Introduction to String Field Theory

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1. INTRODUCTION

There are several possible string field theories corresponding to the different types of strings. One can choose among bosonic, supersymmetric, or heterotic strings, and open or closed strings. In addition one has a choice between a gauge invariant or a gauge fixed formulation of the theory. At the present time, these theories are in various stages of development.

In these lectures I will try to provide an introduction to string field theory. I have chosen to focus on a gauge invariant, bosonic open string field theory. This has the advantage of being general enough to illustrate many of the new features of string field theory, while being specific enough to keep the analysis simple. It is also one of the most developed theories at the moment. Even with this restriction, there are several actions which have been proposed. Much of our general discussion will apply equally well to all candidate theories, however when it is necessary to become more specific, we will focus on an approach due to Witten\textsuperscript{1} and the related purely cubic action\textsuperscript{2}.

In a sentence, string field theory is a second quantized field theory of strings. To understand this better, it is perhaps instructive to compare string theory with ordinary point particle theories. Consider first a massless relativistic particle in flat spacetime. At the classical level, this is described by a worldline $X^\mu(\tau)$ extremizing the action:

$$S = \frac{1}{2} \int d\tau g^{-\frac{1}{2}} \dot{X}^\mu \dot{X}_\mu$$

where $g(\tau)$ is a one dimensional metric. This action is invariant under reparametrizations of $\tau$. The equation of motion (in the gauge $g = \text{constant}$) is simply $\ddot{X}^\mu(\tau) = 0$. The gauge invariance of the action leads to the constraint

$$P_\mu P^\mu = 0$$
where the momentum is \( P_\mu = g^{-\frac{1}{2}} \dot{X}_\mu \). So the solutions are simply null geodesics.

At the first quantized level, the particle is described by a wave function \( \psi \). \( \psi \) is initially a function on the space of configurations of the particle at one moment of \( \tau \) (which is just ordinary spacetime \( X^\mu \)) and an external time \( T \). These wave functions must satisfy Schrödinger's equation:

\[
\frac{\partial \psi}{\partial \tau} = H \psi
\]

and also the operator version of the constraint:

\[
P^2 \psi = -\nabla^2 \psi = 0
\]

But \( H = P^2 \), so the Hamiltonian is constrained to vanish. Thus Schrödinger's equation implies that \( \psi \) is independent of \( T \). This means that there is no explicit time dependence in \( \psi \). Time is already included in \( X^0 \). The constraint is sometimes called the physical state condition. One also has the propagator giving the amplitude for the particle to propagate from one point in spacetime \( X_1^\mu \) to another \( X_2^\mu \) which can be expressed in terms of a path integral (ignoring gauge fixing):

\[
<X_1^\mu | X_2^\mu > = \int \mathcal{D}X^\mu \mathcal{D}g e^{-S[X^\mu, g]}
\]

over all paths starting at \( X_1^\mu \) and ending at \( X_2^\mu \) and over all metrics. (Since \( g \) is not dynamical it does not appear in the specification of states on the left hand side. Its value at the endpoints is integrated over.)

At the second quantized level, the wave functions \( \psi(X^\mu) \) become quantum field operators \( \phi(X^\mu) \) which create or destroy particles. The first quantized constraint equation becomes the second quantized linearized equation of motion:

\[
\nabla^2 \phi = 0
\]
The non-linear terms in the equation of motion describe how several particles interact. They are not determined by the first quantized theory which of course describes a single particle. They are a new physical input which can be added subject to general principles such as renormalizability, gauge invariance, etc.

Now let us repeat the same three levels of description for a relativistic string in a flat spacetime. At the classical level the string is described by a worldsheet $X^\mu(\sigma, \tau)$ extremizing the action:

$$ S = \frac{1}{2} \int d\sigma d\tau \sqrt{-q} q^{ab} \partial_a X^\mu \partial_b X_\mu $$

where $q_{ab}$ is a metric on the worldsheet. This action is invariant under reparameterizations of $\sigma$ and $\tau$ and also conformal rescalings of $q_{ab}$. The equation of motion (in the conformal gauge $q_{ab} \propto \eta_{ab}$) is just the two dimensional wave equation:

$$ \partial^2 X^\mu = 0 $$

The gauge invariance leads to the constraint

$$ T_{ab} = 0 \quad (1.1) $$

where $T_{ab} = \frac{\delta S}{\delta q^{ab}}$.

At the first quantized level, the string is described by a wave function $\psi[X^\mu(\sigma)]$ which is a function on the space of string configurations at one moment of worldsheet time $\tau$. Schroedinger's equation again implies that there is no explicit time dependence. However, $\psi$ must satisfy the constraints:

$$ L_n |\psi\rangle = 0 \quad n > 0 $$

$$ (L_0 - 1) |\psi\rangle = 0 \quad (1.2) $$

where $L_n$ are the Fourier modes of the constraint (1.1) and are called the Virasoro operators. We cannot impose (1.2) for all $n$ because of the well known anomaly
in the Virasoro algebra. The propagator giving the amplitude for a string in one configuration $X_1^\mu(\sigma)$ to propagate to $X_2^\mu(\sigma)$ can be expressed in terms of the functional integral (ignoring gauge fixing)

$$< X_1^\mu(\sigma)|X_2^\mu(\sigma) > = \int D X^\mu Dq_{ab} e^{-S[X^\nu, q_{ab}]}$$

where the integral is over all $X^\mu(\sigma, r)$ which start at $X_1^\mu(\sigma)$ and end at $X_2^\mu(\sigma)$ and over all metrics. This first quantized string is very conveniently described using BRST quantization. As we will see, in this approach one introduces anti-commuting ghost fields $c$ and $b$ and considers wave functions on the extended space $\psi[X^\mu(\sigma), c(\sigma)]$. There is an operator $Q$ called BRST operator which has the convenient property that the infinite set of constraints (1.2) is replaced by the single constraint $Q|\psi > = 0$.

Finally, at the second quantized level, the wave function $\psi$ becomes a quantum string field operator $A[X^\mu(\sigma), c(\sigma)]$ which creates or destroys entire strings. The linearized equation of motion is simply

$$QA = 0.$$ 

As before, one then adds interactions subject to general principles. The resulting theory is string field theory and the subject of these lectures.

Having said what string field theory is, I must now say a few words about why it is needed. The fact is that most researchers in string theory today are not concerned with any of the three levels of string theory described above! Instead they work with a procedure for calculating string scattering amplitudes perturbatively, using only first quantized methods, without the full machinery of string field theory.* Almost all discussions of conformal field theory, Riemann surfaces, and

* This can also be done for point particle theories. However the resulting formalism is more cumbersome than ordinary quantum field theory and so is usually avoided. One difference between this and string theory is that, for the point particle theories, the diagrams involved are not smooth one dimensional manifolds.
vertex operators fall in this category. Since these first quantized methods involve
the beautiful geometry of Riemann surfaces and their moduli space, some people
have suggested that this might be sufficient for a complete theory. However, this
seems unsatisfactory for at least three reasons.

First, this procedure is inherently perturbative: a Riemann surface of genus $g$
corresponds to a $g^{\text{th}}$ order contribution to a scattering amplitude. Although
one could in principle calculate to arbitrary order in perturbation theory, one still
runs into problems. For example, there is no reason to expect the string coupling
constant to be small in nature. In fact, various arguments\textsuperscript{4,5} indicate that the cou-
ing constant is probably of order unity, which makes the perturbation expansion
highly questionable. More seriously, one can show that the perturbation expan-
sion must break down for any value of the coupling constant\textsuperscript{6}. This is because
an exact relation (partial wave unitarity) is violated by the tree level amplitudes
for large values of the external momentum. Thus some higher loop amplitudes
must become large to compensate. Another problem with an inherently pertur-
bative procedure is that string theory may contain qualitatively new effects which
can only be seen non-perturbatively. This is by now a familiar feature of ordi-
nary quantum field theories. But most importantly, the main reason one wants
a quantum theory of gravity is not to calculate graviton scattering perturbatively
in flat spacetime. Rather, it is to gain a better understanding of the structure of
spacetime at short distances, the origin of the universe, and the evaporation of
black holes. In other words, one needs a better description of gravity in strong
field regions where classical general relativity breaks down. These are certainly
non-perturbative regions.

The second reason these perturbative methods fall short of a complete theory
is that they require a choice of background spacetime metric (and possibly other
fields) to perturb about. In the full theory these background fields should be dynamical. Although there is a way of selecting which backgrounds are classically allowed (by requiring conformal or superconformal invariance) there is no way of choosing among these classical alternatives. One needs a more fundamental description to discover solutions to the full quantum equations of motion.

Finally, a more aesthetic objection is the following. One has the strong impression that there is some fundamental principle underlying string theory analogous to the way differential geometry is the basis for general relativity or gauge invariance is the basis for QCD. It is difficult to discover this fundamental principle in a perturbative framework. One needs a more fundamental description.

