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Quantum Yang-Mills Theory On Arbitrary Surfaces

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Abstract

We study quantum Yang-Mills theory on two-dimensional surfaces. Using path integral methods we derive general and explicit expressions for the partition function and expectation values of homologically trivial and non-trivial Wilson loops on closed surfaces of any genus, as well as for the kernels on manifolds with handles and boundaries.

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1 Introduction and survey of results

The purpose of this paper is to present general and explicit formulae for the partition function of Yang-Mills theory on arbitrary (orientable) two-dimensional surfaces as well as for the expectation values of homologically trivial and non-trivial Wilson loops. Our derivation is based on the ability to compute explicitly the path integral on the disc (and more generally on spheres with boundaries) and has been explained in [1] in the much simpler case of Maxwell theory. The kernels and partition functions on surfaces with any number of handles and boundaries will then be derived from the rules for glueing manifolds and joining boundaries appropriate for two-dimensional Yang-Mills theory.

Previous studies of quantum Yang-Mills theory on topologically non-trivial surfaces have been performed by Rajeev [2] for the cylinder and by Fine [3] for the two-sphere, and in these particular cases our general formulae reproduce the known results.

In order to state our results we have to introduce some notation. G is a compact connected and simply-conenected Lie group and we fix an invariant positive definite scalar product (trace) on the Lie algebra of G as well as the corresponding normalized Haar measure dg, $\int_G dg = 1$. $\hat{G}$ is the discrete set of equivalence classes of irreducible unitary representations of G, and for $\lambda \in \hat{G}$ we denote by $d(\lambda)$ the dimension of the representation λ, by χ_λ the corresponding character (normalized by $\chi_\lambda(1) = d(\lambda)$), and by $c(\lambda)$ the quadratic Casimir invariant of λ. For the properties of characters used in this paper see e.g. [4]. $\Sigma_{g,s}$ is an oriented two-dimensional surface of genus g with s boundary components ($\Sigma_{g,0} = \Sigma_g$). The action of Yang-Mills theory on a surface Σ is

$$S = \frac{1}{2e^2}\int_\Sigma F_A * F_A \qquad (1.1)$$

where $F_A = dA + \frac{1}{2}[A, A]$ is the curvature of a conection on a (necessarily trivial) G bundle over Σ, $*$ is the Hodge duality operator with respect to some metric on Σ, e is the coupling constant, and a trace and wedge product are understood in (1.1). Finally, for $X \subset \Sigma$ we denote by $A(X) = \int_X *1$ the area of X.

With this notation we have the result that the partition function of G-Yang-Mills theory on Σ_g is

$$Z(\Sigma_g) = \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g} e^{-e^2 c(\lambda) A(\Sigma_g)/2} \quad . \tag{2}$$

For a contractible loop γ on Σ_g the expectation value of the trace of the Wilson loop $Pe^{\oint_\gamma A}$ in the representation $\mu \in \hat{G}$ is

$$\langle \chi_\mu (Pe^{\oint_\gamma A}) \rangle_{\Sigma_g} = \sum_{\lambda \in \hat{G}} \sum_{\rho \in \lambda \otimes \mu} d(\lambda)^{1-2g} d(\rho) e^{-e^2 (c(\lambda) A(D) + c(\rho) A(D'))/2} \tag{3}$$

where D is the disc enclosed by γ, $\partial D = \gamma$, and D' is its complement in Σ_g. The formuale for homotypically non-trivial but homologically trivial and homologically non-trivial loops are slightly more complicated and will be given in section 4. Equations (2) and (3) are immediate consequences of a general formula for the kernel $K(\Sigma_{g,n})$ on a surface $\Sigma_{g,n}$ with boundary. $K(\Sigma_{g,n})$ is a functional of the boundary conditions imposed at the n boundary circles of $\Sigma_{g,n}$. As the only gauge invariant degree of freedom of a gauge field A on a circle S is the conjugacy class of its holonomy $Pe^{\oint_S A} \in G$, the kernel can be considered as a function $K(\Sigma_{g,n})(g_1, \ldots, g_n)$ on $G \times \ldots \times G$ invariant under conjugation in each entry. Explicitly $K(\Sigma_{g,n})$ is

$$K(\Sigma_{g,n})(g_1, \ldots, g_n, A(\Sigma_{g,n})) = \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g-n} \chi_\lambda(g_1) \ldots \chi_\lambda(g_n) e^{-e^2 c(\lambda) A(\Sigma_{g,n})/2} \tag{4}$$

Here we have indicated explicitly the metric (area) dependence of $K(\Sigma_{g,n})$ in our notation. For reasons explained in [1] two-dimensional Yang-Mills theory is almost a topological field theory in the sense that the partition function and correlation functions of metric independent operators (Wilson loops) will depend on the metric only through terms of the form $e^2 A(\Sigma)$, $e^2 A(D)$ (in particular no derivatives of the metric appear).

