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CHAOS CAUSED BY A TOPOLOGICALLY MIXING MAP *

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ABSTRACT

In the present paper we show that for a topologically mixing map there exists a subset consisting of considerably many points in its domain, called chaotic subset, for which orbits of all points display time dependence greatly more erratic than for a scrambled subset, i.e., if a continuous map $f : X \to X$ is topologically mixing, where X is a separable locally compact metric space containing at least two points, then for any increasing sequence $\{p_i\}$ of positive integers there exists a c-dense subset C of X satisfying the condition for any continuous map $F : A \to X$, where A is a subset of C, there is a subsequence $\{q_i\}$ of the sequence $\{p_i\}$ such that $\lim_{i\to\infty} f^{q_i}(x) = F(x)$ for every $x \in A$. As an application we show that the interval maps having a chaotic (or scrambled) subset with full Lebesgue measure is dense in the space consisting of all topologically mixing (transitive, respectevly) maps.

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1 Introduction

Originally, Li and Yorke [4] showed the chaotic phenomenon of orbits for interval self-maps. We now draw the following definition from [4].

Definition 1.1 *Suppose $f : X \longrightarrow X$ where X is a metric space with metric d. A subset C of X is said to be* scrambled *if for any two points $x, y \in C$, with $x \neq y$,*

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad and \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

Remark 1.2 *Some authors following Li and Yorke [4] define scrambled sets with one more condition that is for any $x \in C$ and for any periodic point p*

$$\limsup_{n\to\infty} d(f^n(p), f^n(x)) > 0$$

In fact, this condition is unnecessary because a scrambled set in the sense of Definition 1.1 contains at most one point, which doesn't satisfy this condition.

Theorem 1.3 ([4]) *If a continuous map $f : I \longrightarrow I$, where $I = [0,1]$, has a periodic point with period 3, then there exists an uncountable scrambled subset of I.*

Xiong [10] discussed the erratic time dependence of orbits for a topologically mixing map and showed the following theorem.

Theorem 1.4 ([10]) *Suppose $f : X \longrightarrow X$ is a continuous map, where X is a compact metric space consisting of infinitely many points. The following conditions are equivalent.*

(1) *f is topologically mixing.*

(2) *If $\{p_i\}$ is an increasing sequence of positive integers and if S is a countable dense subset of X, then there exists a c-dense subset C of X such that*

(a) *for any $s \in S$ there is a subsequence $\{q_i\}$ of the sequence $\{p_i\}$ such that*

$$\lim_{i\to\infty} f^{q_i}(x) = s \qquad \textit{for every } x \in C$$

(b) *for any $n > 0$, for any n different points $y_1, \dots, y_n$ in C and for any map $\phi : \{y_1, \dots, y_n\} \longrightarrow X$ there exists a subsequence $\{t_i\}$ of the sequence $\{p_i\}$ such that*

$$\lim_{i\to\infty} f^{t_i}(y_j) = \phi(y_j) \qquad \textit{for } j = 1, \dots, n.$$

It is easy seen that the time dependence of orbits of points in a set C satisfying conditions (a) and (b) of Theorem 1.4 is greatly more complicated than in a scrambled set.

On the other hand Shihai Li (unpublished) and Xiong[11] point out the following

Theorem 1.5 ([**11**]) *Suppose $f : I \longrightarrow I$ is a continuous map, where $I = [0,1]$. Then the following conditions are equivalent.*

(1) *The map f has a periodic point with period which is not a power of 2.*

(2) *There is an integer $n > 0$ and a closed subset X of I invariant in respect to f^n such that $f^n|X$ is topologically mixing.*

Obviously, combined operation of Theorem 1.4 and Theorem 1.5 improved the Theorem 1.3 in some sense.

The first aim of the present paper is to strengthen Theorem 1.4. In Section 2 we will define the chaotic subsets (see Definition 2.6) for which the behaviour of orbits more complicated than for those subsets which satisfies the conditions (a) and (b) of Theorem 1.4 and will show that for a topologically mixing map (or, a weakly topologically mixing map) there exists a chaotic c-dense F_σ-subset. (See Theorem 2.8.) On the other hand, we will show that for a topologically mixing map the chaotic set can not be very large from a topological point of view. (See Theorem 2.12).

In recent years many authors, including Smital [8,9], Misiurewicz [6], and Bruchner and Hu [2] studied the measure of scrambled sets of interval self-maps.

Theorem 1.6 ([**6**]) *There exists a continuous map $f : I \longrightarrow I$, where $I = [0,1]$, for which there is a scrambled subset of I with Lebesgue measure 1.*

An interval self-map is said to be chaotic almost everywhere if there exists a scrambled set with full Lebesgue measure. In Section 4 of this paper we will show that the interval self-maps chaotic almost everywhere in a more severe sense exist 'almost everywhere' in the space consisting of all topologically mixing interval self-maps. (See Theorem 4.5.) And we will also show that the interval self-maps chaotic almost everywhere in Misiurewicz's sense exist 'almost everywhere' in the space consisting of all transitive interval self-maps. (See Theorem 4.6.)

2 Main Results

Throughout this section X denotes a topological space.