Of course there is no guarantee that string field theory is the right approach to these questions. We do not yet know the best logical framework for a fundamental description of string theory. However field theory is the most familiar framework and quantum field theory has worked extremely well for theories without gravity e.g. Weinberg-Salam model, QCD, etc. It is therefore natural to try to develop string theory along these same lines. Even if it turns out not to be the ultimate description of string theory (which, personally, I find rather likely), it at least provides a framework in which non-perturbative effects can in principle be investigated. Hopefully, this will lead us closer to the ultimate description. An alternative approach to a fundamental description of string theory has been developed by Friedan and Shenker.

The outline for these lectures is as follows. In Section 2 we discuss the free (non-interacting) string field theory. The interacting theory is developed in Section 3. One of the main tests of a string field theory is that it should reproduce the standard string scattering amplitudes. This is discussed in Section 4. The issue of background dependence and a method for removing it via a purely cubic action is
described in Section 5.
2. FREE STRING FIELD THEORY

Since the BRST operator plays such an important role in constructing a gauge invariant string field theory, we begin this section with a brief summary of BRST quantization. This is a general procedure which can be applied to any theory with a gauge invariance. Here we consider just the first quantized string. Recall that the usual covariant quantization of a string in a flat background leads to the constraints

\[ L_n |\psi > = 0 \quad n > 0 \]
\[ (L_0 - 1) |\psi > = 0 \]  

(2.1)

where \(L_n\) are the Virasoro operators. These operators generate conformal reparameterizations of the worldsheet. In the BRST approach, one introduces a nilpotent operator \(Q\) which generates the same transformation on \(X^\mu\) but now with an anticommuting gauge parameter. Since the original action \(S_0\) is gauge invariant it commutes with \(Q\). One can now obtain a BRST invariant gauge fixing term \(S_1\) by taking essentially any non-gauge invariant expression and anticommuting with \(Q\). The total action \(S_0 + S_1\) has no local gauge invariance but is equivalent to \(S_0\) on the physical subspace consisting of states satisfying \(Q |\psi > = 0\). Rather than describe this gauge fixing in detail, let me just illustrate the relation between the familiar constraints (2.1) and \(Q |\psi > = 0\).

To start, we introduce a pair of anticommuting ghost fields \(c^a(\sigma, \tau)\) and \(b_{ab}(\sigma, \tau)\). The first is a worldsheet vector and the second is a symmetric traceless tensor. (These fields can be thought of as Fadeev Papov ghosts introduced in the two dimensional functional integral.) Even though they each have two components, the open string boundary conditions result in one set of modes \(c_n\) and \(b_{n}\) for all \(n\).
which satisfy $c_n^\dagger = c_{-n}$, $b_n^\dagger = b_{-n}$ and

$$\{c_m, b_n\} = \delta_{m+n,0}$$

$$\{c_m, c_n\} = \{b_m, b_n\} = 0$$ \hspace{1cm} (2.2)

The BRST operator $Q$ is given explicitly by

$$Q = \sum_{n=-\infty}^{\infty} (L_n - \delta_{n,0})c_{-n} - \frac{1}{2} \sum_{m,n=-\infty}^{\infty} (m - n) : c_{-m}c_{-n}b_{m+n} :$$ \hspace{1cm} (2.3)

The pure ghost piece has been chosen so that $Q^2 = 0$. (This turns out to require 26 spacetime dimensions). Notice that it is a sum of terms, each of which contains at least one annihilation operator. So if $|\psi\rangle$ has no ghost excitations, then $Q|\psi\rangle = 0$ implies

$$\sum_{n=0}^{\infty} (L_n - \delta_{n,0})c_{-n}|\psi\rangle = 0$$

Since $c_{-n}|\psi\rangle$ are linearly independent, we recover eq. (2.1). So for states with no ghost excitations, the single constraint $Q|\psi\rangle = 0$ is equivalent to the infinite set of constraints (2.1). What about states with ghost excitations? Since $Q^2 = 0$, any state of the form $|\psi\rangle = Q|\epsilon\rangle$ satisfies the constraint. One can show that if you quotient out by the states of this form, then the resulting space $\frac{\text{Ker}Q}{\text{Im}Q}$ is in one-to-one correspondence with the light cone gauge states of the string$^{10,11,12}$. In other words, by taking this quotient one removes not only the states with ghost excitations but also the spurious (null) states which remain in the usual covariant quantization. Notice that the quotient of the kernel of $Q$ by the image of $Q$ is exactly the form of a cohomology group. So in this approach the physical states of the string are in one-to-one correspondence with cohomology classes of $Q$.\* 

\* In the literature one often sees this statement qualified by adding a restriction on the "ghost number" of the states. (This will be defined shortly.) However if the boundary conditions on $\epsilon$ are weak enough, one can show that the only nontrivial cohomology of $Q$ is at one value of the ghost number$^{11}$. 


Mathematically, one can ask what space is $Q$ measuring the cohomology of. The answer is that $Q$ is not measuring the usual cohomology of a particular manifold. One can also define the cohomology of a Lie algebra. This depends on a choice of representation. In the trivial representation, this cohomology agrees with that of the Lie group, but in general it is different. It turns out that $Q$ is measuring the cohomology of the Virasoro algebra in the Fock space representation\textsuperscript{12}.

Why is the BRST approach such a convenient starting point for string field theory? It is certainly not needed for a gauge fixed theory. Starting from the light cone gauge first quantized string, one can construct a field theory of just the physical degrees of freedom without including ghosts\textsuperscript{13}. However if one tries to construct a gauge invariant theory starting from the covariant constraints (2.1) then one is led to an action whose kinetic energy term is nonlocal. To understand this, recall that a single string describes an infinite number of pointlike particles. Correspondingly, a single string field $A$ on the infinite dimensional space of string configurations is equivalent to an infinite number of ordinary fields on spacetime with increasing mass and spin. These are called the component fields. Now it is rather difficult to write down gauge invariant equations for massive fields. From the string field theory viewpoint it turns out that a natural way to obtain a gauge invariant action is to introduce a certain projection operator. This unfortunately gives a nonlocal kinetic energy term to many of the component fields. This nonlocality can be removed by the addition of extra component fields called Stueckelberg fields which are not dynamical. The structure of these fields was worked out by Banks and Peskin\textsuperscript{14}. The advantage of the BRST approach is that these extra fields are automatically included as coefficients of the ghost oscillators as we will see shortly. The BRST approach actually provides more than the minimal number of extra fields required for spacetime locality. However the field equation and the
gauge transformation take such a simple form in the BRST approach that it is easier to work with this larger space of fields.

Before discussing the field equation, let me describe in detail how one expands a string field $A[X(\sigma), c(\sigma)]$ in terms of an infinite number of component fields on spacetime. Consider first the case in which $A$ is a classical string field i.e. simply a function on the space of string configurations. Then $A$ can be viewed as a wavefunction for the first quantized string and hence as a state in the first quantized Fock space. It can therefore be expanded in the usual Fock basis of creation operators acting on the vacuum. Let $\alpha_n^\mu$ be the creation and annihilation operators for $X^\mu$, and $c_n, b_n$ be the corresponding operators for the ghosts. These satisfy (2.2) and

$$[\alpha_n^\mu, \alpha_n^\nu] = m\delta_{m+n,0} \eta^{\mu\nu}$$

Since the ghost zero modes satisfy $\{e_0, b_0\} = 1$, there exists a two dimensional space annihilated by all the positive $n$ oscillators. Let $\downarrow$ denote the state annihilated by $b_0$ i.e. $b_0|\downarrow> = 0$. Then $c_0|\downarrow>\equiv |\uparrow>$ is linearly independent. Clearly $|\downarrow> = b_0|\uparrow>$ and so $<\downarrow |\downarrow> = <\uparrow | b_0 b_0 |\uparrow> = 0$. Similarly $<\uparrow |\downarrow> = 0$. The non-zero inner product is $<\downarrow |\uparrow> = <\uparrow |\downarrow> = 1$.

It is convenient to define an operator called the ghost number operator. Roughly speaking, it counts the number of $c$ excitations minus the number of $b$ excitations in a state. More precisely,

$$N_G = \sum_{n=1}^{\infty} (c_{-n}b_n - b_{-n}c_n) + \frac{1}{2}(c_0 b_0 - b_0 c_0)$$

(2.4)

Since $|\uparrow> = c_0|\downarrow>$, the ghost number of $|\uparrow>$ must be one more than $|\downarrow>$. In (2.4), the ghost number of these vacuum states is assigned symmetrically so $|\downarrow>$

* Note that although indices will be suppressed, both components of $c^a(\sigma)$ appear as arguments of the string field. One can also choose a representation in which $A$ is a function of $b^a_{\dot{b}}(\sigma)$, or even one component of $c^a(\sigma)$ and an (anti)commuting component of $b^a_{ab}(\sigma)$. 
has ghost number $-\frac{1}{2}$ and $\mid\uparrow\rangle$ has ghost number $+\frac{1}{2}$. It is convenient to restrict our string field $A$ to have ghost number $-\frac{1}{2}$. This is possible because $-\frac{1}{2}$ is the only ghost number for which $Q$ has nontrivial cohomology. Thus a string field with this ghost number includes all the physical component fields.

Since we want the component fields to be real, the string field $A$ must satisfy a reality condition. At this stage several different conditions could be adopted. However with a view toward the interacting theory developed in the next section, we now require\(^{\dagger}\)

$$A^*[X(\sigma), e(\sigma)] = A[X(\pi - \sigma), -e(\pi - \sigma)]$$

(2.5)

where * denotes complex conjugation.