In section 2 we will calculate explicitly the kernel on the disc $D = \Sigma_{0,1}$ from the path integral. This requires a number of technicalities to be handled and our computation will be based on a spectral representation for the delta function imposing the boundary conditions in the path integral, use of the Schwinger-Fock gauge to have the Nicolai map of [5, 1] at our disposal, and a fermionic path integral representation for the trace of a Wilson loop. In section 3 we generalize that calculation to the cylinder. At that point it becomes clear that in principle the calculation can be extended further to spheres with three and more boundaries, but as things then become a little murky we proceed in a different and much simpler way. We rederive the result for the cylinder directly from that for the disc, basically by deforming the disc to a rectangle and identifying two opposite sides. This procedure, which is possible due to the almost topological nature of two-dimensional Yang-Mills theory, is the prototype of the operations which then allow us to determine immediately the kernels on surfaces with an arbitrary number of handles and boundaries. In section 4 we derive general formulae for the expectation values of Wilson loops and give some examples. We conclude with additional remarks on possible generalizations and open problems, some of which will be dealt with in [6] to which we also refer for some of the details we are not able to go into in this letter.

2 The wave function on the disc

Let $\gamma = \partial D$ be the boundary of a disc D. Choosing the boundary condition to be $Pe^{\oint_\gamma A} = g_1 \in G$ (modulo conjugation, i.e. gauge transformations of A) our task is to compute the path integral

$$K(D)(g_1, A(D)) \equiv \int_\Phi e^{-S_q} \delta(Pe^{\oint_\gamma A}, g_1) \tag{5}$$

where S_q is the BRST invariant quantum action on D (for some gauge condition $G(A) = 0$) and $\int_\Phi$ symbolically denotes the path integral over all fields (gauge fields, multipliers and ghosts). The precise nature of the delta function entering (5) will be specified shortly (topic d) below). Let us now assemble the techniques that will go into computing (5).

a) It will be convenient to replace the path ordered exponential in (5) by a quantum mechanics amplitude and we shall use a not so widely known fermionic path integral representation for the trace of a Wilson loop, namely

$$\chi_\lambda(Pe^{\oint_\gamma A}) = \int D\eta D\bar{\eta}\, e^{i(\int_0^1 dt[\bar{\eta}^i(t)\dot{\eta}^i(t) - A^a_\mu(\gamma(t))\dot{\gamma}^\mu(t)\bar{\eta}^i(t)\lambda^a_{ik}\eta^k(t)])}\bar{\eta}^i(1)\eta^k(0)\delta_{ik} \tag{6}$$

where η^k and $\bar{\eta}^k$, $k = 1, \ldots, d(\lambda)$ are Grassmann variables (with the obvious generalization to traces of the form $\chi_\lambda(Pe^{\oint_\gamma A}g)$, $g \in G$). The shortest (although not most illuminating) proof of (6) we know uses the fact that the fermion propagator in one dimension is

$$\int D\eta D\bar{\eta} e^{i(\int_0^1 dt \eta^i(t)\dot{\bar{\eta}}^i(t))} \bar{\eta}^i(s)\eta^j(0) = \delta^{ij}\theta(s) \ .$$

Together with the change of variables

$$\begin{aligned} \eta^i(t) &\rightarrow [Pe^{\int_0^t A^a_\mu \lambda^a \dot{\gamma}^\mu ds}]_{ij}\eta^j(t) \\ \bar{\eta}^i(t) &\rightarrow \bar{\eta}^j(t)[Pe^{-\int_0^t A^a_\mu \lambda^a \dot{\gamma}^\mu ds}]_{ji} \end{aligned}$$

this can be seen to imply (6). A proof more in the spirit of quantum mechanics can be found in [7].

b) In [5, 1] we have shown that for the partition function and correlation functions of operators expressible in terms of F_A there exists a change of variables (Nicolai map) which essentially trivializes the path integral. In order to have this Nicolai map available here (cf. c) below) we will use the so-called Schwinger-Fock gauge $G^a(A) = x^\mu A^a_\mu = 0$. The reason for this is that in this gauge the connection A^a_μ can be expressed in terms of $F^a_{\mu\nu}$ via

$$A^a_\mu(x) = \int_0^1 ds x^\nu F^a_{\nu\mu}(sx) \ , \tag{7}$$

and this allows us in particular to replace the connection A appearing in (5) and (6) by equation (7).

