NIKHEF

NATIONAAL INSTITUUT VOOR KERNFYSICA EN HOGE-ENERGIEFYSICA

NIKHEF-H/91-05

Relativistic Time Dilation in an External Field

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March 25, 1991

Abstract

We demonstrate that relativistic time dilation is a dynamical effect: it depends on interactions. As a result, even stationary particles may experience time dilation, for example in an external field. This happens in particular when there are extra degrees of freedom attached to a particle like spin and a magnetic moment. We illustrate this phenomenon with examples from QED and QCD.

*Research supported by the Stichting F.O.M.

1 Introduction

It is often suggested, that time dilation in special relativity is a purely kinematic effect, meaning that one can always perform a Lorentz transformation to one's local inertial co-ordinate system such that the proper time of the physical system to be studied (e.g. a particle) coincides with the laboratory time. This suggestion is incorrect: time dilation is a dynamical effect [1], and only for the most simple systems (scalar particles which are perfectly spherically symmetric in the instantaneous rest system) can time dilation always be removed by changing one's inertial co-ordinate system [2]. This is particularly clear for particles with spin, which carry an amount of intrinsic rotation not removable by any Lorentz transformation.

For such particles space-time transformations other than the Lorentz tranformations can be relevant in describing the intrinsic properties of the particle with respect to an observer in an inertial laboratory frame of reference, especially in the presence of external fields. Let us emphasize, that this statement does *not* imply, that the Lorentz covariance of the laws of particle dynamics is violated. On the contrary, Lorentz covariance of the dynamical equations is *necessary* to guarantee that observers in different inertial systems can always compare their results consistently by applying the Lorentz-Poincaré transformation mapping one system of co-ordinates to the other [2, 3]. However one should keep in mind that, since not all particles are rotation-free in the rest system, the Lorentz-Poincaré transformations cannot always be used to pass from the particle's intrinsic (possibly non-inertial) reference frame to an arbitrary inertial system, at least not in the presence of external fields.

It is the purpose of the present paper to show that spinning particles in special relativity indeed have non-standard time-transformation properties: in an external field the transformation between proper time and laboratory time is not always a Lorentz transformation[1]. In order to show this it is not

[1]As a historical note, it is amusing that in the paper published in 1902 Lorentz originally introduced a somewhat more general kind of transformation [1]; from those transformations he then deduced in the paper of 1904 the standard Lorentz transformations by appealing to his equation of motion for a charged point particle. It was Poincaré who coined the name Lorentz transformations for this restricted set forming the group SO(3,1) [3].

necessary to introduce new laws of physics. In fact, in this paper we only use well-established theories like classical and quantum electro-dynamics to prove our conjecture, which is easily confirmed by studying simple systems such as a particle in a constant magnetic field. The ultimate check on the results we establish would of course be a direct experimental test. Although the effects we predict are generally rather small, we believe the possibility of such experimental tests is not entirely academic.

This paper is organised as follows. In sect. 2 we summarize a number of well-known results form classical and quantum electrodynamics which are used in this paper. We express the classical time dilation effect in terms of the energy rather than the velocity of the particle, leading to the conclusion that time dilation is indeed a dynamical rather than a kinematical effect. This is illustrated by a discussion of various possible relativistic particle dynamics. In sect. 3 we show how the behaviour of classical spinning particles may be understood from the simple example of a rotating spinless charged particle in a constant magnetic field. In sect. 4 we then analyze in full detail the quantum theory of both spinless and spinning particles in a magnetic field, and derive the result anticipated from the earlier example, discussing in particular the contribution of spin. In sect. 5 we rederive the time-dilation formula describing the decay of a charged particle in an external field directly from the quantum theory without recourse to our earlier classical result. We compute the order of magnitude of the time-dilation effects expected and finally summarize the conclusions reached in sect. 6.

Throughout this paper we use Pauli metric $diag(+,+,+,+)$, with imaginary time components of four-vectors and tensors, so as to avoid having to distinguish between upper and lower indices. The Einstein summation convention is used for repeated indices as usual. Finally we choose natural units such that $c = 1, \hbar = 1$.