Definition 2.1 *Suppose $f : X \longrightarrow X$ is a continuous map. f is said to be* transitive *if there is an integer $n > 0$ such that*

$$f^n(U) \cap V \neq \emptyset$$

provided that U and V are non-empty open subsets of X. f is said to be weakly topologically mixing *if $f \times f$ is transitive, i.e., there is an integer $n > 0$ such that*

$$f^n(U_1) \cap V_1 \neq \emptyset \quad \textit{and} \quad f^n(U_2) \cap V_2 \neq \emptyset$$

provided that U_1, U_2, V_1 and V_2 are non-empty open subsets of X. f is said to be topologically mixing *if there is an integer $N > 0$ such that*

$$f^n(U) \cap V \neq \emptyset \qquad \text{for every } n \geq N$$

provided that U and V are non-empty subsets of X.

It is clear that a topologically mixing map is weakly topologically mixing and a weakly topologically mixing map is transitive.

One can find some elementary properties of transitive, weakly topologically mixing, and topologically mixing maps in [5] and [7], including the following

Proposition 2.2 *Suppose $f : X \longrightarrow X$ is a continuous map where X is a compact metric space. Then f is transitive if and only if there exists a point $x \in X$ such that $\omega(x, f) = X$, where $\omega(x, f)$ is the ω-limit set of x in respect to the map f.*

Suppose $f : X \longrightarrow X$ is a continuous map. For any two subsets U and V let

$$\aleph(U, V) = \{n > 0 | f^n(U) \cap V \neq \emptyset\}$$

Lemma 2.3 *If $f : X \longrightarrow X$ is transitive, then $\aleph(U, V)$ is infinite, provided that U and V are non-empty open subsets of X.*

Proof. If $n \in \aleph(U, V)$, then there is $m > 0$ such that

$$m \in \aleph(U \cap f^{-n}(V), U \cap f^{-n}(V))$$

It is easy seen that $m + n \in \aleph(U, V)$. □

Lemma 2.4 *Suppose $f : X \longrightarrow X$ is weakly topologically mixing. If U_1, U_2, V_1 and V_2 are non-empty open subsets of X, then there exist two non-empty open subsets U and V of X such that*

$$\aleph(U, V) \subset \aleph(U_1, V_1) \cap \aleph(U_2, V_2)$$

Therefore, for $n > 0$ if $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$ are non-empty subsets of X, then

$$\bigcap_{i=1}^{n} \aleph(U_i, V_i)$$

is infinity.

Proof. Since f is weakly topologically mixing, we can choose an integer $n > 0$ from the set $\aleph(U_1, U_2) \cap \aleph(V_1, V_2)$. Let

$$U = f^{-n}(U_2) \cap U_1 \quad \text{and} \quad V = f^{-n}(V_2) \cap V_1$$

For any $s \in \aleph(U, V)$ we have

$$\begin{aligned} f^s(U_1) \cap V_1 &\supset f^s(U) \cap V \neq \emptyset \\ f^s(U_2) \cap V_2 &\supset f^s(f^n(U)) \cap f^n(V) \\ &\supset f^n(f^s(U) \cap V) \neq \emptyset \end{aligned}$$

and it follows that $s \in \aleph(U_1, V_1) \cap \aleph(U_2, V_2)$. □

We need the following

Lemma 2.5 *Suppose $f : X \longrightarrow X$ is continuous. Then*

(1) *If f is topologically mixing, then for two finite families $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_\ell\}$ of open subsets of X there is an integer $N > 0$ such that*

$$f^n(A_i) \cap B_j \neq \emptyset \qquad \text{for every } n > N$$

provided that A_i and B_j are non-empty.

(2) *If f is weakly topologically mixing, then for any two finite families $\{A_1, \ldots, A_n\}$ and $\{B_1, \ldots, B_\ell\}$ of open subsets of X and for any $N > 0$ there is $n > N$ such that*

$$f^n(A_i) \cap B_j$$

provided that A_i and B_j are non-empty.

Proof. (1) is trivial and (2) comes from Lemma 2.4. □

Definition 2.6 *Suppose $f : X \longrightarrow X$ is a continuous map. Suppose $\{p_i\}$ is a given increasing sequence of positive integers.*

A subset C of X is said to be chaotic in respect to the sequence $\{p_i\}$ *if for any subset A of C and for any continuous map $F : A \longrightarrow X$ there is a subsequence $\{q_i\}$ of the sequence $\{p_i\}$ such that*

$$\lim_{i \to \infty} f^{q_i}(x) = F(x) \qquad \text{for every } x \in A$$

A subset C of X chaotic in respect to the sequence $1, 2, 3, \ldots$ if briefly said to be chaotic.

Proposition 2.7 *Suppose $f : X \longrightarrow X$ is a continuous map, where X is a metric space with metric d. Suppose C is a subset of X chaotic in respect to a given increasing sequence $\{p_i\}$, then for any two points $x, y \in C$, with $x \neq y$,*

$$\liminf_{i \to \infty} d(f^{p_i}(x), f^{p_i}(y)) = 0 \quad \text{and} \quad \limsup_{i \to \infty} d(f^{p_i}(x), f^{p_i}(y)) = |X|$$

where $|X|$ is the diameter of X.

Especially, C is a scrambled subset.