We can now show the standard expansion of $A$ into its component fields. Rather than working with the zero mode momentum $\alpha_0 = P$, it is convenient to retain the dependence on the zero mode coordinate $x_0$. Then a general string field can be expanded:

$$\mid A \rangle = \mid \varphi(x_0) + iA_{\mu}(x_0)\alpha^\mu_{-1} + B(x_0)b_{-1}e_0 + \cdots \mid \downarrow \rangle$$

where we have included all terms in the first two levels. Note that $b_{-1}$ requires $e_0$ in order that $A$ have ghost number $-\frac{1}{2}$. (We do not include a term proportional to $c_{-1}b_0$ since $b_0 \mid \downarrow \rangle = 0$.) The coefficients in the above expansion $\varphi, A_{\mu}, B, \text{etc.}$, are ordinary (scalar, vector, and tensor) fields on spacetime. They are the component fields of $A$. The $i$ is needed so that the reality condition (2.5) is satisfied for real component fields. For a classical string field, the component fields are classical.

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* There is nontrivial cohomology at $+\frac{1}{2}$ also if one restricts $e$ to be bounded in time.
† This condition is appropriate for the theory without Chan-Paton factors. Otherwise $A$ carries two group indices and complex conjugation also transposes the indices.
fields on spacetime. A quantum string field can be expanded in exactly the same way except now the component fields become quantum field operators.

It is important not to confuse the Fock space of the first and second quantized string. There is, of course, an enormous difference. In the string field theory, a Fock space of states can be constructed for each of the component fields and the full Fock space is obtained by taking their (infinite) tensor product. The first quantized states in the above expansion simply provide a convenient basis in which to expand the string field.

Let me now digress a moment to mention a philosophical point. The above procedure seems to yield a unique decomposition of $A$ into component fields. But string theory is supposed to be a unified theory of all these fields. One does not unify a theory simply by taking a collection of objects, giving it a new name, and writing down an equation for it! A truly unified theory is one in which the decomposition of the fundamental field into component fields depends on extra input e.g. a choice of gauge, a choice of observer, etc. For example, Maxwell unified electricity and magnetism by showing that if one observer sees a purely electric field, then an observer moving with respect to the first will see a combination of electric and magnetic fields. In string theory, the extra input is a choice of basis for functions on string space. It is convenient to choose a basis which diagonalizes the kinetic energy operator. (In principle this should be possible even for non-trivial backgrounds.) However, other choices of basis functions could be chosen and the component fields would change. The invariant object is the string field itself.

We now turn to the linearized field equation satisfied by $A$. As mentioned in the previous section, this is the same as the first quantized physical state condition:

$$Q_A = 0$$  \hspace{1cm} (2.6)
Recall that an important property of $Q$ is that it is nilpotent: $Q^2 = 0$. It thus follows that if $A_1$ is any solution to (2.6), and $\varepsilon$ is any string field, then $A_2 = A_1 + Q\varepsilon$ is also a solution. This is interpreted as the linearized gauge invariance of string field theory. We now have the following result:

**Claim:**  
1. $QA = 0$ corresponds to an infinite set of gauge invariant linear field equations for the component fields which includes Maxwell's equations.

2. $\delta A = Q\varepsilon$ corresponds to the associated spacetime gauge transformations.

It is important to note that this is a highly non-trivial result. The BRST operator $Q$ was originally obtained by considering the reparameterization invariance on the two dimensional worldsheet. The fact that the same operator yields spacetime gauge transformations is a truly remarkable fact. This is just one of many indications of the subtle interplay between spacetime and worldsheet geometry in string theory.

I now wish to verify the above claim explicitly for the case of the low mass fields. The expression for the BRST operator in modes is given in (2.3). Recall that the Virasoro operators are

$$L_n = \sum_{m=-\infty}^{\infty} \frac{1}{2} :\alpha_{n-m} : \alpha_m :$$

Note that $Q$ preserves level number, i.e. it is the sum of terms with equal numbers of creation and annihilation operators. This leads to the important simplification that we do not have to consider all the component fields simultaneously. $QA = 0$ implies that $Q$ must annihilate each level separately.

To begin, consider just the $\varphi$ term in the expansion of $A$. Since there are no ghost creation operators, the second term in $Q$ vanishes. The first term also vanishes except for $n = 0$. Since $\alpha_0^\mu = P^\mu = \frac{1}{i} \nabla^\mu$ we find that $QA = 0$ implies

$$\nabla^2 \varphi + 2\varphi = 0$$
which is just the equation for the tachyonic field.

Acting on the $A_\mu$ term we again find that the second term in $Q$ vanishes. We thus obtain

$$Q_i A_\mu \alpha^\mu_{-1} | \downarrow > = \frac{i}{2} P^2 A_\mu \alpha^\mu_{-1} c_0 + \alpha_0 \cdot \alpha_{-1} c_0 | A_\mu \alpha^\mu_{-1} | \downarrow >$$

$$= \frac{i}{2} P^2 A_\mu \alpha^\mu_{-1} c_0 + i P_\mu A^\mu c_{-1} | \downarrow >$$

(2.7)

We must add to this $Q$ acting on the $B$ term (which has the same level)

$$Q B b_{-1} c_0 | \downarrow > = [\alpha_{-1} \cdot \alpha_0 c_1 - 2 c_{-1} c_1 b_0] B b_{-1} c_0 | \downarrow >$$

$$= [P_\mu B c^\mu_{-1} c_0 + 2 B c_{-1}] | \downarrow >$$

(2.8)

Adding (2.7) and (2.8) we obtain a sum of the linearly independent states $\alpha^\mu_{-1} c_0 | \downarrow >$ and $c_{-1} | \downarrow >$. Requiring that the coefficients of these independent terms vanish yields the two equations

$$\frac{i}{2} P^2 A_\mu + P_\mu B = 0$$

$$i P_\mu A^\mu + 2 B = 0$$

The second equation implies $B = -\frac{i}{2} P_\mu A^\mu$ which when substituted into the first yields

$$\nabla^\mu \nabla_{[\mu} A_{\nu]} = 0$$

This is of course the familiar source-free Maxwell equations! So we have recovered Maxwell's equations from the strange looking equation $Q A = 0$. Note that $B$ is not a dynamical field. It is an auxiliary field which is needed for gauge invariance.

Let us now consider the string field gauge transformation $\delta A = Q e$. Since $A$ has ghost number $-\frac{1}{2}$ and $Q$ increases ghost number by one (it has one more $c$ oscillator than $b$), the gauge parameters $e$ must have ghost number $-\frac{3}{2}$. The simplest state with this ghost number is

$$e = -\lambda(x_0) b_{-1} | \downarrow >$$
(The minus sign is for convenience.*) Note that this is already a level one state. Since there is no state with ghost number $-\frac{3}{2}$ and level zero, there is no gauge invariance for the tachyon, as expected. Applying $Q$ to $\epsilon$ we obtain

$$Q\epsilon = -[(\frac{1}{2}a_0^2 + b_{-1}c_1 - 1)e_0 + \alpha_{-1}\alpha_0c_1]\lambda b_{-1}|\downarrow>$$

$$= -[\frac{1}{2}P^2\lambda e_0b_{-1} + P_\mu\lambda \alpha_{-1}^\mu]|\downarrow>$$

Comparing with the expansion for $A$ and equating coefficients of linearly independent terms yields

$$\delta A_\mu = \nabla_\mu \lambda$$

and

$$\delta B = -\frac{1}{2}\nabla^2 \lambda$$

The first is of course just the usual Maxwell gauge invariance. The second is determined by the earlier equation for $B$ in terms of $A_\mu$.

Knowing that the linearized equation of motion for the string field is $QA = 0$, the quadratic action must have the general form

$$S = \int D\sigma Dc(\sigma)AQA$$

Of course this infinite dimensional integral must be suitably defined. Using the identification of the classical string field with a state in the first quantized Hilbert space, we can give the precise definition:

$$S = <A|Q|A>$$

Since $Q$ is a hermitean operator, $Q^\dagger = Q$, we find that $\frac{\delta S}{\delta A} = 0$ implies $QA = 0$.

How does one fix the linearized gauge invariance of string field theory? It turns out that a suitable gauge choice is $b_0A = 0$. Since $b_0|\downarrow> = 0$, this has the effect

* In order for $A$ to satisfy the reality condition (2.5), $\epsilon$ must satisfy a similar condition but with an overall minus sign on the right hand side. Hence $\lambda$ is real and no $i$ is required.
of setting all terms with $c_0$ in the expansion of $A$ to zero. At the massless level, this condition is equivalent to $B = 0$ which implies $\nabla_\mu A^\mu = 0$. In other words, $b_0 A = 0$ is the string analog of the Lorentz gauge.