2 Preliminaries

A main theme of this paper is the behaviour of relativistic charged particles in external fields, in particular (but not exclusively) electromagnetic fields. The classical Lagrangian for a particle with mass M and charge q in a field described by the vector potential A_μ is of the form

$$L = \frac{1}{2} M \dot{x}_\mu^2 + q A_\mu(x) \dot{x}_\mu + \Delta L(x,\theta). \tag{1}$$

Here $x_\mu(\tau)$ are the wordline co-ordinates of the particle parametrized by the proper time τ, the overdot denotes a derivative with respect to τ, and $\Delta L(x,\theta)$ represents terms in the Lagrangian which are a function of the wordline co-ordinates and any other dynamical variables (denoted collectively by θ) which are necessary to describe the particle's observable properties.

The canonical momentum of the particle as derived from a Lagrangian of the form (1) is

$$p_\mu = M \dot{x}_\mu + q A_\mu. \tag{2}$$

Now the Lagrangian (1) is invariant modulo a total derivative under electromagnetic gauge transformations $A_\mu \to A_\mu + \partial_\mu \Lambda$, with $\Lambda(x)$ an arbitrary scalar function of the space-time co-ordinates. Clearly, the canonical momentum (2) is not invariant under such a gauge transformation. Therefore we introduce a gauge invariant momentum

$$\pi_\mu = p_\mu - q A_\mu. \tag{3}$$

In our classical model this is just the kinetic part of the momentum:

$$\pi_\mu = M \dot{x}_\mu. \tag{4}$$

In quantum mechanics, the canonical momentum is the operator which satisfies the standard commutation relations

$$[x_\mu, p_\nu] = i\,\delta_{\mu\nu}, \qquad [p_\mu, p_\nu] = 0. \tag{5}$$

In the co-ordinate representation this implies that p_μ is defined by $-i$ times the ordinary partial derivative ∂_μ. In contrast, the gauge invariant momentum π_μ becomes $-i$ times the *covariant* derivative $D_\mu = \partial_\mu - iqA_\mu$, satisfying non-standard commutation relations

$$[\pi_\mu, \pi_\nu] = iqF_{\mu\nu}, \tag{6}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the usual electromagnetic field strength tensor.

Taking the time component of the canonical momentum now gives a relation between intervals dt of laboratory time and the corresponding interval $d\tau$ of proper time,

$$dt = d\tau\,\frac{E - q\phi}{M}, \tag{7}$$

where E is the canonical energy and ϕ the scalar potential.

This relation is very general, and is the fundamental one we employ. It is quite clear from this formula, that any quantity which contributes to the energy E in an observable way[2], also contributes to the time dilation and hence time dilation is a *dynamical* effect, rather than a *kinematic* one.

To illustrate this point, let us consider two possible dynamical situations, both involving a charged point particle and both realizable entirely within the frame work of special relativity. First we assume that the particle is spinless and satisfies the gauge invariant energy-momentum relation

$$\pi_\mu^2 + M^2 = 0, \tag{8}$$

defining the particle's mass shell. In terms of the velocity and canonical momentum components this relation implies

$$\begin{aligned} \vec{p} - q\vec{A} &= \frac{M\vec{v}}{\sqrt{1-\vec{v}^{\,2}}}, \\ E - q\phi &= \frac{M}{\sqrt{1-\vec{v}^{\,2}}}. \end{aligned} \tag{9}$$

From the last line we read off the relation between laboratory and proper time:

$$dt = \frac{d\tau}{\sqrt{1-\vec{v}^{\,2}}}. \tag{10}$$

These results are all standard.

Now consider the alternative situation of a particle with electric and magnetic dipole moments $(\vec{d},\vec{\mu})$, represented covariantly by an anti-symmetric tensor $D_{\mu\nu}$ [4] with

[2]By an observable contribution we mean a term in E which is not compensated by an equal contribution to $q\phi$.

$$D_{ij} = \varepsilon_{ijk}\,\mu_k, \qquad D_{i4} = id_i, \qquad (i,j,k) = 1,2,3. \tag{11}$$

A Lorentz and gauge invariant energy-momentum relation is [5]

$$\pi_\mu^2 + M^2 - MD\cdot F = 0, \tag{12}$$

where $D \cdot F = D_{\mu\nu}F_{\mu\nu}$. This is the kind of energy-momentum relation one expects from the Dirac equation for a charged spin-1/2 fermion. Now the components of the canonical four-momentum can be expressed in terms of the velocity as

$$\begin{aligned} \vec{p} - q\vec{A} &= M\vec{v}\left(\frac{1 - D\cdot F/M}{1-\vec{v}^2}\right)^{1/2}, \\ E - q\phi &= M\left(\frac{1 - D\cdot F/M}{1-\vec{v}^2}\right)^{1/2}, \end{aligned} \tag{13}$$