c) In correlation functions of the form

$$\langle \mathcal{O} \rangle = \int_\Phi e^{-S_q} \mathcal{O}(F_A) \tag{8}$$

one can perform a change of variables

$$\begin{aligned} \zeta^a(A) &= F^a_A \ , \\ \omega^a(A) &= G^a(A) \ , \end{aligned} \tag{9}$$

whose Jacobian cancels precisely against the ghost determinant (implicit in (8)) in any BRST invariant (e.g. ζ-function) regularization [5]. As we are working on the disc where the Schwinger-Fock gauge is well defined globally, so is our Nicolai map (in particular we do not have to worry about the moduli of flat connections, cf. the discussions in [5, 1]). Performing the trivial integral over ω and the multiplier field enforcing the gauge condition, (8) reduces to the remarkably (and perhaps deceptively, cf. the discussion following equation (26) below) simple expression

$$\langle O \rangle = \int D\xi e^{-\frac{1}{2e^2}\int_D \xi * \xi} O(\xi) \ . \tag{10}$$

d) As a consequence of the considerations in a) and b) (equations (6) and (7)) the path integral (5) we originally set out to calculate is also of the form (10) provided that we can use (6) in the representation of the delta function appearing in (5). Thus the last ingredient we need to actually calculate (5) is the specification of the delta function which is some delta function on the group G. It is possible to use the delta function of $L^2(G)$, given in the spectral representation by

$$\delta(g,h) = \sum_{\lambda \in \hat{G}} d(\lambda) \chi_\lambda(g^{-1}h) \ . \tag{11}$$

With this choice of delta function the path integral (5) is not manifestly conjugation invariant but, as the result turns out to be, use of (11) is sufficient for our present purposes. We can however build in conjugation invariance from the outset by using the delta function $\bar{\delta}$ on the space $L^2(G)^G$ of conjugation invariant functions (class functions),

$$\bar{\delta}(g,h) = \sum_{\lambda \in \hat{G}} \chi_\lambda(g^{-1}) \chi_\lambda(h) \tag{12}$$

related to $\delta(g,h)$ by

$$\bar{\delta}(g,h) = \int_G dg' \delta(g, g' h g'^{-1}) \tag{13}$$

as a consequence of the relation

$$\int dg \chi_\lambda(x g y g^{-1}) = d(\lambda)^{-1} \chi_\lambda(x) \chi_\lambda(y) \ . \tag{14}$$

In the case of surfaces with more than one boundary component the use of (12) actually becomes mandatory if one wants to work with the gauge fixed path integral and retain conjugation invariance. Of course nothing prevents one from using (for the cylinder, say) $\delta(P_1, g_1)\delta(P_2, g_2)$ in the path integral over all gauge fields (here P_k is the holonomy along the k-th boundary). In that case, however, the issue of gauge fixing requires more care. If one implements the Faddeev-Popov procedure the gauge group will not simply factor out of the path integral. Rather, at the boundaries it will act via conjugation on the P_k and whence, via (13), turn $\delta(P_k, g_k)$ into the conjugation invariant delta function $\tilde{\delta}(P_k, g_k)$ in the now gauge fixed path integral, which brings us back to the first alternative. We have in any case achieved our goal of expressing the delta function in (5) in terms of traces involving the Wilson loop, allowing us to make use of the fermionic representation (6).

Putting all this together the (now Gaussian) integral over ξ is easily performed and the task of calculating (5) reduces to that of evaluating the fermionic integral

$$\int D\eta D\bar{\eta} e^{i\int_0^1 dt\eta^k(t)\dot{\eta}^k(t) - \frac{1}{2}e^2 A(D)\int_0^1 dt(\bar{\eta}\lambda^a\eta)(t)(\bar{\eta}\lambda^a\eta)(t)} \bar{\eta}^i(1)\eta^k(0)$$

where $(\bar{\eta}\lambda^a\eta)(t) = \bar{\eta}^i(t)\lambda^a_{ik}\eta^k(t)$. In the calculation for the cylinder we will encounter a slight generalization of this, namely

$$\int D\eta D\bar{\eta} e^{\int_0^1 dt[i\eta^k(t)\dot{\eta}^k(t) + \rho^a(t)(\bar{\eta}\lambda^a\eta)(t)] - \frac{1}{2}e^2 A(D)\int_0^1 dt(\bar{\eta}\lambda^a\eta)(t)(\bar{\eta}\lambda^a\eta)(t)} \bar{\eta}^i(1)\eta^k(0) . \tag{15}$$

These integrals can be calculated order by order in perturbation theory, but in [6] we will give an alternative derivation which is comparable in simplicity to the result

$$(15) = e^{-e^2 A(D)c(\lambda)/2}[Pe^{\int_0^1 dt\rho(t)}]^{ik} . \tag{16}$$

Combining this with (6) and (11) or (12) we finally arrive at our first main result, the equation for the kernel (wave function) on the disc (5),

$$K(D)(g_1, A(D)) = \sum_{\lambda\in\hat{G}} d(\lambda)\chi_\lambda(g_1)e^{-e^2 c(\lambda)A(D)/2} . \tag{17}$$

This kernel can be used to obtain information in several directions. Postponing a discussion of its role in the determination of kernels on more complicated surfaces to the next section, our more immediate concern here is with the infinite cylinder $S^1 \times \mathbf{R}$ and the two-sphere S^2.