Proof. Let $A = \{x, y\}$, let $F_1 : A \longrightarrow X$ be a constant map. It comes from the definition that there is a subsequence $\{q_i\}$ of the sequence $\{q_i\}$ such that

$$\lim_{i\to\infty} f^{q_i}(x) = \lim_{i\to\infty} f^{q_i}(y)$$

Therefore,

$$\liminf_{i\to\infty} d(f^{p_i}(x), f^{p_i}(y)) = 0$$

Given $\epsilon > 0$. Take arbitrarily two points $a, b \in X$ such that

$$|X| - \epsilon \leq d(a, b) \leq |X|$$

Define $F_2 : A \longrightarrow X$ by $F_2(x) = a$ and $F_2(y) = b$. It is clear that there is a subsequence $\{t_i\}$ of the sequence $\{p_i\}$ such that

$$\lim_{i\to\infty} f^{t_i}(x) = a \quad \textit{and} \quad \lim_{i\to\infty} f^{t_i}(y) = b$$

Therefore,

$$|X| - \epsilon \leq \limsup_{i\to\infty} d(f^{p_i}(x), f^{p_i}(y)) \leq |X|$$

Hence,

$$\limsup_{i\to\infty} d(f^{p_i}(x), f^{p_i}(y)) = |X| \quad \square$$

The following Theorem 2.8 is the main result of the present paper.

Theorem 2.8 *Suppose $f : X \longrightarrow X$ is a continuous map, where X is a separable locally compact metric space containing at least two points. Then*

(1) *f is weakly topologically mixing if and only if there is a chaotic c-dense F_σ-subset of X.*

(2) *f is topologically mixing if and only if for any increasing sequence of positive integers there is a c-dense F_σ-subset of X chaotic in respect to this sequence.*

Proof. It is clear that the sufficiency parts of (1) and (2) come from the following Proposition 2.9 and the necessity parts of (1) and (2) come from Lemma 2.5 and the following Proposition 2.10. $\square$

Proposition 2.9 *Suppose $f : X \longrightarrow X$ is a continuous map.*

(1) *If there is a chaotic dense subset C of X, then f is weakly topologically mixing.*

(2) *If for any increasing sequence $\{p_i\}$ of positive integers there is a dense subset C of X chaotic in respect to the sequence $\{p_i\}$, then f is topologically mixing.*

Proof. (1) Suppose U_1, U_2, V_1 and V_2 are non-empty open subsets of X and choose $x_1 \in U_1 \cap C$, $x_2 \in U_2 \cap C$, $y_1 \in V_1$ and $y_2 \in V_2$. Let $A = \{x_1, x_2\}$ and define $F : A \longrightarrow C$ by $F(x_1) = y_1$ and $F(x_2) = y_2$. By the definition of chaotic subset there is an increasing sequence $\{q_i\}$ of positive integers such that

$$\lim_{i\to\infty} f^{q_i}(x_j) = y_j \qquad \textit{for } j = 1, 2$$

Hence,

$$f^{q_i}(U_j) \cap V_j \neq \emptyset \qquad \textit{for } j = 1, 2$$

whenever q_i large enough. This proves that f is weakly topologically mixing.

(2) Assume that f is not topologically mixing. Then there are two non-empty open subsets U and V of X such that

$$f^{p_i}(U) \cap V = \emptyset$$

for some increasing sequence $\{p_i\}$ of positive integers. By the assumption of (2) there is a dense subset C chaotic in respect to the sequence $\{p_i\}$. Choose $x \in U \cap C$ and $y \in V$. Then

$$\lim_{i\to\infty} f^{q_i}(x) = y$$

for some subsequence $\{q_i\}$ of the sequence $\{p_i\}$. Therefore,

$$f^{q_i}(U) \cap V \neq \emptyset$$

for q_i large enough. This is a contradiction. $\square$

Proposition 2.10 *Suppose $f : X \longrightarrow X$ is a continuous map, where X is a separable locally compact metric space containing at least two points. Suppose $\{p_i\}$ is an increasing sequence of positive integers. If for any two* *finite families $\{A_1, \ldots, A_m\}$ and $\{B_1, \ldots, B_\ell\}$ of open subsets of X and for any $N > 0$ there is some $p > N$ in the sequence $\{p_i\}$ such that*

$$f^p(A_i) \cap B_j \neq \emptyset$$

provided that A_i and B_j are non-empty, then there is a c-dense F_σ-subset of X which is chaotic in respect to the sequence $\{p_i\}$.

We leave the proof of this proposition in Section 3.

In the remaining part of this section we show that for a topologically mixing map any chaotic subset couldn't be very "large" from a topological point of view.

Lemma 2.11 *Suppose $f : X \longrightarrow X$ is a topologically mixing map, where X is a Hausdorff space containing at least two points. If a subset C of X satisfying that there is $x_0 \in X$ and an increasing sequence $\{q_i\}$ of positive integers such that*

$$\lim_{i\to\infty} f^{q_i}(x) = x_0 \qquad \text{for every } x \in C$$

then C is of first category (i.e., C is an union of countably many nowhere dense subsets).