Then we obtain for the relation between laboratory and proper time

$$dt = d\tau\left(\frac{1 - D\cdot F/M}{1-\vec{v}^2}\right)^{1/2}. \tag{14}$$

According to this last result, time dilation takes place even for a particle at rest in a non-zero external field. It shows clearly the importance of the expression (7) for the classical time dilation effect, and at the same time the restricted validity of eq.(10), which describes only the kinematical contribution to the effect. Of course, it remains to investigate to what extent alternative energy-momentum relations like (12) decribe the behaviour of real particles. This question is addressed in the following sections. Here we just note the possibility and consistency of such models, and the resulting change in the time dilation formula. Obviously, terms in the energy representing other physical quantities than the electromagnetic dipole moments can also contribute to the time dilation effect. However, in this paper we consider the dipole moment as a prime example of the general idea.

3 Classical particle in a magnetic field

For small values of the velocity $\vec{v}$ and weak fields, we can expand the square root in the last equation (13) to obtain

$$E = M + \frac{1}{2}M\vec{v}^2 + q\phi - \left(\vec{\mu}\cdot\vec{B} + \vec{d}\cdot\vec{E}\right) + \ldots \tag{15}$$

Thus the square root involving $D \cdot F$ represents a relativistic generalization of the usual expression for the energy of an electric and magnetic dipole in an external field. This shows that the energy-momentum relation (12) is a perfectly reasonable one from the physical point of view.

Now this relativistic generalization is not the only one possible: any function of the invariant quantity $D \cdot F$ which has the first-order behaviour (15) is acceptable. In order to convince ourselves that this relativistic generalization is also a *physically correct* generalization, we now derive this expression for a system where the dynamics is fully known and explicitly solvable: the motion of a spinless (non-radiating) charged particle in a constant magnetic field.

The classical equation of motion for a spinless charged particle in a magnetic field is given by the Lorentz force law

$$\frac{d}{dt}\frac{M\vec{v}}{\sqrt{1-\vec{v}^2}} = q\vec{v}\times\vec{B}. \tag{16}$$

Choosing the positive z-direction to be that of the magnetic field: $\vec{B} = (0,0,B)$, $B \geq 0$, the solution of this equation decribes a motion which is a superposition of two components: a rotation in the plane perpendicular to $\vec{B}$ (the x-y plane) with constant angular velocity

$$\omega = \frac{qB}{M}\sqrt{1-\vec{v}^2}, \tag{17}$$

the cyclotron frequency, and a translation parallel to the field with constant linear velocity

$$v_z = u_z\sqrt{1-\vec{v}^2}, \tag{18}$$

u_z being the corresponding constant four-velocity component. The full velocity in the laboratory frame is

$$\vec{v} = (\omega r \sin \omega t,\ \omega r \cos \omega t,\ v_z) \tag{19}$$

Substitution of this result back into equation (17) gives the result

$$\omega = \frac{qB}{M} \left(\frac{1 - v_z^2}{1 + (qBr/M)^2} \right)^{1/2}. \tag{20}$$

Comparing the two expressions for ω then leads to the relation

$$\frac{1}{1 - \vec{v}^2} = \frac{1 + (qBr/M)^2}{1 - v_z^2}. \tag{21}$$

The numerator on the right hand side of this equation involving the quantity $(qBr/M)^2$ requires some explanation. Its interpretation is the following. Define the gauge invariant magnetic dipole moment

$$\vec{\mu} = \frac{q}{2M} \vec{r} \times \vec{\pi}. \tag{22}$$

In particular, its component in the direction of $\vec{B}$ is

$$\mu_z = -\frac{q}{2M} \frac{M \omega r^2}{\sqrt{1 - \vec{v}^2}} = -\frac{q^2 B r^2}{2M}. \tag{23}$$

The minus sign here is due to Lenz's law: the induced moment opposes the magnetic field which creates it. Now eq.(21) can be rewritten in the form

$$\frac{1}{1 - \vec{v}^2} = \frac{1 - 2\vec{\mu} \cdot \vec{B}/M}{1 - v_z^2}. \tag{24}$$