Via the boundary conditions the path integral on a manifod M with boundary B (possibly with operator insertions in the interior of M) becomes a functional of the fields on B. According to the general rules establishing the correpondence between the path integral and operator formalisms of field theory this functional can be regarded as a time-dependent state in the canonical Hilbert space of the theory on $B \times \mathbf{R}$ satisfying the Schrödinger equation (or, in the case of Euclidean path integrals, the corresponding heat equation). In the case at hand this can easily be verified explicitly. According to [2] the Hilbert space of Yang-Mills theory is the space $L^2(G)^G$ of class functions, spanned by the characters χ_λ, $\lambda \in \hat{G}$, and the Hamiltonian operator is (proportional to) the Laplacian Δ_G on the group manifold G. Using the fact that

$$\Delta_G \chi_\lambda = c(\lambda)\chi_\lambda \tag{18}$$

we see that the wave function (17) indeed satisfies

$$(\partial_{A(D)} + \frac{e^2}{2}\Delta_G)K(D)(g, A(D)) = 0 \tag{19}$$

with the initial condition

$$\lim_{A(D)\to 0} K(D)(g, A(D)) = \delta(1, g) \equiv \delta(g)$$

(for a discussion of the identification of the area as the time coordinate see [1]). Thus $K(D)$ is nothing other than the well-known heat kernel on G (cf. [8, 9, 10]) which we have derived here from Yang-Mills theory on the disc.

$K(D)$ can also be used to compute the partition function of Yang-Mills theory on S^2 as well as expectation values of Wilson loops. Considering S^2 as the union of two discs,

$$S^2 = D \cup_\gamma D' \ , \qquad \partial D = \partial(-D') = \gamma \ ,$$

we see that we can write $Z(S^2)$ as

$$Z(S^2) = \int_G dg K(D)(g, A(D))K(D')(g^{-1}, A(D')) \tag{20}$$

(the inverse g^{-1} being due to the opposite orientation of $\partial D'$). That dg is the correct measure to use can be seen from the change of variables $A \to Pe^{\oint_\gamma A}$

on γ. Using the orthonormality

$$\int_G dg\chi_\lambda(g)\overline{\chi_\mu(g)} = \int_G dg\chi_\lambda(g)\chi_\mu(g^{-1}) = \delta_{\lambda,\mu} \tag{21}$$

of the characters this becomes

$$Z(S^2) = \sum_{\lambda\in\hat{G}} d(\lambda)^2 e^{-e^2 c(\lambda)A(S^2)/2} , \tag{22}$$

where $A(S^2) = A(D) + A(D')$.

If we want to compute the expectation value $\langle\chi_\mu(Pe^{\oint_\gamma A})\rangle_{S^2}$ we split S^2 along γ into two discs D and D' and put a Wilson loop on the boundary of D before glueing D and D' together again. In equations this amounts to computing

$$\langle\chi_\mu(Pe^{\oint_\gamma A})\rangle_{S^2} = \int_G dgK(D)(g,A(D))\chi_\mu(g)K(D')(g^{-1},A(D')) . \tag{23}$$

In order to calculate this we make use of one further property of characters, namely that

$$\chi_\lambda(g)\chi_\mu(g) = \chi_{\lambda\otimes\mu}(g) \equiv \sum_{\rho\in\lambda\otimes\mu,\rho\in\hat{G}} \chi_\rho(g) . \tag{24}$$

Then we find

$$\langle\chi_\mu(Pe^{\oint_\gamma A})\rangle_{S^2} = \sum_{\lambda\in\hat{G}}\sum_{\rho\in\lambda\otimes\mu} d(\lambda)d(\rho)e^{-e^2(c(\lambda)A(D)+c(\rho)A(D'))/2} \tag{25}$$

for the unnormalized expectation value of a Wilson loop on S^2. Equations (22) and (25), special cases of the general formulae (2) and (3), agree with the results of Fine [3].

We end this section with one important remark concerning the partition function (20). Equation (10), valid for correlation functions on the disc after the change of variables (9), may lead one to believe that the partition function $Z(S^2)$ is simply

$$Z(S^2) = \int D\xi e^{-\frac{1}{2e^2}\int_{S^2}\xi*\xi} . \tag{26}$$

Although correct if interpreted correctly this expression is too implicit to be of much practical use as the integration domain of ξ has not been specified.