Proof. Choose two non-empty open subsets U and V of X such that $x_0 \in U$ and $U \cap V = \emptyset$. For any $n > 0$ let

$$C_n = \{x \in C | f^{q_i}(x) \in U, i \geq n\}$$

It is clear that

$$C = \bigcap_{n=1}^{\infty} C_n$$

Assume that C_n is not nowhere dense for some $n > 0$. Then there exists a non-empty open subset W contained in $\overline{C}_n$. Therefore, there is $N > 0$ such that

$$f^s(W) \cap V \neq \emptyset$$

for every $s > N$. Choose an integer m from the sequence $\{q_i\}$ such that

$$m > \max\{N, q_n\}$$

Then

$$f^m(W) \subset f^m(\overline{C}_n) \subset \overline{f^m(C_n)} \subset \overline{U}$$

It follows that

$$f^m(W) \cap V = \emptyset$$

This is a contradiction. □

Theorem 2.12 *Suppose $f : X \longrightarrow X$ is a topologically mixing map, where X is a Hausdorff space containing at least two points. If C is a subset of X chaotic in respect to some increasing sequence $\{p_i\}$ of positive integers, then C is of first category.*

Especially, a chaotic subset is of first category.

Proof. Choose arbitrarily $x_0 \in X$ and let $F : C \longrightarrow X$ be the map taking the constant value x_0. Then there is a subsequence $\{q_i\}$ of the sequence $\{p_i\}$ such that

$$\lim_{i\to\infty} f^{q_i}(x) = x_0$$

for every $x \in C$. By Lemma 2.11 the subset C is of first category. □

The following easy consequences of Theorem 2.8 may be interesting.

Corollary 2.13 *If there exists a weakly topologically mixing self-map defined on a separable locally compact metric space containing at least two points, then every non-empty open subset of this space contains uncountably many points.* □

Corollary 2.14 *Suppose $f : X \longrightarrow X$ is a continuous map, where X is a separable locally compact metric space containing at least two points.*

(1) *If f is weakly topologically mixing, then for any continuous map $g : X \longrightarrow X$ there exists an increasing sequence $\{q_i\}$ of positive integers such that the following formula*

$$\lim_{i\to\infty} f^{q_i}(x) = g(x)$$

holds on a c-dense subset of X.

(2) *If f is topologically mixing, then for any continuous map $g: X \longrightarrow X$ and for any increasing sequence $\{p_i\}$ of positive integers there exists a subsequence $\{q_i\}$ of the sequence $\{p_i\}$ such that the following formula*

$$\lim_{i\to\infty} f^{q_i}(x) = g(x)$$

holds on a c-dense subset of X. □

3 Proof of Proposition 2.10

Throughout this section (X,d) denotes a separable locally compact metric space. For any subset A of X let $|A|$ denote the diameter of A, i.e.,

$$|A| = \sup\{d(x,y)|x,y \in A\}$$

Lemma 3.1 *There exists a countable base $\mathcal{U} = \{U_0, U_1, U_2, \ldots\}$ of X such that*

(1) *$U_0 = X$, and $U_n \neq \emptyset$ for every $n \geq 1$;*

(2) *$\overline{U}_n$ is compact whenever $n \geq 1$; and*

(3) *for any $i, j > 0$, the family*

$$\{U_n \in \mathcal{U} \mid U_n \cap U_i \neq \emptyset, |U_n| > \frac{1}{j}\}$$

is finite.

Proof. Let $\{V_n|n > 0\}$ be a countable open cover of X such that $\overline{V}_n$ is compact for every $n > 0$. Let $W_n = \bigcup_{j=1}^n V_j$. Let $N_1 = 1$ and define, inductively,

$$N_{n+1} = \min\{j|V_j \supset \overline{V}_{N_n}\} + N_n$$

Let $L_n = V_{N_n}$. It is easy to verify that $\{L_n|n > 0\}$ is an open cover of X, $\overline{L}_n$ is compact, and $\overline{L}_n \subset L_{n+1}$ for every $n > 0$.

Let $L_{-1} = L_0 = \emptyset$, $K_n = \overline{L}_{n+1} - L_n$ and $J_n = L_{n+2} - L_{n-1}$. We have

(a) K_n is compact and $K_n \subset J_n$,

(b) $\{K_n \mid n \geq 0\}$ and $\{J_n \mid n \geq 0\}$ are covers of X, and

(c) if $k \geq n+3$, then $J_n \cap J_k = \emptyset$.

For any $j > 0$ choose an open cover $\mathcal{V}_j$ of X such that V is non-empty, $\overline{V}$ is compact, and $|V| < 1/j$ for every $V \in \mathcal{V}_j$.

Fixed $n \geq 0$. For each $j > 0$ the open cover $\mathcal{V}_j$ has a finite subfamily $\mathcal{W}_{n,j}$ covers K_n. Then $\{W \cap J_n \mid W \in \mathcal{W}_{n,j}\}$ is a finite open cover of K_n which refines $\mathcal{V}_j$. It is easy to verify that

(d) $\mathcal{U}_j = \bigcup_{n=0}^\infty \{W \cap J_n \mid W \in \mathcal{W}_{n,j}\}$ is a countable open cover of X which refines $\mathcal{V}_j$, and

(e) for given $U \in \mathcal{U}_{j_0}$ the family $\{V \in \mathcal{U}_j \mid V \cap U \neq \emptyset\}$ is finite for every $j > 0$.