From distances large compared to the orbital radius r and for times long compared to the period of the orbit, the particle behaves effectively as a particle moving along the z-axis with constant velocity v_z and carrying a magnetic moment $\vec{\mu}$. At the same time, for a spinless particle as considered here the energy-momentum and the time dilation factor are given by eqs.(9) and (10). However, by eq.(24) these give results which in the absence of a net electric dipole moment are now identical with eqs.(13) and (14):

$$\vec{\pi} = \frac{M\vec{v}}{\sqrt{1-\vec{v}^{\,2}}} = M\vec{v}\left(\frac{1-2\vec{\mu}\cdot\vec{B}/M}{1-v_z^2}\right)^{1/2},$$

$$E = \frac{M}{\sqrt{1-\vec{v}^{\,2}}} = M\left(\frac{1-2\vec{\mu}\cdot\vec{B}/M}{1-v_z^2}\right)^{1/2}, \tag{25}$$

and

$$dt = \frac{d\tau}{\sqrt{1-\vec{v}^{\,2}}} = d\tau\left(\frac{1-2\vec{\mu}\cdot\vec{B}/M}{1-v_z^2}\right)^{1/2}. \tag{26}$$

This shows that relation (12) holds effectively in this case as well:

$$\pi_z^2 - E^2 + M^2 - 2M\,\vec{\mu}\cdot\vec{B} = 0, \tag{27}$$

where $\vec{\mu}$ is understood to be the gauge invariant expression (22).

4 Quantum theory

We have seen that equations (12)-(14) hold effectively for a classical particle with an induced orbital magnetic moment in an external field. In this section we show, that the same result also holds for quantum mechanical particles with intrinsic spin and magnetic moment. We consider the case of a spin-1/2 Dirac fermion in a constant magnetic field[3], described by the potentials

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}, \qquad \phi = 0. \tag{28}$$

By this choice of potentials we have implicitly fixed the gauge for the electromagnetic gauge transformations. The one-particle wave functions are determined in QED by the Dirac equation

$$(\gamma\cdot D + M)\psi = 0. \tag{29}$$

Multiplying by $(-\gamma\cdot D + M)$ then gives a second order differential equation which is a generalization of the Klein-Gordon equation:

[3]See for example ref.[6], ch.2

$$\left(-D_\mu^2 + M^2 + iq\sigma\cdot F\right)\psi = 0. \tag{30}$$

We observe, that with the identification

$$\pi_\mu = -iD_\mu, \qquad D_{\mu\nu} = -i\frac{q}{M}\sigma_{\mu\nu}. \tag{31}$$

this is the operator analogue of eq.(12). This is the reason for our earlier remark, that eq.(12) is the energy-momentum relation expected for a Dirac fermion. In the absence of an electric field it is easy to diagonalize the operator $\sigma\cdot F$, resulting in an equation

$$\left(\partial_t^2 - \vec{D}^2 + M^2 - 2qBs_z\right)\psi = 0, \tag{32}$$

with $s_z = \pm 1/2$ for a Dirac fermion. Note that with $s_z = 0$ the same relation also holds for a charged scalar particle described by the Klein-Gordon equation.

For stationary positive-energy states we consider solutions of the form

$$\psi(\vec{x},t) = e^{i(kz+m\varphi-E_{km}t)}\,R_{km}(r), \tag{33}$$

where z, $r = \sqrt{x^2+y^2}$ and φ define a three-dimensional cylindrical coordinate system. The equation for the radial wave functions of the stationary states then becomes

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{m^2}{r^2} + \left(\frac{qB}{2}\right)^2 r^2 - (m+2s_z)qB + k^2 + M^2 - E_{km}^2\right)R_{km}(r) = 0. \tag{34}$$

This equation can be solved completely by observing that $R_{km}(r)$ is the wave function of a two-dimensional harmonic oscillator in a state with orbital angular momentum $L_z = m$. Therefore the energies are labeled by the principal quantum number n and are given by

$$E_{kmn}^2 = M^2 + k^2 + qB(n - m + 1 - 2s_z), \tag{35}$$

with n is a non-negative integer, whilst m is restricted to values

$$m = -n, -n+2, ..., n-2, n. \tag{36}$$

As a byproduct we note, that for $s_z = 0, \pm 1/2$ we always have

$$n - m + 1 - 2s_z \geq 0. \tag{37}$$

Therefore an external magnetic field always increases the mass gap in the particle spectrum, unless $m = n$ and $s_z = 1/2$, in which case it remains unchanged.