In particular it is far from obvious how this "trivial" integral could lead to the result (22). To see what is required recall that in the Abelian case [1] the partition function in the topological sector k is given by (26) with a delta function insertion specifying the integration domain (topological sector),

$$Z_k(S^2) = \int D\xi e^{-\frac{1}{2e^2}\int_{S^2}\xi*\xi}\delta(\int_{S^2}\xi - 2\pi k) \ . \tag{27}$$

Even if no constraint on the topological sector is imposed the partition function is not just (26) but rather (27) with $\delta(\int_{S^2}\xi - 2\pi k)$ replaced by the periodic delta function $\delta^P(\int_{S^2}\xi) = \sum_k \delta(\int_{S^2}\xi - 2\pi k)$ so that

$$Z(S^2) = \int D\xi e^{-\frac{1}{2e^2}\int_{S^2}\xi*\xi}\delta^P(\int_{S^2}\xi) = \sum_k Z_k(S^2) \ . \tag{28}$$

This insertion of $\delta^P(\int\xi)$ expresses nothing other than the fact that one is not integrating over arbitrary two-forms but over curvatures of connections, and clearly an analogous condition should find its way into (26) in the non-Abelian case as well.

The required constraint can be deduced from inserting the definition (5) of the kernel on the disc, and not the result (17), into (20). Doing this one finds (the connection A should again be thought of as being expressed in terms of ξ via (7))

$$Z(S^2) = \int D\xi e^{-\frac{1}{2e^2}\int_{S^2}\xi*\xi}\delta(Pe^{-\oint_{\partial D}A}Pe^{-\oint_{\partial D'}A}) \ , \tag{29}$$

where (cf.(11)) $\delta(g) \equiv \delta(1,g)$. This constraint expresses the obvious requirement that the holonomies of A along ∂D and $\partial(-D')$ be equal. As (20) can also be written as

$$Z(S^2) = \int dg \int dg' K(D)(g, A(D))K(D')(g', A(D'))\tilde{\delta}(g,g')$$

(this is easy to check using (21)) one can also relax this condition to the requirement that the holonomies be equal up to conjugation.

As (29) is independent of the splitting of S^2 into D and D' one would hope to be able to express this constraint in a form which does not make any reference to a particular splitting (as in (28)), and with a slight sleigh of hand this can be done. Via the non-Abelian Stokes' theorem (see e.g. [11]) the path ordered exponential entering (29) can be written as

$$Pe^{-\oint_{\partial D} A} = \mathcal{P}e^{-\int_D \xi} \tag{30}$$

where the right hand side is a surface ordered exponential involving path dependent curvatures whose precise definition will not be required for our considerations (moreover, in the Schwinger-Fock gauge the path dependent curvature reduces to the ordinary curvature and the surface ordering $\mathcal{P}$ to the usual path ordering). We therefore obtain

$$Z(S^2) = \int D\xi e^{-\frac{1}{2\epsilon}\int_{S^2}\xi * \xi}\delta(\mathcal{P}e^{-\int_D \xi}\mathcal{P}e^{-\int_{D'}\xi}) \ . \tag{31}$$

As this is still independent of the choice of D and (with our conventions regarding orientation) $\int_{S^2}\xi = \int_D \xi + \int_{D'}\xi$, it is suggestive to rewrite (31) as

$$Z(S^2) = \int D\xi e^{-\frac{1}{2\epsilon}\int_{S^2}\xi * \xi}\delta(\mathcal{P}e^{-\int_{S^2}\xi}) \ , \tag{32}$$

where we keep in mind that this is defined by (31) for some (arbitrary) splitting of S^2. This is the obvious analogue of (28), the delta function on G having replaced the delta function δ^P on $U(1)$, and is also the correct interpretation and precise meaning of the somewhat vague expression (26). We do not know, however, if the object $\mathcal{P}e^{-\int_{S^2}\xi}$ can be given a splitting invariant meaning outside the path integral (32).

3 Extension to $\Sigma_{g,n}$

It is possible to derive the kernel of the cylinder directly from that of the disc. The same method (of glueing surfaces and joining boundaries) will then allow us to derive immediately the kernels of arbitrary surfaces $\Sigma_{g,n}$. But as a check on that procedure (and to convince the reader that the above method is both simpler and faster) we will first calculate the kernel of the cylinder using the techniques of the previous section.

The boundary conditions for the path integral specify the conjugacy classes of the holonomies $P_k(A) \equiv e^{\oint_{\gamma_k} A}$ along the boundaries γ_1 and γ_2

of the cylinder. As discussed in section 2 these are most conveniently implemented by inserting the conjugation invariant delta functions $\tilde{\delta}(P_k(A), g_k)$ into the gauge fixed path integral. The issues we will briefly have to readdress here are the use of the Schwinger-Fock gauge and the related question of what the required Nicolai map now looks like (corresponding to the issues b and c of section 2).