It comes immediately from (d) and (e) that

$$\mathcal{U} = \{X\} \cup (\bigcup_{j=1}^\infty \mathcal{U}_j)$$

is a countable base of X with the required properties. □

Lemma 3.2 *Suppose $\mathcal{U} = \{U_0, U_1, U_2, \ldots\}$ is a countable base of X satisfying the conditions (1), (2) and (3) listed in Lemma 3.1. Suppose $\{U_{p_i}\}$ is a sequence in $\mathcal{U}$ such that $U_{p_i} \cap U_{j_0} \neq 0$ for every $i > 0$, where $j_0 > 0$. If $\lim_{i\to\infty} p_i = \infty$, then*

$$\lim_{i\to\infty} |U_{p_i}| = 0$$

Proof. Given $\epsilon > 0$. By the Condition (3) of Lemma 3.1

$$T = \max\{j > 0 \mid U_j \cap U_{j_0} \neq \emptyset \ \textit{and} \ |U_j| > \epsilon\}$$

is a finite number. Take $N > 0$ such that $p_i > T$ whenever $i > N$. It comes that $|U_{p_i}| < \epsilon$ whenever $i > N$. □

To prove Proposition 2.10 we need more preliminaries.

For any $n > 0$, let

$$N_n = \{0, 1, \cdots, n\}$$

endowed with the discrete topology, and let

$$\Sigma = \Pi_{i=1}^\infty N_i = N_1 \times N_2 \times \ldots$$

endowed with the product topology, which is metrizable. The metric δ is defined by

$$\delta(a,b) = \frac{1}{2^i}$$

where $i = \min\{n > 0 \mid a_n \neq b_n\}$ for every $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots)$ in Σ. The space (Σ, δ) is compact.

For any $n > 0$ let Φ_n be the family of all maps from $N_1 \times N_2 \times \cdots \times N_n$ into N_n. Φ_n is a finite family consisting of $(n+1)^{(n+1)!}$ maps, say,

$$\phi_1^{(n)}, \phi_2^{(n)}, \ldots, \phi_{(n+1)^{(n+1)!}}^{(n)}$$

in which the map $\phi_1^{(n)}$ is defined by

$$\phi_1^{(n)}(a_1, a_2, \cdots, a_n) = a_n$$

for every $(a_1, a_2, \cdots, a_n) \in N_1 \times N_2 \times \cdots \times N_n$.

We denote by a_n or by $(a)_n$ the n^{th} coordinate of $a \in \Sigma$, and let $[a]_n = (a_1, a_2, \cdots, a_n)$. In this section $k_1, k_2, \ldots$ is a fixed sequence of integers defined by $k_1 = 1$ and $k_n = k_{n-1} + n^{n!}$ for every $n > 1$.

Define a map $\psi : \Sigma \longrightarrow \Sigma$ by

$$\begin{aligned}\psi(a_1, a_2, \ldots) \;=\; & (\phi_1^{(1)}(a_1), \ldots, \phi_4^{(1)}(a_1), \phi_1^{(2)}(a_1, a_2), \ldots, \phi_{3^{3!}}^{(2)}(a_1, a_2), \ldots, \\ & \phi_1^{(n)}(a_1, \ldots, a_n), \ldots, \phi_{(n+1)^{(n+1)!}}^{(n)}(a_1, \ldots, a_n), \ldots\ldots)\end{aligned}$$

i.e., if $b = \psi(a_1, a_2, \cdots,)$ and if $k_n \leq i < k_{n+1}$, then

$$b_i = \phi_{i-k_n+1}^{n}(a_1, a_2, \cdots, a_n)$$

Note that $\delta(\psi(a), \psi(b)) \leq \delta(a, b)$, for every $a, b \in \Sigma$. We have

Remark 3.3 *ψ is continuous.*

Defined another map $\lambda : \Sigma \longrightarrow \Sigma$ by $\lambda(a_1, a_2, \ldots) = (a_{k_1}, a_{k_2}, \ldots)$. It is easy to verify that

Remark 3.4 *$\lambda \circ \psi$ is the identity of Σ. Therefore, ψ is injective.*

For any $n > 0$ let $M_n = \{a \in \Sigma \mid [a]_n = (0, \ldots, 0, n)\}$, let $W_n = \psi(M_n)$ and $W = \bigcup_{n=1}^{\infty} W_n$. The following remark is also easy to prove.

Remark 3.5 *Every M_n (or W_n) is a subset both open and closed in Σ consisting of uncountable points and M_n's (or W_n's) are disjoint.*

Proof of Proposition 2.10 Choose a countable base $\mathcal{U} = \{U_0, U_1, \ldots\}$ of X satisfying the conditions (1), (2) and (3) stated in Lemma 3.1.