Inserting the energy eigenvalues in the time dilation formula (7) gives the result, that even for $k = 0$ there is a time dilation effect

$$dt = d\tau \sqrt{1 + (qB/M^2)(n - m + 1 - 2s_z)}. \tag{38}$$

This is the quantum equivalent of the classical result (26) with $v_z = 0$; the term proportional to $(n-m+1)$ in the square root represents the contribution from the orbital part of the magnetic moment and the term proportional to $2s_z$ the contribution of spin. Clearly the magnitude of the time dilation effect depends on the orientation of the spin with respect to the field. In particular, for the decay time of an instable particle in an external field we find that the relative difference in the decay times is

$$R \equiv 2\,\frac{\Delta t_- - \Delta t_+}{\Delta t_- + \Delta t_+} \approx \frac{qB}{M^2}, \tag{39}$$

where $\Delta t_\pm$ is the decay time for the states with $s_z = \pm 1/2$ in the direction of the magnetic field. From these considerations we infer that in quantum theory the contribution of spin to time dilation indeed appears together with that of orbital angular momentum under the square root, in agreement with the result we derived for classical scalar particles. Note also that the magnetic moment connected with spin is qs_z/M, corresponding to a gyromagnetic factor $g = 2$, as expected for fermions at this level of treatment (i.e. negelecting radiative corrections).

5 Discussion

Of course the approach taken above is somewhat hybrid, in the sense that even though we computed the energy eigenvalues exactly in the one-particle relativistic quantum theory, we used the classical time dilation formula (7).

This is justified by noting that also in quantum theory time is a c-number parameter and not an operator, and that the quantum expectation value

$$< E - q\phi > \tag{40}$$

transforms in the same way under Lorentz transformations as the classical gauge invariant energy. However, it is possible to provide an independent derivation of this result which starts directly from the quantum theory and does not take recourse to expressions borrowed from classical electrodynamics.

First we must provide an observable quantity measuring time in quantum theory. For this we take the inverse decay width Γ of an unstable particle which decays exponentially, for example a muon:

$$|\psi|^2(t) = e^{-\Gamma t}|\psi|^2(0), \tag{41}$$

where $|\psi|^2(t)$ is the probability of the original particle still being present at time t. Now if the classical time dilation formula (7) is to carry over to quantum theory, we must show that the decay width in a magnetic field B depends inversely on the energy:

$$\Gamma_{kmn} = \Gamma_0 \frac{M}{E_{kmn}}, \tag{42}$$

where Γ_0 is the decay width for the free particle at rest in the absence of external fields; thereby we then establish eq.(38).

To prove eq.(42), we recall that exponential decay is described effectively by adding the width Γ_{kmn} as an imaginary part to the energy:

$$E_{kmn} \to \mathcal{E}_{kmn} = E_{kmn} - \frac{i}{2}\Gamma_{kmn}. \tag{43}$$

For the free particle this amounts to adding $-i\Gamma_0/2$ to the mass M. In order to maintain formal Lorentz invariance and real three-dimensional rotation invariance, we then take $\psi(\vec{x},t)$ to be a solution of the Klein-Gordon-Dirac equation (30) with complex mass and complex energy eigenvalues; this then allows us to compute the imaginary part of the energy in terms of Γ_0. In particular, taking as before solutions of the form

$$\psi(\vec{x},t) = e^{i(kz+m\varphi-\mathcal{E}_{kmn}t)} R_{kmn}(r), \tag{44}$$

which are normalizable at fixed time t, we find a relation analogous to eq.(35):

$$\begin{aligned}
\mathcal{E}^2_{kmn} &= \left(E_{kmn} - \frac{i}{2}\Gamma_{kmn}\right)^2 \\
&= \left(M - \frac{i}{2}\Gamma\right)^2 + k^2 + qB(n - m + 1 - 2s_z) .
\end{aligned} \tag{45}$$

Equating the imaginary parts in the two expressions for $\mathcal{E}^2_{kmn}$ then gives the desired result (42). This way we have established the time dilation formulae (7) and (38) in relativistic quantum theory in an independent way.