To show that $b_0 A = 0$ is a good gauge fixing condition for the string field, one would like to prove that there are no gauge transformations $\varepsilon$ which preserve this condition. Unfortunately, this is not the case. Any $\varepsilon$ of the form $Q\eta$ where $\eta$ has ghost number $-\frac{5}{2}$ does not change $A$ at all. In other words, there is a gauge invariance of the gauge invariance! These transformations cannot be fixed by any condition on $A$ but require a gauge condition on $\varepsilon$. We thus impose $b_0\varepsilon = 0$. But since $\eta$ can be $Q$ on some field of ghost number $-\frac{7}{2}$ this structure clearly continues indefinitely. Thus the gauge condition must be imposed at each ghost number.*

We conclude this section with the following table which summarizes the relationship between Maxwell theory and free string field theory:

<table>
<thead>
<tr>
<th>Gauge Field</th>
<th>Maxwell Theory</th>
<th>Free String Field Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field equation</td>
<td>$\nabla_\mu F_{\mu\nu} = 0$</td>
<td>$QA = 0$</td>
</tr>
<tr>
<td>Gauge transformation</td>
<td>$\delta A_\mu = \nabla_\mu \lambda$</td>
<td>$\delta A = Q\varepsilon$</td>
</tr>
<tr>
<td>Lorentz gauge</td>
<td>$\nabla_\mu A^\mu = 0$</td>
<td>$b_0 A = 0$</td>
</tr>
</tbody>
</table>

This table can be interpreted in two ways. Maxwell theory provides both an analog for the full free string field theory, and the precise content of its massless level.

* In the quantum theory, this gauge fixing is accompanied by ghost string fields which have ghost number different from $-\frac{1}{2}$. Their component fields are the usual Fadeev Papov ghost fields associated with the component fields of $A$. This hierarchical gauge invariance leads to a "ghosts for ghosts" mechanism. The net result is that the final gauge fixed action involves a string field which is not restricted to have ghost number $-\frac{1}{2}$ but contains all ghost numbers.
3. INTERACTING STRING FIELD THEORY

In the previous section we described the free string field theory. We now want to include interactions. This corresponds to a new physical input and cannot be derived from the first quantized theory of a single string. Recall that the basic assumption is that strings interact by simply splitting and joining. The worldsheets are thus smooth two dimensional manifolds and higher order quantum corrections correspond to higher genus two-manifolds. It has sometimes been suggested that this is the only possible interaction of strings or that this is the only possibility consistent with spacetime locality. This may be the case, but it is far from obvious. Naively, one might consider fundamental four string interactions of the type shown in Fig. 1. Since the interaction occurs at one point, this is also local in spacetime. It is possible that a string theory based on interactions of this type (or more complicated generalizations) suffers from various types of inconsistencies and the only viable string interaction is indeed the standard one. But this has not yet been proven. At the present time the smooth worldsheet picture is adopted because of its simplicity and because it predicts the observed couplings of massless fields.

![Diagram of a string interaction](image)

Fig. 1 A string interaction which is local in spacetime but different from the standard interaction (see Fig. 2).
We must now find a string field theory interaction which reproduces the smooth worldsheets for perturbative scattering. Since all interactions can be built up from a basic one involving three strings (see Fig. 2), one expects a simple cubic term in the action. However the usual pointwise multiplication of three string fields $A^3$ gives the wrong interaction. As an operator, $(A[X(\sigma)])^3$ would annihilate one string and create two more in exactly the same configuration. The worldsheet would then look like Fig. 3. We thus obtain the important result that a local interaction on string space corresponds to a non-local interaction in spacetime. To obtain the usual string interaction one needs a non-local product on string space. All of the proposed string field theory actions involve such a product.

Fig. 2 (left) The standard three string interaction. Fig. 3 (right) A local product in string space does not reproduce smooth worldsheets.

As we will discuss in the next section, to calculate string scattering amplitudes perturbatively from the field theory, one considers string Feynman diagrams. These are worldsheets created from propagators and vertices. This corresponds to worldsheets with a preferred time slicing. Since the worldsheets in the Polyakov

* This expectation must be checked by verifying that the single cubic interaction reproduces the standard scattering amplitudes. For example, in the light cone gauge string field theory, one finds that additional quartic interactions are required$^{13,15}$. 

approach do not have this extra structure, one must make some arbitrary choices. The different string field theory interactions which have been proposed are essentially the result of different choices of slicing the worldsheet. One possibility is shown in Fig. 4. This is a covariant generalization of the light cone gauge interaction. The resulting field theory has been extensively investigated\textsuperscript{16}. A more symmetric slicing was proposed by Witten and is shown in Fig. 5. In the following, I will concentrate on the field theory resulting from this slicing\textsuperscript{1}. I will divide my discussion of Witten's string field theory into two parts. In the first half of this section I will describe the basic operations ignoring the ghosts. In the second half, the ghost dependence will be discussed. We will see that the ghosts play an important role in the structure of the theory.

Fig. 4 (left) Horizontal time slices of this worldsheet gives an interaction similar to the light cone gauge interaction. Fig. 5 (right) A more symmetrical slicing proposed by Witten.

The first step is to define a (non-local) product on string fields which captures the joining of strings pictured in Fig. 5. Recall that the string field $A$ is a function on the space of parameterized strings $X^\mu(\sigma) \in \sigma \in [0, \pi]$. Consider each string as consisting of a left half $X^\mu_L(\sigma) \in \sigma \in [0, \frac{\pi}{2}]$ and a right half $X^\mu_R(\sigma) \in \sigma \in [\frac{\pi}{2}, \pi]$. Then
the interaction corresponds to the right half of one string matching up with the left half of another string. The remaining two half strings combine to form the third string. This leads us to define the following product on string fields

$$(A \ast B)(X_L, X_R) \equiv \int DYA[X_L, Y]B[\bar{Y}, X_R]$$

(3.1)

where the $\sigma$ dependence of $X_L, X_R$, and $Y$ is suppressed. $Y(\sigma)$ is a right half string defined for $\sigma \in [\frac{\pi}{2}, \pi]$ and $\bar{Y}(\sigma) = Y(\pi - \sigma)$ is a left half string. Notice that this product is analogous to infinite dimensional matrix multiplication with $X_L$ and $X_R$ labelling the rows and columns. Thus it is not commutative, but it is (formally) associative.*

The decomposition of the string into a right half and left half is invariant under reparameterizations of the string that leave the midpoint fixed. This gives rise to a set of global symmetries (usually denoted $K_n$) of the theory we are constructing. However it is clearly not invariant under reparameterizations that move the midpoint. Thus the star product defined above is not invariant under all reparameterizations. This is perhaps unappealing, but it is not fatal. The original motivation for a reparameterization invariant action for the first quantized string was not just based on aesthetic reasons. There are negative norm states arising from oscillations of $X^0$ in the original Hilbert space. If the action was not reparameterization invariant, there would be no constraints to project out these unphysical states. In the field theory, we have fixed reparameterization invariance, but we will retain a second quantized gauge invariance which will achieve the same purpose of allowing us to remove (gauge away) negative norm states. It remains an interesting question whether or not there exists a more reparameterization invariant string field theory. If so, then Witten's theory may be recovered as a

* It turns out that associativity fails for some (large) string fields which are outside the open string Fock space\textsuperscript{17}.
partially gauge fixed version of this more fundamental theory\textsuperscript{18,19}. (Note that the obvious attempt to generalize Witten’s interaction by choosing a point other than the midpoint yields an interaction which is not symmetric among the three strings.)

Using the star product, one would like the interaction to be \( A \ast A \ast A \). But what do we mean by \( \int \)? The star product of three strings is shown in Fig. 6. In order to recover Witten’s interaction which is cyclically symmetric, the integral must sew the last two ends of the string together. So for a general string field we define

\[
\int A \equiv \int \mathcal{D}Y A[\bar{Y}, Y]
\]

(3.2)

This is analogous to taking the trace in the infinite dimensional matrix interpretation.

\[\text{Fig. 6 The star product of three strings.}\]

In terms of \( \ast \) and \( \int \), the usual first quantized inner product between two states \( A \) and \( B \) can be expressed as:

\[
\langle A|B \rangle = \int \mathcal{D}X(\sigma) A^*[X(\sigma)]B[X(\sigma)]
\]

\[
= \int \mathcal{D}X(\sigma) A[X(\pi - \sigma)]B[X(\sigma)]
\]

\[
= \int \mathcal{D}X_L \mathcal{D}X_R A[\bar{X}_R, \bar{X}_L]B[X_L, X_R]
\]

\[
= \int A \ast B
\]
where we have used the reality condition (2.5). Note that this can be interpreted as saying that the infinite dimensional matrix $A[X_L, X_R]$ is hermitean. In particular $< A|B >$ is always real. Thus the free action described in the last section can be re-expressed as

$$S_{\text{free}} = \int A \ast QA$$

and the full action is

$$S = \int (A \ast QA + \frac{2}{3} g A \ast A \ast A)$$

(3.3)

where $g$ is a coupling constant. The factor of $\frac{2}{3}$ is for convenience. We will see shortly that $S$ is invariant under the non-linear gauge transformation

$$\delta A = Qe + g(A \ast e - e \ast A)$$

(3.4)

This should look familiar. We saw in the free theory that $A$ was analogous to a vector potential. If one thinks of $Q$ as an exterior derivative and $\ast$ as a wedge product, then (3.4) is analogous to the usual non-abelian gauge invariance of Yang-Mills theory and $S$ is analogous to the integral of the three dimensional Chern-Simons term. The equation of motion is

$$F = QA + g A \ast A = 0$$

(3.5)

This would be trivial in Yang-Mills theory, but it is not trivial here since $Q$ is actually quadratic in spacetime derivatives as we saw in the previous section. It is natural to ask why we did not use the usual Yang-Mills type action: $\int F^2$. The point is that the only product we have on string fields is the star product which is analogous to the wedge product on forms. Thus $\hat{S} \equiv \int F \ast F$ is analogous to the topological invariant $\int F_{\mu\nu} F_{\rho\sigma}$. One can show that $\frac{\delta \hat{S}}{\delta A} = 0$ for all field configurations. We will soon see that the analog of Chern-Simon terms of other
dimensions automatically vanish (due to ghost number) and hence do not provide viable actions.

