/2}) - \zeta^{1/2}P_r^{1/2}\, I_1'(\zeta^{1/2}P_r^{1/2})}{I_0(\zeta^{1/2}P_r^{1/2})\left[I_1(\zeta^{1/2}P_r^{1/2}) - \zeta^{1/2} I_1'(\zeta^{1/2})\right]}$$

$$const_2^{(1)} = \frac{\zeta^{1/2}}{I_0(\zeta^{1/2}P_r^{1/2})} \cdot \frac{\left[I_1(\zeta^{1/2}P_r^{1/2}) I_1'(\zeta^{1/2}) - P_r^{1/2}\, I_1'(\zeta^{1/2}P_r^{1/2}) I_1(\zeta^{1/2})\right]}{I_1(\zeta^{1/2}P_r^{1/2}) - \zeta^{1/2} I_1'(\zeta^{1/2})}. \tag{17}$$

Now the prime on $I_n(z)$ indicates differentiation with respect to the argument z.

Employing the known relations

$$I_{v-1}(z) = I_v'(z) + \frac{v}{z} I_v(z), \quad I_{v+1}(z) = I_v'(z) - \frac{v}{z} I_v(z) \tag{18}$$

the solutions to Eqs.(10) and (11) can be obtained similarly. The results are

$$rU^{(2)} = \frac{1}{2}\,\frac{G_r\,\theta_1}{i\sigma}\,\frac{\cdot 1}{P_r^{1/2}(P_r-1)}\left[const_2^{(2)} \cdot r^2 - const_1^{(2)} \cdot I_2(\zeta^{1/2} r) + \frac{I_2(\zeta^{1/2}P_r^{1/2} r)}{I_1(\zeta^{1/2}P_r^{1/2})}\right]$$

$$rV^{(2)} = -\frac{1}{4}\,\frac{G_r\,\theta_1}{i\sigma}\,\frac{1}{P_r^{1/2}(P_r-1)}\left[2\,const_2^{(2)} \cdot r^2 - const_1^{(2)} \cdot \zeta^{1/2} r\, I_2'(\zeta^{1/2} r) + \right.$$

$$\left. + \zeta^{1/2}P_r^{1/2} r\, \frac{I_2'(\zeta^{1/2}P_r^{1/2} r)}{I_0(\zeta^{1/2}P_r^{1/2})}\right] \tag{19}$$

and

$$rU^{(n\pm1)} = \frac{1}{2}\,\frac{G_r\,\theta_n}{i\sigma}\,\frac{1}{P_r^{1/2}(P_r-1)}\left[const_2^{(n\pm1)}\cdot r^{(n\pm1)} - const_1^{(n\pm1)}\cdot I_{n\pm1}(\zeta^{1/2}r) + \frac{I_{n\pm1}(\zeta^{1/2}P_r^{1/2}r)}{I_n(\zeta^{1/2}P_r^{1/2})}\right]$$

$$rV^{(n\pm1)} = -\frac{1}{4}\,\frac{G_r\,\theta_n}{i\sigma}\,\frac{1}{P_r^{1/2}(P_r-1)}\left[(n\pm1)const_2\cdot r^{n\pm1} - const_1^{(n\pm1)}\cdot\zeta^{1/2}r\,I_{n\pm1}(\zeta^{1/2}r) + \right.$$

$$\left. + \zeta^{1/2}P_r^{1/2}r\,\frac{I'_{n\pm1}(\zeta^{1/2}P_r^{1/2}r)}{I_n(\zeta^{1/2}P_r^{1/2})}\right], \tag{20}$$

where the constants in Eqs.(19) and (20) may be expressed as in Eq.(17).

With the help of Eqs.(16)–(20) we can finally solve for axial velocity components in Eqs.(12)–(14). The procedure is standard and the results are

$$W^{(1)} = -\frac{1}{2}\,K^{(0)}Re\,\frac{G_r\,\theta_0}{i\sigma}\,\frac{1}{P_r^{1/2}(P_r-1)}\left\{\frac{const_2^{(1)}}{i\sigma}\left[\frac{I_1(\zeta^{1/2}r)}{I_1(\zeta^{1/2})} - r\right] + \right.$$

$$+ \frac{1}{2}\,const_1\,I'_1(\zeta^{1/2})\left[\frac{I_1(\zeta^{1/2}r)}{I_1(\zeta^{1/2})} - r\,\frac{I'_1(\zeta^{1/2}r)}{I'_1(\zeta^{1/2})}\right] +$$