It is of course of interest to estimate the magnitude of the time dilation effect due to a magnetic moment in an external field. From eqs.(38), (39) we observe that an appreciable effect requires

$$\vec{\mu} \cdot \vec{B} \gtrsim 10^{-2} M \tag{46}$$

For a particle like the muon this means we need fields of the order of 10^{12} - 10^{14} T. Such fields evidently cannot be created in the laboratory. There are only two conceivable situations where fields of this order of magnitude might arise: nuclear magnetic fields and the fields of neutron stars. In another paper [5] we have computed the effect of magnetic hyperfine splitting due to nuclear fields on the life time of a muon in an atomic bound state. The effects turn out to be small: the effective field is at most of the order of 10^{10} T. Although decay rates of muons can be measured rather accurately, and such a small effect might be observable in principle, competing effects in atomic muon decay such as muon capture by the nucleus may make it difficult to clearly establish the time dilation due to the nuclear field. Since the fields of neutron stars are roughly of the same order of magnitude or less than nuclear fields, astrophysical effects will be equally small.

However, time dilation being a dynamical effect it is in principle not confined to electromagnetic interactions. For instance, in strong interaction physics we expect quarks to experience a time dilation

$$\Delta t = \Delta\tau \, \frac{< E - g\phi >}{M}, \tag{47}$$

where g is the QCD coupling constant and $\phi = A_4^i \lambda_i/2$ is the matrix-valued scalar potential of the color field, with λ_i the hermitean Gell-Mann matrices of SU(3).

As an example, consider the D-mesons D^+ and D^{*+}, which both have quark content $c\bar{d}$. The D^+ has spin-parity $J^P = 0^-$ and a mass of 1870 MeV, whilst the D^{*+} has $J^P = 1^-$ and a mass of 2010 MeV. Hence we can think of these particles as pair of more or less degenerate bound states separated by a color hyperfine splitting

$$\Delta E = 140\,\mathrm{MeV} \tag{48}$$

Now due to weak interactions the c-quark in the D^+ meson is instable, and it decays after an average life time of 1.062×10^{-12} sec. To the extend to which it is allowed to think of the D-meson as a c-quark moving in the color-field of a (stable) $\bar{d}$-quark, we expect the weak decay of the c-quark in the D^{*+}-state to be delayed by

$$\Delta t^* \approx \Delta t \left(1 + \frac{\Delta E}{M}\right), \tag{49}$$

where ΔE is given in (48) and M is an effective mass for the c-quark which we take to be of order

$$M \approx 1700\,\mathrm{MeV}. \tag{50}$$

Then the weak life time of the D^{*+} is about 8% longer than that of the D^+:

$$\Delta t^* \approx 1.9 \times 10^{-12}\mathrm{sec}. \tag{51}$$

Of course this calculation is not very sophisticated, but it might give a correct order of magnitude for the effect, which is certainly larger than in the case of electromagnetic interactions. Unfortunately the weak life time of the D^{*+} is not known and presumably difficult to measure because the D^{*+} decays almost exclusively via *strong* interactions to a D^+; still this example illustrates the general idea.

6 Summary

In this paper we have shown that relativistic time dilation is a dynamical effect, rather than a kinematic one. This means that other degrees of freedom than mass and charge contributing to the energy of the particle may also affect the amount of time dilation, as measured for instance in the decay of instable particles. As an example we have considered in detail the contribution to time dilation of spin and the magnetic moment of charged particles in a magnetic field. Similar effects may be found in strong interaction physics, where there is a contribution to the gauge invariant energy from the color degrees of freedom of quarks.

It should be stressed that what is at stake here is not special relativity as such: the systems we have considered are described by well-established classical and quantum theories which are known to be fully covariant. The issue is rather, that a clock carried by a particle and co-moving with the particle does not necessarily measure inertial time as defined by an observer in the laboratory frame. This is especially clear in our example of the rotating classical point particle with an orbital magnetic moment. It is also reminiscent of the concept of *Zitterbewegung* in the Dirac theory of fermions.

In all of our examples we have considered only particles in given static external field. We have completely neglected the back reaction of the particles on the field, because we donot expect the results to be affected by this approximation. We intend to study this problem in a future publication.

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