The main ingredient left out of the above discussion has been the ghosts. In fact an important role is played by the ghost number \( N_G \) and its violation. Recall that \( N_G \) is roughly a measure of the number of \( c \) excitations minus the number of \( b \) excitations and is given explicitly in eq. (2.4).

There is a very general topological argument which shows that the standard string interaction must increase ghost number by \( \frac{3}{2} \). This is not specific to Witten's choice of slicing the worldsheet and applies to any interaction which reproduces the smooth worldsheet picture. If one calculates the divergence of the ghost number current \( j_b = c^a b_{ab} \), then one finds in general an anomalous non-zero answer. Integrating over the worldsheet one can show that

\[
\Delta N_G = -3\chi
\]

where \( \chi \) is the Euler number of the worldsheet. Now for a two manifold \( M \) with boundary the complete expression for the Euler number is

\[
\chi = \frac{1}{4\pi} \left[ \int_M R \sqrt{g} \, d^2\sigma + 2 \int_{\partial M} K ds + 2 \sum_{\text{corners}} (\pi - \theta_j) \right] \quad (3.6)
\]

where \( R \) is the scalar curvature of the worldsheet, \( K \) is the extrinsic curvature of the boundary away from the corners, and \( \theta_j \) is the interior angle of the \( j^{th} \) corner. This last term can be thought of as the result of \( \delta \)-function contributions in the extrinsic curvature. Before I apply this formula to string interactions, let me illustrate it with the following example.

Consider a unit sphere which has been cut along the equator. Then for the top half, \( R = 2, K = 0 \) and there are no corners, so \( \chi = 1 \) which is correct for the hemisphere. But this is a topological invariant and does not change if we
continuously deform the surface. Of course its contributions from different terms in (3.6) will change. If we flatten the hemisphere to a disk, then $R = 0, K = 1$, and there are no corners so again $\chi = 1$. Finally, if we straighten out the edges to a square, $R = 0, K = 0, \theta_j = \frac{\pi}{2}$ which again yields $\chi = 1$.

Now the worldsheet of a single string is a long rectangle. Like the square it has $\chi = 1$. According to the above formula, a contribution of $\frac{1}{2}$ comes from the two corners on top and the other $\frac{1}{2}$ from the two corners on the bottom. If we associate these contributions to the ghost number of the asymptotic states, then we see that the interior has no contribution to $\chi$. So we conclude that there is no violation of ghost number for a single string. Now consider the standard three string interaction (Fig. 2). This is topologically equivalent to a rectangle and so again has $\chi = 1$. As before the asymptotic regions each contribute $\frac{1}{2}$ to the Euler number which is associated with the ghost number of the asymptotic states. So the interaction region must contribute $-\frac{1}{2}$ to $\chi$. Since $\Delta N_g = -3\chi$ we discover that the standard string interaction violates ghost number by $\frac{3}{2}$. Notice that this result is purely topological and is independent of exactly how we draw the interaction region, how we slice the worldsheet, etc.

We now have to incorporate this violation of ghost number into our definition of $\ast$ and $\int$. In eq. (3.1) we defined $\ast$ in terms of a path integral over half strings. However, it is more convenient and more precise to define $\ast$ as the limit of a two dimensional functional integral. To avoid complicated edge effects from corners which would result in a complicated propagator, Witten straightens out all edges of the worldsheet keeping the asymptotic corners at 90°. This is not possible with a metric that is everywhere flat. However if one allows a $\delta$-function curvature

* Since the Euler number is a global property of a manifold, it usually is not possible to divide it up into local contributions. Nevertheless the argument given above seems justified since the different contributions in eq. (3.6) are so widely separated.
singularity at one point \( p \) (corresponding to the common midpoint of the three strings) of strength \( \mathcal{R} = -2\pi \delta(p) \), then it is possible (Fig. 7). To define the star product, we take a thin strip of width \( \delta \) including \( p \) and set

\[
(A \ast B)(X, c) = \lim_{\delta \to 0} \int DXDcDb e^{-S_{AB}}
\]

where \( X(\sigma) \) and \( c(\sigma) \) are fixed on the final boundary. On the initial two boundaries they are integrated over weighted by the string fields \( A \) and \( B \). What we have shown is that under the star product the ghost number is not conserved but satisfies:

\[
N_G(A \ast B) = N_G(A) + N_G(B) + \frac{3}{2}
\]

Fig. 7 The worldsheet for Witten's interaction has a curvature singularity at the point \( p \).

We now consider integration. Instead of eq. (3.2), to be more precise we again define it as the limit of a two dimensional functional integral. Consider a worldsheet of length \( \delta \) and identify \( X_L \) with \( X_R \) at one end. We define

\[
\int A = \lim_{\delta \to 0} \int DXDcDb e^{-S_{A}}
\]

There is again a curvature singularity at the midpoint \( p \) of strength \( \mathcal{R} = 2\pi \delta(p) \), so \( \chi = \frac{1}{2} \) and \( \Delta N_G = -\frac{3}{2} \) for integration. Since the functional integral is zero
unless the total ghost number adds up to zero, we conclude that if $A$ has ghost number different from $\frac{3}{2}$

$$\int A = 0$$

The structure of this string field theory bears a remarkable similarity to ordinary forms on a three dimensional manifold. To make this more explicit, it is convenient to think of string fields as forms with rank $= N_C + \frac{3}{2}$. This is consistent with the interpretation of $\ast$ as a wedge product since the star product of an $m$ form and an $n$ form has ghost number

$$(m - \frac{3}{2}) + (n - \frac{3}{2}) + \frac{3}{2} = m + n - \frac{3}{2}$$

and so is an $m + n$ form. It is also consistent with the interpretation of $Q$ as an exterior derivative since $Q$ increases ghost number and hence the rank of the form by one. Recall that gauge parameters have ghost number $-\frac{3}{2}$ and hence are zero forms. The physical string field has ghost number $-\frac{1}{2}$ and hence is a one form. Since ghost number measures the number of ghost oscillators, and the ghosts anticommute, the star algebra is graded. The even elements have $N_C = -\frac{3}{2} + 2n$ and the odd elements have $N_C = -\frac{1}{2} + 2n$.

The fact that the integral vanishes unless $A$ has ghost number $\frac{3}{2}$ and hence is a three form is also analogous to forms on a three manifold. Notice that since our physical string fields have ghost number $-\frac{1}{2}$, the ghost number of $A \ast QA$ is $-\frac{1}{2} + \frac{3}{2} + 1 - \frac{1}{2} = \frac{3}{2}$ and the ghost number of $A \ast A \ast A = -\frac{3}{2} + 3 = \frac{3}{2}$. So the three dimensional Chern-Simons action does not automatically vanish. But higher dimensional Chern-Simons terms would. In this sense, the action is unique.

It is natural to ask what is so special about the number 3. How did this "threeness" enter string field theory? The answer can be traced back to the fact
that the ghost $c^a$ satisfies the equation for a conformal Killing field and there are three conformal Killing fields on the disk.

One property of forms that is not shared by the star algebra is (graded) commutativity. The wedge product of forms commutes up to a sign. But there is no simple relation between $A \star B$ and $B \star A$. Inside the integral, however, the star product is commutative:

\[ \int A \star B = \int B \star A \] (3.9)

This is a simple consequence of eqs. (3.1) and (3.2). Note that this equation holds for all string fields with no restrictions on ghost number or reality. (However the integral is equal to the Fock space inner product only if the reality condition is satisfied.)

We now discuss two properties of $Q$, $\star$, and $\int$ which further illustrate the analogy with forms and exterior derivatives. Let $A$ and $B$ denote string fields of arbitrary ghost number. Then:

\[ \int QA = 0 \] (3.10)

\[ Q(A \star B) = QA \star B + (-1)^A A \star QB \] (3.11)

where $(-1)^A$ is $-1$ if $A$ is Grassmann odd and $+1$ if it is Grassmann even. These two properties can be established by the following argument. Let me draw the worldsheet used in the definition of the integral schematically as in Fig. 8 where the curvature singularity is denoted by a dot. There are two types of boundaries. The short side has width $\delta$ and has standard open string boundary conditions. The long side has boundary conditions where $X^\mu(\sigma)$ and $c(\sigma)$ are fixed. In the

* This is sometimes written with an extra minus sign depending on the ghost number of $A$ and $B$. However it is easy to check that this sign is $+1$ whenever the integral is non-zero. This is also true for forms on a three manifold.
functional integral one integrates over these boundary conditions weighted by the string field $A$.