$$\left. + \frac{1-i}{(2\sigma)^{1/2}(P_r-1)}\left[\frac{I_1(\zeta^{1/2}P_r^{1/2}r)}{I_0(\zeta^{1/2}P_r^{1/2})} - \frac{I_1(\zeta^{1/2}P_r^{1/2})}{I_0(\zeta^{1/2}P_r^{1/2})}\cdot\frac{I_1(\zeta^{1/2}r)}{I_1(\zeta^{1/2})}\right]\right\}$$

$$W^{(2)} = -\frac{1}{4}\,K^{(0)}Re\,\frac{G_r\,\theta_1}{i\sigma}\,\frac{1}{P_r^{1/2}(P_r-1)}\left\{\frac{const_2^{(2)}}{i\sigma}\left[\frac{I_2(\zeta^{1/2}r)}{I_2(\zeta^{1/2})} - r^2\right] + \right.$$

$$+ \frac{1}{2}\,const_1^{(2)}\,I'_2(\zeta^{1/2})\left[\frac{I_2(\zeta^{1/2}r)}{I_2(\zeta^{1/2})} - r\,\frac{I'_2(\zeta^{1/2}r)}{I'_2(\zeta^{1/2})}\right] +$$

$$\left. + \frac{1-i}{(2\sigma)^{1/2}(P_r-1)}\left[\frac{I_2(\zeta^{1/2}P_r^{1/2}r)}{I_1(\zeta^{1/2}P_r^{1/2})} - \frac{I_2(\zeta^{1/2}P_r^{1/2})}{I_1(\zeta^{1/2}P_r^{1/2})}\cdot\frac{I_2(\zeta^{1/2}r)}{I_2(\zeta^{1/2})}\right]\right\}$$

$$W^{(n\pm1)} = -\frac{1}{4}\,K^{(0)}Re\,\frac{G_r\,\theta_n}{i\sigma}\,\frac{1}{P_r^{1/2}(P_r-1)}\left\{const_2^{(n\pm1)}\left[\frac{I_{n\pm1}(\zeta^{1/2}r)}{I_{n\pm1}(\zeta^{1/2})} - r^{n\pm1}\right] + \right.$$

$$+ \frac{1}{2}const_1^{(n\pm1)}\,I'_{n\pm1}(\zeta^{1/2})\left[\frac{I_{n\pm1}(\zeta^{1/2}r)}{I_{n\pm1}(\zeta^{1/2})} - r\,\frac{I'_{n\pm1}(\zeta^{1/2}r)}{I'_{n\pm1}(\zeta^{1/2})}\right]$$

$$\left. + \frac{1-i}{(2\sigma)^{1/2}(P_r-1)}\left[\frac{I_{n\pm1}(\zeta^{1/2}P_r^{1/2}r)}{I_n(\zeta^{1/2}P_r^{1/2})} - \frac{I_{n\pm1}(\zeta^{1/2}P_r^{1/2})}{I_n(\zeta^{1/2}P_r^{1/2})}\,\frac{I_{n\pm1}(\zeta^{1/2}r)}{I_{n\pm1}(\zeta^{1/2})}\right]\right\} \tag{21}$$

and the solutions are now complete.

4. DISCUSSIONS

In the previous three sections, we have formulated and solved for a model to simulate the effect of ultrasound heating on blood flow. Here we give a quantitative discussion of the results.

Indeed we shall consider only limiting values of the temperature distribution, since the velocity components follow a similar pattern.

Therefore, when $\sigma \to 0$, we can show that Eq.(15) reduces to

$$\Theta^{(n)} \sim \theta_n r^n . \tag{22}$$

If however, $\sigma \to \infty$ we find it necessary to define a variable η which is of order $0(1)$. Thus

$$\left(\frac{1}{2}\sigma\right)^{1/2} (1-r) = \eta ,$$

and r is therefore very close to unity. In this case Eq.(15) becomes

$$\Theta^{(n)} \sim \theta_n \, e^{-(1+i)P_r^{1/2}\eta} . \tag{23}$$

Eq.(22) indicates that the temperature is fairly well distributed throughout the blood vessel between the axis and the wall. This is what one may expect in a low frequency beam which has average penetration thereby allowing the tissue for effective absorption. This is not the case in ultrasound.

When $\sigma \to \infty$, which is appropriate for ultrasound, we find from Eq.(23) that the ultrasonic effect is restricted to a thin boundary layer near the wall and of thickness $\sigma^{-1/2}$. Even though the ultrasound beam decays exponentially, and oscillatorily, from the wall; which decay is rapid since the Prandtl number (P_r) for blood is about 25, prolong application of the ultrasound is likely to cause damage to the walls of blood vessels. Further deductions will be made in Part II where the effects of oscillation of the left ventride is also taken into consideration.

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