The natural analogue of the Schwinger-Fock gauge on the cylinder C (coordinates r and φ) is the radial gauge $A^a_r = 0$. In that gauge a gauge field $A_\varphi(r,\varphi)$ can be expressed in terms of the curvature (cf. (7)) and the value of A_φ at $r = 0$,

$$A_\varphi(r,\varphi) = A_\varphi(0,\varphi) + \int_0^r dr' F_{r\varphi}(r',\varphi) \ . \tag{33}$$

In particular, both gauges have the property that $F_A = dA$ globally, as in the Abelian case. Thus we can parametrize gauge equivalence classes of connections in terms of the curvature and the holonomy at $r = 0$, i.e. $P_1(A)$. The corresponding Nicolai map leads to

$$K(C)(g_1,g_2) = \int D\xi dP_1 e^{-\frac{1}{2e^2}\int_C \xi * \xi} \sum_{\lambda,\mu} \chi_\lambda(P_1^{-1})\chi_\lambda(g_1)\chi_\mu(P_2^{-1})\chi_\mu(g_2) \tag{34}$$

with P_2 expressed in terms of ξ and $A_\varphi(0,\varphi)$ via (33). Using the fermionic representation (6) for $\chi_\lambda(P_2)$ the ξ integral will give rise to equation (15) (with $\rho^a = A^a_\varphi(0,\varphi)$). Thus, using (16), the remaining integral to compute is simply

$$\begin{aligned} K(C)(g_1,g_2,A(C)) &= \sum_{\lambda,\mu} \chi_\lambda(g_1)\chi_\mu(g_2) \int dP_1 \chi_\lambda(P_1^{-1})\chi_\mu(P_1) e^{-e^2 c(\mu) A(C)/2} \\ &= \sum_{\lambda\in\hat{G}} \chi_\lambda(g_1)\chi_\lambda(g_2) e^{-e^2 c(\lambda) A(C)/2} \ . \end{aligned} \tag{35}$$

From (35) we can deduce the partition function of the torus (by joining the boundaries),

$$\begin{aligned} Z(\Sigma_1) &= \int_G dg K(\Sigma_{0,2})(g, g^{-1}, A(\Sigma_1)) \\ &= \sum_{\lambda\in\hat{G}} e^{-e^2 c(\lambda) A(\Sigma_1)/2} \ . \end{aligned} \tag{36}$$

To check this result we calculate the partition function directly from the Hamiltonian point of view as $tr(e^{-\beta H})$. Here β is the length of the cylinder and the trace is taken over the Hilbert space of Yang-Mills theory on a circle (the 'initial' boundary of the cylinder) which is $L^2(G)^G = span_{\mathbf{C}}\{\chi_\lambda, \lambda \in \hat{G}\}$. From the heat equation satisfied by (35) (in both entries seperately) we deduce (with R the radius of the cylinder) that the Hamiltonian is

$$H = e^2 R\pi\Delta_G \ .$$

As $\Delta_G\chi_\lambda = c(\lambda)\chi_\lambda$ (18)) we find

$$tr(e^{-\beta H}) = \sum_{\lambda\in\hat{G}} e^{-e^2 R\pi\beta c(\lambda)}$$

which is precisely (36).

In principle this calculation can be generalized to spheres with $n > 2$ boundaries (in particular, thinking of these as discs with $n-1$ holes or cylinders with $n-2$ holes one knows that there is a gauge with the property $F_A = dA$ and the corresponding Nicolai map). However, the analogue of (15,16) will become rather unpleasant, and so will therefore the analogue of going from (34) to (35). For that reason we will now give an alternative derivation of (35) based on nothing but the kernel for the disc (17) and the fact that $K(C)$ can depend only on the holonomies along the boundaries and the area $A(C)$. We deform the disc to a rectangle with the same area with edges a, b, c and d, and write the holonomy g_1 around the boundary of D as $g_1 = g_a g_b g_c g_d$ (this is possible as the holonomy is a path ordered exponential and can therefore be written as the product of the group elements obtained from going along a, then along b, etc.). Identifying the edges a and c (with opposite orientation) now amounts to setting $g_a = g_c^{-1}$ and integrating. Using (14) and (17) we find