For any two families α and β of subsets of X let

$$\alpha \vee \beta = \{A \cap B \mid A \in \alpha, B \in \beta\}$$

and let

$$f^{-1}(A) = \{f^{-1}(A) \mid A \in \alpha\}$$

For any $i > 0$ let $\alpha_i = \{U_0, U_1, \cdots, U_i\}$. Define a sequence $\beta_1, \beta_2, \ldots$ of families of open subsets of X and a subsequence $\{r_i\}$ of the given sequence $\{p_i\}$ as follows. Let $\beta_1 = \alpha_1$ and r_1 be an element in the sequence $\{p_i\}$ such that $U \cap f^{-r_1}(V) \neq \emptyset$ for every non-empty $U \in \beta_1 \cap \alpha_2$ and for every non-empty $V \in \alpha_1$. For any $i > 1$ define, inductively,

$$\beta_i = \alpha_i \cup \{\beta_{i-1} \vee \alpha_i \vee f^{-r_i}(\alpha_i)\}$$

and $r_i > r_{i-1}$ be an element in the sequence $\{p_i\}$ such that $U \cap f^{-r_i}(V) \neq \emptyset$ for every non-empty $U \in \beta_i \vee \alpha_{i+1}$ and for every non-empty $V \in \alpha_i$.

To any $i > 0$ and any non-empty subset $V \subset X$ we assign an integer

$$\nu_i(V) = \max\{j \in N_{i+1} \mid U_j \cap V \neq \emptyset\}$$

Therefore, $U_{\nu_i(V)} \in \alpha_{i+1}$ and $U_{\nu_i(V)} \cap V \neq \emptyset$.

Fixed $n > 0$. For any $a \in \Sigma$ and for any $i > 0$ we assign to $[a]_i$ an open subset $\mu_n([a]_i)$ of X as follows. Let

$$\mu_n([a]_1) = \cdots = \mu_n([a]_n) = U_n$$

and define, inductively,

$$\mu_n([a]_i) = \mu_n([a]_{i-1}) \cap U_{\nu_i(\mu_n([a]_{i-1}))} \cap f^{-r_i}(U_{a_i})$$

for every $i > n$.

It is easy to see that $\mu_n([a]_i) \in \beta_i$ for every $i \geq n$. Given $n > 0$ and $a \in W_n$. The Conclusions (1) and (2) come from the definition immediately.

Conclusion 1 *$\mu_n([a]_i) \neq \emptyset$ for every $i > 0$.*

Conclusion 2 *We have*

$$U_n = \mu_n([a]_1) = \cdots = \mu_n([a]_n) \supset \mu_n([a]_{n+1}) \supset \mu_n([a]_{n+2}) \supset \cdots$$

Given $N > 0$. Since $\overline{U}_n$ is compact, there exist $j_s > \cdots > j_2 > j_1 > N$ such that $\bigcup_{k=1}^{s} U_{j_k} \supset \overline{U}_n$. Since $\mu_n([a]_i) \subset U_n$, we have $\mu_n([a]_i) \cap U_{j_\ell} \neq \emptyset$ for some j_ℓ. Therefore, $\nu_i(\mu_n([a]_i)) \geq j_1 > N$ whenever $i > j_s$. This shows

$$\lim_{i\to\infty} \nu_i(\mu_n([a]_i)) = \infty$$

and it follows from Lemma 3.2 that

$$\lim_{i\to\infty} | U_{\nu_i(\mu_n([a]_i))} | = 0$$

Hence, we have

Conclusion 3 $\lim_{i\to\infty} | \mu_n([a]_i) | = 0$.

Conclusion 4 $\bigcap_{i=1}^{\infty} \overline{\mu_n([a]_i)}$ *is a singleton contained in* $\overline{U}_n$.

We now define a map $\mu_n : \Sigma \longrightarrow \overline{U}_n$ by

$$\mu_n(a) = \bigcap_{i=1}^{\infty} \overline{\mu_n([a]_i)}$$

for every $a \in \Sigma$. It is clear that for any $i > n$

$$\mu_n(a) \in \overline{\mu_n([a]_i)} \subset \overline{f^{-r_i}(U_{a_i})} = f^{-r_i}(\overline{U}_{a_i})$$

so, we have

Conclusion 5 *For every* $i > n$,

$$f^{r_i}(\mu_n(a)) \in \overline{U}_{a_i}$$

Given $\epsilon > 0$. Choose $M > 0$ such that $| \mu_n([a]_M) | < \epsilon$ and choose $\epsilon_1 > 0$ small enough such that $a_i = b_i$ for $i = 1, 2, \cdots, M$, provided $\delta(a,b) < \epsilon_1$. It follows that if $\delta(a,b) < \epsilon_1$ then both $\mu_n(a)$ and $\mu_n(b)$ are in $\overline{\mu_n([a]_M)}$ and $d(\mu_n(a), \mu_n(b)) \leq \epsilon$. This proves

Conclusion 6 $\mu_n : \Sigma \longrightarrow \overline{U}_n$ *is continuous.*

Suppose $a, b \in \psi(\Sigma)$ with $a \neq b$. There is $n_0 > 0$ such that $(\lambda(a))_{n_0} \neq (\lambda(b))_{n_0}$. Choose arbitrarily $i_0, j_0 > 0$ such that $\overline{U}_{i_0} \cap \overline{U}_{j_0} \neq \emptyset$ and let $m_0 = \max\{n_0, i_0, j_0\}$. It is clear that there is $\phi^{(m_0)} \in \Phi_{m_0}$ such that $\phi^{(m_0)}([\lambda(a)]_{m_0}) = i_0$ and $\phi^{(m_0)}([\lambda(b)]_{m_0}) = j_0$. Therefore, there is $m > 0$ such that $a_m = i_0$ and $b_m = j_0$. It follows that $f^{r_m}(\mu_n(a)) \in \overline{U}_{i_0}$ and $f^{r_m}(\mu_n(b)) \in \overline{U}_{j_0}$, so that $\mu_n(a) \neq \mu_n(b)$. It follows that