Fig. 8 (left) A schematic picture of the worldsheet used for integration. Fig. 9 (right) A similar picture of the worldsheet used for the star product.

Now $Q$ is the integral of a BRST current

$$j_a = c^b(T^X_{ab} + \frac{1}{2}T^G_{ab})$$

where $T^X_{ab}$ and $T^G_{ab}$ are the stress energy tensors for the $X$ field and the ghost fields. (A total derivative term which does not affect $Q$ has been dropped.) This current is conserved even on a curved worldsheet. Therefore the integral of $j_a$ over any closed contour must vanish. Consider the contour denoted by the dashed line in Fig. 8. On the short side $j_\sigma = 0$ by the open string boundary conditions. On the long side one obtains $\int j_r = Q$ acting on the state $A$. Since this must vanish for any finite width $\delta$, we conclude

$$\int QA = 0$$

for all string fields $A$.

Similarly, let me draw the worldsheet for the star product schematically as in Fig. 9 where a dot denotes the curvature singularity at the midpoint of the three
strings. The short sides again have width $\delta$ and open string boundary conditions.

If we integrate the BRST current over the closed contour denoted by the dashed line, then we must get zero. But this is just the sum of three terms involving $QA, QB,$ and $Q(A \ast B)$. Since the sum must vanish, we obtain eq. (3.11). (The relative signs are determined by Grassmanianity)

Using the properties (3.9 - 3.11), it is easy to verify that the action (3.3) is invariant under the non-linear gauge invariance (3.4).

Recall that we originally described the free action in terms of the first quantized expectation value of the BRST operator:

$$S_{\text{free}} = \langle A | Q | A \rangle$$

and later showed this was equivalent to

$$S_{\text{free}} = \int A \ast QA$$

Since oscillators are better defined than functional integrals, it is natural to ask if one can go the other way. Rather than expressing the free action in terms of $\ast$ and $\int$, can one express the interaction $\int A \ast A \ast A$ in terms of oscillators? The answer is yes, and the precise form was derived independently by several groups\textsuperscript{20,21,22,23}. Consider, more generally, $\int A \ast B \ast C$. We need a map which takes three states into a number. In other words, we need an element of the tensor product of the three string Hilbert spaces. Let us denote this element by $|V_3\rangle$. Then

$$\int A \ast B \ast C = \langle A | \langle B | < C | V_3 \rangle$$

The state $|V_3\rangle$ takes the following form. Since it must have total ghost number $\frac{3}{2}$ we can write it as

$$|V_3\rangle = V | \uparrow \rangle_1 | \uparrow \rangle_2 | \uparrow \rangle_3$$
where \( | \uparrow >_r \) is the up vacuum with ghost number \( +\frac{1}{2} \) in the \( r^{th} \) string Hilbert space. The operator \( V \) has ghost number zero and is constructed from the creation operators \( \alpha^r_m, \ r = 1, 2, 3 \) associated with the three strings. More precisely:

\[
V = \exp \left( \sum_{r,s=1}^{3} \sum_{m,n=0}^{\infty} \alpha^r_m \alpha^s_n N^{rs}_{mn} + \text{ghosts} \right)
\]

The coefficients \( N^{rs}_{mn} \) are the Fourier coefficients of the Neumann function on the worldsheet corresponding to the scattering of three strings. This Neumann function can be calculated using conformal mapping techniques.

This oscillator expression contains the term \( e^{-\sum_{r=1}^{3} K'(p'^2)} \) for some positive constant \( K \). Since this is not polynomial in spacetime derivatives, it can be viewed as a nonlocal interaction. Thus, even though the vertex was designed to reproduce the smooth worldsheet picture, it still gives rise to a non-local interaction in terms of component fields! This can perhaps be viewed as resulting from the fact that the component fields are functions of the zero mode coordinate, and Witten's three string vertex does it not require these coordinates to agree\(^24\). The effects of this nonlocality have recently been investigated\(^25\).

The star product of two fields can also be expressed in terms of \( |V_3> \)

\[
|A \ast B >_3 =_1 < A|_2 < B||V_3>
\]

Note that the star product does not preserve level number. The star product of two tachyons is a state containing all component fields. Physically this is expected because there are interactions among all component fields.

Recall that in the free theory, restricting the field equation to the massless level yields precisely Maxwell's equation. In the interacting theory, restricting the field equation (3.5) to the massless level does not yield the Yang-Mills equation (even when Chan-Paton factors are included). This can be seen in two ways. First, since
the action is just the Chern-Simons term, there is no cubic interaction in the field equation. Second, as we have just remarked, restricting the field equation to the massless level still yields an equation coupling all the component fields together. From the standpoint of string theory, the Yang-Mills equation is an approximate equation which is valid only at low energy.

One can also give an explicit expression for the interaction vertex using two dimensional conformal field theory. In this approach one works directly with the Neumann functions rather than their Fourier coefficients. In this formalism, it is easier to prove local operator identities such as the overlap equations

\[ [X^{(r)}(\sigma) - X^{(r+1)}(\sigma)]|V_3| = 0 \]
\[ [P^{(r)}(\sigma) + P^{(r+1)}(\sigma)]|V_3| = 0 \]
\[ [c^{(r)}(\sigma) + c^{(r+1)}(\sigma)]|V_3| = 0 \]
\[ [b^{(r)}(\sigma) - b^{(r+1)}(\sigma)]|V_3| = 0 \]

where \( \sigma \in [\frac{\pi}{2}, \pi] \), and \( r \) labels the three strings (\( r = 4 \) and \( r = 1 \) are identified). These equations express how the strings join at the vertex and can be used as an alternative definition of \( |V_3\rangle \). Similar equations hold for certain composite operators such as the stress energy tensor and the BRST current. Having established these local overlap equations, properties such as (3.11) follow immediately from summing over \( r \) and integrating over \( \sigma \).*

* For some composite operators there are subtleties associated with divergences near the string midpoint.
4. SCATTERING AMPLITUDES

At the very least, a string field theory should be able to reproduce the perturbative scattering amplitudes calculated from first quantized methods. In this section I will describe briefly how this comes about.

To calculate scattering amplitudes perturbatively using Feynman diagrams, one must first fix the gauge. Otherwise the kinetic energy operator cannot be inverted to yield a propagator. Recall that in the free theory we said a possible gauge choice was $b_0 A = 0$. This implies that

$$|A >= \tilde{A}| \downarrow >= \tilde{A} b_0 | \uparrow >$$

where $\tilde{A}$ has no ghost zero mode oscillators. In this gauge, the quadratic part of the action becomes

$$< A|Q|A >= <\uparrow | \tilde{A} b_0 Q b_0 \tilde{A}| \uparrow >$$

But $\{Q, b_0\} = L_0^X - 1 + L_0^G \equiv \Delta$ and $\Delta$ is just the worldsheet Hamiltonian. Since

$$<\uparrow | b_0 | \uparrow > = 1$$

we have

$$< A|Q|A >= <\uparrow | \tilde{A} b_0 \Delta \tilde{A}| \uparrow >$$

$$= <\tilde{A}|\Delta|\tilde{A} >$$

where the last matrix element is taken in a reduced Hilbert space without the ghost zero modes. Now $\Delta$ can be inverted as usual

$$\frac{1}{\Delta} = \int_0^\infty dr e^{-r\Delta}$$

Since $\Delta$ is the Hamiltonian, $e^{-r\Delta}$ just propagates a string freely for time $r$. This builds up a flat worldsheet of width $\pi$ and length $r$. It is reasonable to conjecture that the Feynman rules for the full theory are simply to tie these flat worldsheets
together three at a time using Witten's vertex. To establish this conjecture one must first fix the gauge in the full interacting theory. This is a difficult problem. Although considerable progress has been made, I do not think the issue has been completely resolved. However most workers in the field seem to agree that these Feynman rules probably do follow from Witten's string field theory. We will now assume this and investigate the scattering amplitudes that result.

Fig. 10 The worldsheet for calculating the three string coupling.

To calculate a three string coupling $A + B \rightarrow C$, one considers the worldsheet shown schematically in Fig. 10 where the legs are infinitely long and $A, B,$ and $C$ are the wave functions of the asymptotic states. The metric on this worldsheet is flat except for a curvature singularity at the midpoint of the three strings. The three string coupling is obtained by doing a functional integral over the $X$ and ghost fields living on this worldsheet. But this is precisely the Polyakov prescription for a particular choice of metric on the worldsheet. To put this in the more standard form, one can conformally map the worldsheet to a disk. The asymptotic states are then mapped to points on the boundary of the disk and give rise to the usual vertex operators. Thus we see that gauge fixing the string field theory leads to a gauge fixed Polyakov approach. So the string field theory yields the right three string couplings. This is not surprising. The vertex was essentially
constructed to do this.