$$\begin{aligned} K(C)(g_b, g_d, A(C)) &= \int_G dg_a K(D)(g_a g_b g_a^{-1} g_d, A(C)) \\ &= \int_G dg_a \sum_{\lambda\in\hat{G}} d(\lambda)\chi_\lambda(g_a g_b g_a^{-1} g_d) e^{-e^2 c(\lambda) A(C)/2} \\ &= \sum_{\lambda\in\hat{G}} \chi_\lambda(g_b)\chi_\lambda(g_d) e^{-e^2 c(\lambda) A(C)/2} \ , \end{aligned} \qquad (37)$$

which is precisely the result (35) obtained above. We emphasize that this procedure is only available because Yang-Mills theory in two dimensions is invariant under deformations preserving the area. Does the same method allow us to calculate the kernel for the 'pair of pants' $\Sigma_{0,3}$? Indeed it does. We simply visualize the cylinder as a rectangle (disc) with a hole and now identify the edges a and c as above. Thus, to obtain $K(\Sigma_{0,3})$ all we have to do is calculate $\int dg_a K(C)(g_a g_1 g_a^{-1} g_2, g_3, A(\Sigma_{0,3}))$ and using (35) and (14) this becomes

$$K(\Sigma_{0,3})(g_1, g_2, g_3, A(\Sigma_{0,3})) = \sum_{\lambda \in \hat{G}} d(\lambda)^{-1} \chi_\lambda(g_1)\chi_\lambda(g_2)\chi_\lambda(g_3) e^{-e^2 c(\lambda) A(\Sigma_{0,3})/2} \ . \tag{38}$$

This result, of which we have given a from-scratch proof here, was also reported by Witten at the 1990 Trieste conference [12].

Knowing the kernel of the 'pants' and the rules for joining boundaries and glueing surfaces it is now a simple matter to deduce from (38) the general formulae

$$K(\Sigma_{g,n})(g_1, \ldots, g_n, A(\Sigma_{g,n})) = \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g-n} \chi_\lambda(g_1) \ldots \chi_\lambda(g_n) e^{-e^2 c(\lambda) A(\Sigma_{g,n})/2} \tag{39}$$

and

$$Z(\Sigma_g) = \sum_{\lambda \in \hat{G}} d(\lambda)^{2-2g} e^{-e^2 c(\lambda) A(\Sigma_g)/2} \tag{40}$$

announced in the introduction. It is rather remarkable that, in a sense, the basic building block of Yang-Mills theory in two dimensions is not the kernel (38) of the 'pants' but rather that of the disc (17). This can be understood as a consequence of the fact that the theory is not only almost topological in the above sense but also a gauge theory.

Note that in (39,40) the power of $d(\lambda)$ is always the Euler number $2-2g-n$ of $\Sigma_{g,n}$. That it is precisely this function of g and n which appears is of course no coincidence. Compatibility of (39,40) with the operations of joining $2b$ boundaries of a surface $\Sigma_{g,n}$,

$$\Sigma_{g,n} \rightarrow \Sigma_{g+b,n-2b} \ ,$$

and of glueing two surfaces $\Sigma_{g,n}$ amd $\Sigma_{g',n'}$ along b boundaries,

$$(\Sigma_{g,n}, \Sigma_{g',n'}) \rightarrow \Sigma_{g+g'+b-1,n+n'-2b} \ ,$$

requires the putative power $p(g,n)$ of $d(\lambda)$ to satisfy

$$
\begin{aligned}
p(g,n) &= p(g+b, n-2b) \\
p(g,n) + p(g',n') &= p(g+g'+b-1, n+n'-2b)
\end{aligned}
\tag{41}
$$

and this fixes $p(g,n)$ uniquely (up to a scale) to be $p(g,n) = 2-2g-n$. The scale can then be determined by computing e.g. the kernel on the disc (17) or the partition function of the two-sphere (22).

4 Expectation values of Wilson loops

With the general formula (39) for $K(\Sigma_{g,n})$ and the rules for glueing surfaces and joining boundaries at our disposal it is rather straightforward to compute expectation values of Wilson loops (the generalization to correlation functions of several non-intersecting loops being immediate). There are three different types of non-selfintersecting loops to consider, contractible (homotopically trivial) loops, non-contractible homologically trivial loops, and homologically non-trivial loops. As it is really homology and not homotopy that matters the first is actually a special case of the second type, but for simplicity we will treat them seperately.