Conclusion 7 $\mu_n \mid \psi(\Sigma)$ *is injective.*

Now define a map $\mu : W \longrightarrow X$ by

$$\mu \mid W_n = \mu_n \mid W_n$$

for every $n > 0$. It follows from Remark 3.5, Conclusions 6 and 7 that

Conclusion 8 $\mu : W \longrightarrow X$ *is a well-defined, continuous and injective map.*

We now show that the subset $C = \mu(W)$ of X is required.

By Remark 3.6 the set W_n is compact, so is $\mu(W_n)$. Hence,

$$C = \mu(W) = \bigcup_{i=1}^{\infty} \mu(W_n)$$

is an F_σ-subset of X.

For any $n > 0$ the set $\overline{U}_n$ contains the subset $\mu(W_n)$ of C, consisting of uncountable points. It follows that C is c-dense.

It remains to prove that C is chaotic.

Suppose $F : A \longrightarrow X$ is a continuous map, where A is a subset of C. For any $a \in \mu^{-1}(A)$ and for any $i > 0$ we define an integer $\xi_i(a)$ by

$$\xi_i(a) = \max\{j \in N_i \mid \overline{\mu_n([a]_{k_i})} \cap A \subset F^{-1}(U_j)\}$$

where n is the unique integer such that $a \in W_n$. Since $U_0 \in \mathcal{U}$, $\xi_i(a)$ is well-defined.

Conclusion 9 *For any two* $a, b \in \mu^{-1}(A)$ *if* $[\lambda(a)]_i = [\lambda(b)]_i$ *then* $\xi_i(a) = \xi_i(b)$.

Conclusion 10 $\lim_{i\to\infty} \xi_i(a) = \infty$ *for every* $a \in \mu^{-1}(A)$.

Conclusion 11 $\lim_{i\to\infty} | U_{\xi_i(a)} | = 0$ *for every* $a \in \mu^{-1}(A)$.

Conclusion 9 is clear by the definition of $\xi_i(a)$. Conclusion 11 follows from Conclusion 10 and Lemma 3.2. It is sufficient to prove Conclusion 10. Suppose $a \in \mu^{-1}(A)$. Since $\mu_n([a]_{k_i}) \supset \mu_n([a]_{k_{i+1}})$, we know that

$$\overline{\mu_n([a]_{k_i})} \cap A \subset F^{-1}(U_j)$$

implies

$$\overline{\mu_n([a]_{k_{i+1}})} \cap A \subset F^{-1}(U_j)$$

Therefore, $\xi_i(a) \leq \xi_{i+1}(a)$.

Given $N > 0$. There is $t > N$ such that $F\mu(a) \in U_t$. By Conclusion 3 We have

$$\lim_{i\to\infty} | \mu_n([a]_i) | = 0$$

so there exists some $L > t$ such that

$$\overline{\mu_n([a]_{k_L})} \cap A \subset F^{-1}(U_t)$$

It follows that $\xi_L(a) \geq t > N$ and so tnat $\xi_i(a) > N$ whenever $i > L$.

By Conclusion 9 for any $i > 0$ we can find a map $\phi^{(i)}_{j_i} \in \Phi_i$ such that $\phi^{(i)}([\lambda(a)]_i) = \xi_i(a)$, for every $a \in \mu(A)$. By the definition of ψ there is an increasing sequence ℓ_i of positive integers such that for every $a \in \mu^{-1}(A)$ and for every $i > 0$

$$a_{\ell_i} = \phi^{(i)}_{j_i}([\lambda(a)]_i) = \xi_i(a)$$

Let $q_i = r_{\ell_i}$. We now claim that the subsequence $\{q_i\}$ of the sequence $\{r_i\}$, which is a subsequence of $\{p_i\}$ is required.

If $x = \mu(a) \in A$, then by Conclusion 5

$$f^{q_i}(x) = f^{r_{\ell_i}}(\mu(a)) \in \overline{U}_{a_{\ell_i}} = \overline{U}_{\xi_i(a)}$$

for every $i > n$, where n is the unique integer such that $a \in W_n$. On the other hand, we know from the definition of μ and the definition of $\xi_i(a)$ that

$$\mu(a) \in \overline{\mu_n([a]_{k_i})} \cap A \subset F^{-1}(U_{\xi_i(a)})$$

Hence, $F(x) = F\mu(a) \in U_{\xi_i(a)}$. It follows from Conclusion 11 that

$$\lim_{i\to\infty} f^{q_i}(x) = F(x) \quad \square$$

4 Application for Interval Maps

Throughout this section I always denotes the closed unit interval $[0,1]$, $C(I)$ the metric space consisting of all continuous maps from I into itself, endowed with the metric ρ defined by

$$\rho(f,g) = \sup_{x\in I} |f(x) - g(x)|$$

for every $f, g \in C(I)$, $\omega(x,f)$ the ω-limit set of a point $x \in I$ in respect to a map $f \in C(I)$, $\mu(A)$ the Lebesgue measure of a subset A of I, and id the identity map of I.