Now consider the scattering $A + B \rightarrow C + D$. As in any quantum field theory, we must sum together several Feynman diagrams as shown in Fig. 11. The length of the internal propagators is $\tau$ and must be integrated from 0 to $\infty$. This illustrates one of the main drawbacks of string field theory compared to the Polyakov approach for calculating string scattering. In the Polyakov approach, a given scattering process involves just one Riemann surface at each order of the loop expansion. String field theory breaks this up into many diagrams. For tree level scattering the situation is not so bad but for higher loops it gets much worse. There are roughly $n!$ Feynman diagrams for a single $n$ loop amplitude.

![Feynman diagrams](image)

**Fig. 11** The $s$, $t$, and $u$ channel string Feynman diagrams.

The main question now is whether the scattering amplitudes calculated in string field theory by summing these Feynman diagrams agree with those calcu-
lated in the Polyakov approach. Recall that in the Polyakov approach the amplitude is expressed in terms of an integral over moduli space, i.e. the space of conformal metrics on a given Riemann surface:

\[ A_{\text{Poly.}} = \int_{\text{moduli space}} \left( \ldots \right) \]

In string field theory, the amplitude is expressed

\[ A_{\text{SFT}} = \sum_{\text{diagrams}} \int \prod_i d\tau_i \left( \ldots \right) \]

where \( \tau_i \) are Schwinger parameters for the \( i^{th} \) propagator. So the question of whether these two amplitudes agree is really two questions: Do the integrands agree? Do the regions of integration agree? Giddings and Martinec have investigated the first question\(^{27}\). By doing a careful analysis of the Polyakov integrand (in particular the role of ghosts) they were able to cast it in the form that would be obtained from the Feynman diagrams. Thus (modulo the conjecture that the Feynman rules can be derived from the field theory) the integrands do, in fact, agree.

The second question is really the question of whether string field theory gives a cell decomposition of moduli space\(^ {30}\). This means the following. String Feynman diagrams are two dimensional surfaces. Changing the length of the internal string keeping everything else fixed changes the conformal metric on the surface. Thus, the \( \tau_i \) are like local coordinates on moduli space. Since one independently integrates each \( \tau_i \) from 0 to \( \infty \), each Feynman diagram with \( n \) internal propagators corresponds to a region of moduli space which is topologically an \( n \)-cube, and called a cell. However moduli space is topologically very complicated. One can show\(^ {31}\) that for large genus \( g \), the Euler number of the moduli space \( \mathcal{M}_g \) of a Riemann surface of genus \( g \) grows like

\[ \chi(\mathcal{M}_g) \sim \frac{2(-1)^g(2g - 1)!}{(2g - 2)!} \]
So a single Feynman diagram cannot cover all of moduli space. But we do not expect it to. The real question is if you add together all the Feynman diagrams at a given loop order, do you cover all of moduli space or are there pieces left out.

Since the cells are disjoint open sets, they can never completely cover a connected manifold like moduli space. There are points in moduli space corresponding to \( \tau = 0 \) which are on the common boundary of two cells (Fig. 12). (Points corresponding to \( \tau = \infty \) are on the boundary of the entire moduli space.) We wish to know whether the Feynman diagrams - including the limiting cases where one or more \( \tau_i \)'s vanish - cover moduli space.

Let me reformulate the question one more time before answering it. String field theory Feynman diagrams correspond to a certain class of Riemann surfaces. Namely, those which can be obtained by taking flat strips and gluing them together with \( \delta \)-function curvature singularities at the midpoints. Now consider an arbitrary two manifold \( M \) with metric \( g_{ab} \). Since on shell scattering amplitudes are independent of conformal rescalings of the metric, we can always make \( g_{ab} \) flat locally. Now we ask: Can one globally rescale \( g_{ab} \) so that it can be represented by

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Fig. 12 A worldsheet with \( \tau = 0 \) can be obtained as the limit of two different Feynman diagrams.
a string field theory Feynman diagram? At first sight this seems highly unlikely. But we have\textsuperscript{32}:

\textit{Remarkable Theorem:} The answer is yes (provided \( M \) has at least one boundary).

So there is in fact a one-to-one correspondence between Feynman diagrams and Riemann surfaces (with at least one boundary). Thus the regions of integration agree and (modulo the question of gauge fixing) the string field theory amplitudes agree with Polyakov's approach to \textit{all orders in perturbation theory}.

What does the condition on the boundary mean? We want to construct the manifold \( M \) out of flat rectangular strips which certainly have a boundary. We cannot identify the edges of the strips, but can only join them three at a time using the Feynman rules. So there is no hope of obtaining a manifold without a boundary. The remarkable fact is that one can obtain everything else. Physically, this has the following interpretation. We have been talking mainly about open strings. However, it is well known that a consistent interacting theory of open strings must include closed strings. What this theorem says is that the open string Feynman diagrams we have been discussing automatically include all closed string contributions to open string scattering!

Let me illustrate this with the following example. Consider the one closed string loop correction to the open string vacuum diagram Fig. 13. This can be represented as a Feynman diagram with three propagators and two vertices Fig. 14. To see that these two diagrams are topologically equivalent, note that Fig. 14 has only one boundary, is orientable (has two distinct sides) and has an Euler number coming from the two curvature singularities of \(-\frac{1}{2} - \frac{1}{2} = -1\). This is the correct Euler number for the punctured torus. The moduli space for the torus is two dimensional. If one cuts out a disk, one adds one more dimension to the
moduli space corresponding to the radius of the disk. (The center of the disk can be moved using symmetries of the torus and hence does not contribute to the dimension of moduli space.) Now the three propagators in Fig. 14 come with the three \( r \) integrals so one maps out an open subset of moduli space. Other diagrams with the propagators joined in different ways cover the rest of moduli space.

Fig. 13 (left) The closed string one loop correction to the open string vacuum diagram. Fig. 14 (right) The same surface represented as a Feynman diagram.

The proof that Feynman diagrams cover all of moduli space is based on the following idea. It is convenient to have a simpler description of the Feynman diagrams. Since the metric is flat except for isolated curvature singularities, we can realize it as a flat space with identifications. More precisely, consider any Feynman diagram with one boundary. Now cut each propagator along the midline. Since this never cuts the boundary, the diagram remains connected and becomes a simple cylinder. The bottom circle is the original boundary and the top circle can be identified to recreate the original surface. (If there was more than one boundary initially, then one obtains several cylinders after cutting along the midlines.)
cylinder, with identifications of its upper boundary, is called the *canonical presentation* of the Feynman diagram. Mathematicians have shown that every Riemann surface has a *unique canonical presentation*\(^{32}\). Hence it can be identified with a unique Feynman diagram.

Since this string field theory includes closed string intermediate states but apparently only open string external states one might worry that unitarity is violated. This is not the case\(^{33}\). The closed string states can be viewed as bound states of open strings. In ordinary field theory, bound states do not arise at any finite order of perturbation theory. But in string field theory this is possible due to the infinite number of degrees of freedom. It is natural to ask if one can describe closed string external states in this theory. (One does not want to explicitly add a closed string field since this would result in double counting closed string contributions to open string scattering.) The answer is yes, but the details are still being investigated. One approach is to extract the closed string external state by factorizing the one loop amplitude on a closed string pole\(^ {33,34}\). This has the unfortunate feature that to calculate the tree level coupling of three closed string states, one must calculate a three loop diagram! Another approach uses the ideas described in the next section to obtain closed string states, but a modified interaction term must be introduced to obtain the correct tree level scattering\(^ {35}\).

We conclude this section with the following observation. We noted earlier that one of the drawbacks of string field theory over the Polyakov approach for computing string scattering is that one must sum together many Feynman diagrams for each order in perturbation theory. However from a practical point of view this may be a blessing in disguise. As we have said, the moduli space for higher genus Riemann surfaces is topologically quite complicated. It turns out to be difficult to find a concrete characterization of this space which is needed to perform integrals
over it. String field theory breaks up this space into simple building blocks which may be easier to work with.
5. BACKGROUND DEPENDENCE

The string field theory we have been describing has many desirable features, but in its current formulation it has one main drawback. The BRST operator $Q$ which appears in the action depends on a flat 26 dimensional spacetime metric $\eta_{\mu\nu}$. Explicitly

$$Q = \int_0^x c^a (T_a^X + \frac{1}{2} T_a^G) d\sigma$$

and

$$T_{ab}^X = \frac{1}{2} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} - \text{trace}$$

This would not be a problem in a theory of open strings only since the metric would not be dynamical. However, as we saw in the last section, Witten's field theory includes closed strings, and closed strings contain gravity. Thus this string field theory should include general relativity in an appropriate limit. But in general relativity the spacetime metric is not fundamentally split into a flat kinematical background and a dynamical perturbation. Furthermore, since string theory is a unified theory of gravity with an infinite number of other fields, we do not expect to be able to even define a spacetime metric in general. The situation is perhaps analogous to a simple Kaluza-Klein theory. From a general solution (which is a Ricci flat $D$ dimensional spacetime) one cannot always extract a four dimensional metric. This is possible only for particular "vacuum" solutions e.g. those taking a product form. Similarly, in string theory, the spacetime should arise only in a low energy classical approximation.

The question thus arises: Can one reformulate the theory so that the fundamental action is independent of a spacetime metric? The metric could then arise from classical solutions to the field equation of the theory. The answer is yes.
and I would now like to explain how this can be achieved.* (A similar result has
also been obtained in a different string field theory.)