Contractible loops

Expectation values of contractible loops on a surface Σ_g can be computed by glueing a disc $\Sigma_{0,1}$ and a $\Sigma_{g,1}$ with a Wilson loop on the boundary. The calculation is exactly the same as that preformed in section 2 for a Wilson loop on the two-sphere (equations (23-25)) with $K(D')$ in (23) replaced by $K(\Sigma_{g,1})$. Using the multiplicative property (24) of characters we thus find

$$
\begin{aligned}
\langle \chi_\mu(Pe^{\oint A}) \rangle_{\Sigma_g} &= \int_G dg K(D)(g, A(D)) \chi_\mu(g) K(\Sigma_{g,1}))(g^{-1}, A(\Sigma_{g,1})) \\
&= \sum_{\lambda \in \hat{G}} \sum_{\rho \in \lambda \otimes \mu} d(\lambda) d(\rho)^{1-2g} e^{-e^2(c(\lambda)A(D) + c(\rho)A(\Sigma_{g,1}))/2}
\end{aligned}
\tag{42}
$$

Non-contractible homologically trivial loops

These types of loops exist on surfaces of genus > 1 and cut a surface $\Sigma_{g'+g}$ into a $\Sigma_{g',1}$ and a $\Sigma_{g,1}$. Thus the only difference to the example above is that we have to replace D in (42) by $\Sigma_{g',1}$. This gives the result

$$\langle\chi_\mu(Pe^{\oint_\gamma A})\rangle_{\Sigma_{g+g'}} == \sum_{\lambda\in\hat{G}}\sum_{\rho\in\lambda\otimes\mu} d(\lambda)^{1-2g'}d(\rho)^{1-2g}e^{-e^2(c(\lambda)A(\Sigma_{g',1})+c(\rho)A(\Sigma_{g,1}))/2} \tag{43}$$

which reduces to (42) for $g' = 0$.

Homologically non-trivial loops

Not unexpectedly the formulae in this case turn out to be slightly more complicated than (42,43). The required operation is now not that of glueing two surfaces together but rather that of joining the two boundaries of a $\Sigma_{g-1,2}$ with a loop in between. In equations this amounts to calculating

$$\begin{aligned}\langle\chi_\mu(Pe^{\oint_\gamma A})\rangle_{\Sigma_g} &= \int_G dg\sum_{\lambda\in\hat{G}} d(\lambda)^{2-2g}\chi_\lambda(g)\chi_\mu(g)\chi_\lambda(g^{-1})e^{-e^2c(\lambda)A(\Sigma_g)/2}\\ &= \sum_{\lambda\in\hat{G}}\sum_{\rho\in\lambda\otimes\mu} d(\lambda)^{2-2g}\delta_{\lambda\rho}e^{-e^2c(\lambda)A(\Sigma_g)/2}\ .\end{aligned} \tag{44}$$

This means that a representation $\lambda \in \hat{G}$ will only contribute to the sum if it appears again in the decomposition of $\lambda\otimes\mu$. Let $m_\mu(\lambda)$ denote the multiplicity of λ in $\lambda\otimes\mu$. Then

$$\langle\chi_\mu(Pe^{\oint_\gamma A})\rangle_{\Sigma_g} = \sum_{\lambda\in\hat{G}} d(\lambda)^{2-2g}m_\mu(\lambda)e^{-e^2c(\lambda)A(\Sigma_g)/2}\ . \tag{45}$$

Using some group theory (45) can be put into a more explicit form but as explicit is not the same as enlightning in this case we will forego this and look at some examples instead. For $SU(2)$ two extreme cases are represented by choosing μ to be a half-integer spin representation or the spin one representation. If μ is half-integer then for no value of λ will λ reappear in $\lambda\otimes\mu$, so that we have the general result that for a homologically non-trivial loop γ

$$\langle\chi_{n+\frac{1}{2}}(Pe^{\oint_\gamma A})\rangle_{\Sigma_g} = 0\ .$$

On the other hand if $\mu = 1$ then $m_\mu(\lambda) = 1\ \forall\lambda\in\hat{G}$ and thus all representations will contribute to the sum in (45),

$$\langle\chi_{\mu=1}(Pe^{\oint_\gamma A})\rangle_{\Sigma_g} = \sum_{\lambda\in\hat{G}} d(\lambda)^{2-2g}e^{-e^2c(\lambda)A(\Sigma_g)/2}\ .$$

The results of this section can of course also be used to calculate correlation functions of several non-intersecting Wilson loops. The intersecting case is more difficult but can be dealt with at the level of the fermionic path integral representation (6). All the considerations and methods of this paper apply to the $e^2 \rightarrow 0$ limit of Yang-Mills theory which, as a topological field theory of the type discussed in [5], has a number of additional interesting properties. Within our framework it is also straightforward to establish the relation between the Yang-Mills action and the energy-functional on the loop group [13, 14] and this allows us to make contact with the methods used by Fine [3] in the case of the two-sphere. These issues will be addressed in [6].

Note added: After having completed our investigations we were informed of a recent preprint by Witten [15] in which two-dimensional Yang-Mills theory is discussed on the basis of relations obtained in the lattice gauge theory approach. Where comparable the results of [15] and those of the present paper, based on the continuum approach, agree. We thank F. Hussain for drawing our attention to Witten's paper.

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