Consider the action

\[ \int \Phi \ast \Phi \ast \Phi \]  

(5.1)

for a string field \( \Phi \) with ghost number \(-\frac{1}{2}\). Notice that this action has no explicit
metric dependence. The equation of motion is simply

\[ \Phi \ast \Phi = 0 \]  

(5.2)

One solution is clearly \( \Phi = 0 \). If this were the only solution, then this action would
not be very interesting. However it turns out that there is a large class of solutions
to this equation. For example, let \( I \) denote the identity of the star algebra i.e.
\( I \ast A = A \ast I = A \) for all \( A \). (Roughly speaking, \( I \) is a delta function which equates
the left and right halves of a string.) Let \( j_a \) denote the BRST current for a string
in a 26 dimensional flat spacetime and set

\[ Q_L = \int_0^{\frac{\pi}{2}} j_0(\sigma) d\sigma \]

Then it turns out that \( Q_0 = Q_L I \) is a solution to (5.2).

To show this, we use the following three properties:

\[ QI = 0 \]  

(5.3)

\[ \{Q, Q_L\} = 0 \]  

(5.4)

\[ Q_R A \ast B + (-1)^A A \ast Q_L B = 0 \]  

(5.5)

* There are subtleties involving violations of associativity and surface terms which I am going to ignore in the following discussion. A more careful analysis leads to the same results with at most minor changes.
In the last property, \( Q_R = Q - Q_L \), \( A \) and \( B \) are arbitrary string fields, and 
\(( -1 )^A = -1 \) if \( A \) has ghost number \(-\frac{1}{2} + 2n\) and \(+1\) otherwise. These properties have been verified explicitly using both the oscillator representation and conformal field theory. Now from (5.5)

\[
Q_L I \ast Q_L I = Q_R Q_L I \tag{5.6}
\]

However using (5.3) and then (5.5) we have

\[
Q_L I \ast Q_L I = -Q_R I \ast Q_L I = Q_L^2 I \tag{5.7}
\]

So

\[
Q_L I \ast Q_L I = \frac{1}{2} QQ_L I = \frac{1}{2}(Q, Q_L) I = 0 \tag{5.8}
\]

This solution has the important property that

\[
\Phi_0 \ast A - (-1)^A A \ast \Phi_0 = QA \tag{5.9}
\]

for an arbitrary string field \( A \). This is easily derived from the three properties above. Now if we expand the string field \( \Phi \) about this solution

\[
\Phi = \Phi_0 + A
\]

and substitute into the field eq. (5.2) we obtain

\[
0 = (\Phi_0 + A) \ast (\Phi_0 + A) = \Phi_0 \ast A + A \ast \Phi_0 + A \ast A
\]

\[
= QA + A \ast A
\]

which is precisely Witten's field equation. Similarly, if we substitute this expansion into the purely cubic action (5.1), we recover Witten's action (3.3). So one

* Strictly speaking, eq.(5.5) has been verified only for \( A \) and \( B \) in the open string Fock space, i.e. fields expressed in terms of a finite number of oscillators. To apply it to \( I \), which involves an infinite number of oscillators, additional regularization is required. In one approach, eqs.(5.6) and (5.7) are not valid but (5.8) is. In another regularization, all equations are true as written.

† We have absorbed the coupling constant \( g \) by rescaling \( A \).
can obtain Witten's theory by simply expanding the purely cubic action about one of its solutions. In effect, one has shifted away the kinetic energy term in Witten's action.

Although we have removed the explicit background dependence in the action, there is still the possibility that there is implicit or anomalous background dependence in the operations $\ast$ and $\int$. For finite width $\delta$, the functional integral expressions (3.7) and (3.8) require an action $S$ which depends on background fields. Formally this dependence is removed in the limit as $\delta \to 0$. Recently, a lattice version of Witten's theory has been constructed in which one can show explicitly that there is no background dependence\(^{40}\). This issue has also been investigated in the continuum theory\(^{24}\).

There is also the question of the dependence on the spacetime topology. If one considers only continuous strings, then the domain of the string field clearly depends on the topology of the spacetime manifold. However there are arguments which suggest that discontinuous strings should also be included. In this case, one can (formally) show\(^{41}\) that the domain of the string field, and hence the cubic action, is essentially independent of the topology of the spacetime.

Since the star product is commutative inside the integral, the purely cubic action is clearly invariant under the gauge transformation

$$\delta \Phi = \Phi \ast \varepsilon - \varepsilon \ast \Phi$$  \hspace{1cm} (5.10)

where $\varepsilon$ is an arbitrary string field of ghost number $-\frac{3}{2}$. If one again expands $\Phi = \Phi_0 + A$ and uses (5.9), this reduces to

$$\delta A = Q\varepsilon + A \ast \varepsilon - \varepsilon \ast A$$  \hspace{1cm} (5.11)

which is just Witten's gauge transformation. Note however that (5.10) is homogeneous in $\Phi$ whereas (5.11) is not homogeneous in $A$. This has the following
important consequence. The natural ground state of the cubic action $\Phi = 0$ is invariant under all gauge transformations $\epsilon$. In particular, if $\epsilon$ has $P_\mu = 0$, this transformation corresponds to a global spacetime symmetry. Thus $\Phi = 0$ is the symmetric ground state. The solution $\Phi_0 = QL I$ is not invariant under all the symmetries, but only those satisfying

$$\Phi_0 \ast \epsilon - \epsilon \ast \Phi_0 = Q\epsilon = 0$$

Thus $\Phi_0$ is a state of broken symmetry. Now in Witten's formulation, the natural ground state $A = 0$ corresponds to the solution $\Phi_0 = QL I$ and hence is not the state of maximum symmetry. This is reflected by the inhomogeneous term in the transformation law for $A$.

There are other solutions to $\Phi \ast \Phi = 0$ besides the one described above. Consider a curved spacetime with possibly other background fields on it. Consider a single string propagating in this background and construct the first quantized BRST operator $Q^B$. If $(Q^B)^2 = 0$, then one can argue that the three properties (5.3 - 5.5) still hold for this new BRST operator and hence $Q^B L I$ is a solution to (5.2). One can show that $(Q^B)^2 = 0$ is equivalent to the condition for the two dimensional sigma model describing propagation of the string to be conformally invariant. Thus we see that for each conformally invariant sigma model, one can construct a solution to the string field equation. Using first quantized methods, it was argued that conformally invariant sigma models correspond to classically allowed background configurations. But there is no way in the first quantized framework to relate one background to another. It is satisfying to have a single equation whose solutions include all of the previously derived classical background configurations.

To lowest order in sigma model perturbation theory (which physically corre-
sponds to the low energy limit), the condition on the metric for $(Q^B)^2 = 0$ is simply Einstein’s equation. This is perhaps the easiest way to recover general relativity from string theory.

Although we have described a large class of solutions to $\Phi \ast \Phi = 0$, these are not the only ones. All of the above solutions have a well defined spacetime metric and possibly other background fields. However, as we have argued, since string theory is a unified theory of all these fields, there should exist solutions for which there is no well defined metric. Some simple examples of such solutions have recently been found\textsuperscript{43}. By studying solutions of $\Phi \ast \Phi = 0$ that have no spacetime interpretation, we may begin to understand how string theory describes the origin of the universe.
POSTSCRIPT

With the exception of some minor changes and updated references, these lecture notes are identical to the ones I wrote a year and a half ago for lectures I gave at Trieste\textsuperscript{44} and the U.S. summer school TASI\textsuperscript{45}. This is due to the fact that the subject of string field theory has not evolved very much during this time, and the progress that has been made has been rather technical and not easily discussed in pedagogical lectures.

Since a superstring field theory is presumably of most physical interest, let me conclude by summarizing its current status. To begin, a light cone gauge superstring field theory has been constructed\textsuperscript{46}. The BRST approach described in Section 2 generalizes to yield a gauge invariant free superstring field theory\textsuperscript{47}. There is also a natural generalization of the interacting theory discussed in Section 3 for superstrings\textsuperscript{48}. The interaction term has been explicitly expressed in terms of first quantized oscillators\textsuperscript{49}. However some of the scattering amplitudes do not reproduce the first quantized calculations\textsuperscript{50}. Possible modifications are still being investigated. A purely cubic action for the superstring field theory (analogous to that described in Section 5) has also been derived\textsuperscript{51,52}. A heterotic string field theory has been constructed only in the light cone gauge\textsuperscript{53}. In order to make this theory gauge invariant, one needs a better understanding of the gauge invariant, bosonic, purely closed string field theory.

Surprisingly, there does not seem to be a simple extension of these ideas to the closed bosonic string. There are two obvious candidates for the interaction term. One is obtained by using the open string interaction for left and right movers separately. The other is obtained by dividing the closed string into a left and right half and requiring half strings to overlap\textsuperscript{54}. These turn out to be equivalent\textsuperscript{55}. But unfortunately, this interaction does not reproduce the Polyakov tree level
closed string scattering amplitudes. An intriguing approach to closed string field theory using only open string fields has been developed by Strominger. Other approaches are also being pursued.

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