Proposition 4.1 ([1]) *If a continuous map $f \in C(I)$ is transitive then one of the following conditions holds:*

(1) *f is topologically mixing.*

(2) *There exists a fixed point $e \in (0,1)$ such that $f([0,e]) = [e,1]$ and $f([e,1]) = [0,e]$.*

Proposition 4.2 *Suppose $f \in C(I)$. Then the following conditions are equivalent.*

(1) *f is topologically mixing.*

(2) *f is weakly topologically mixing*

Proof. (1)$\Longrightarrow$(2) Trivial.

(2)$\Longrightarrow$(1) Suppose The Condition (2) holds. It is clear that f is transitive. By the definition of weakly topologically mixing maps there exists $n > 0$ such that

$$f^n((0,1)) \cap (0,e) \neq \emptyset \quad \text{and} \quad f^n((0,e)) \cap (e,1) \neq \emptyset$$

for arbitrary point $e \in (0,1)$. This contradicts the condition (2) of Proposition 4.1. $\square$

We need a stronger form of Proposition 4.1.

Proposition 4.3 *Suppose $f \in C(I)$. Then f is transitive if and only if one of the following conditions holds:*

(i) *f is topologically mixing.*

(ii) *There is a fixed point $e \in (0,1)$ such that*

$$f([0,e]) = [e,1] \quad \textit{and} \quad f([e,1]) = [0,e]$$

and both $f^2 \mid [0,e]$ and $f^2 \mid [e,1]$ are topologically mixing.

Proof. **Necessity.** By Proposition 4.1 it is sufficient to prove that if f is transitive and if there is a fixed point $e \in (0,1)$ such that

$$f([0,e]) = [e,1] \quad \text{and} \quad f([e,1]) = [0,e]$$

then both $f^2 \mid [o,e]$ and $f^2 \mid [e,1]$ are topologically mixing. Choose $x \in I$ such that $\omega(x,f) = I$. Without loss of generality, suppose $x \in [0,e]$. Then

$$\omega(x,f^2) \subset [0,e] \quad \text{and} \quad \omega(f(x),f^2) \subset [e,1]$$

Since $I = \omega(x,f) = \omega(x,f^2) \cup \omega(f(x),f^2)$, we have

$$\omega(x,f^2) = [0,e] \quad \text{and} \quad \omega(f(x),f^2) = [e,1]$$

Therefore, both $f^2 \mid [0,e]$ and $f^2 \mid [e,1]$ are transitive. By Proposition 4.2 they are topologically mixing, because e is a fixed point of f^2.

Sufficiency. If the condition (i) holds, it is clear that f is transitive. Suppose (ii) holds. Choose a point $x \in [0,e]$ such that $\omega(x,f^2) = [0,e]$. Then

$$\begin{aligned} \omega(x,f) &= \omega(x,f^2) \cup \omega(f(x),f^2) \\ &= \omega(x,f^2) \cup f(\omega(x,f^2)) \\ &= [0,e] \cup [e,1] = I \end{aligned}$$

This shows that f is transitive. □

Proposition 4.4 ([**3**],[**2**]) *If $C \in I$ is a c-dense F_σ-subset of first category, then there is a homeomorphism $h : I \longrightarrow I$ such that*

(i) $\mu(h(C)) = 1$, *and*

(ii) $\rho(h,id) < \epsilon$.

Theorem 4.5 *Suppose $f \in C(I)$ is topologically mixing. Then there exists a topologically mixing map $g \in C(I)$, satisfying the following conditions:*

(i) $\rho(f,g) < \epsilon$, *and*

(ii) *For the map g there is a chaotic subset C of I with $\mu(C) = 1$.*
Therefore, C satisfies the condition for any two points $x, y \in C$, with $x \neq y$,

$$\liminf_{n\to\infty} |g^n(x) - g^n(y)| = 0 \quad \textit{and} \quad \limsup_{n\to\infty} |g^n(x) - g^n(y)| = 1$$

Especially the subset C is a scrambled subset.

Proof. Given $\epsilon > 0$. Since the map f is uniformly continuous, choose $\delta > 0$ such that for any $x, y \in I$,

$$| f(x) - f(y) | < \frac{\epsilon}{2}$$

whenever $| x - y | < \delta$.

Since f is topologically mixing, by Theorem 2.8 there is a chaotic c-dense F_σ-subset C_1 of I. It follows from Proposition 4.4 that there is a homeomorphism $h \in C(I)$ such that $\mu(h(C_1)) = 1$ and $\rho(h,id) < \min\{\delta, \epsilon/2\}$.

Let $g = h \circ f \circ h^{-1} \in C(I)$. It is easy to see that for the map g the subset $C = h(C_1)$ is chaotic and $\mu(C) = 1$, i.e., the map g satisfies the Condition (ii). Since g and f are topologically conjugate, g is topologically mixing.

It remains to prove that $\rho(f,g) < \epsilon$. We do this as follows:

$$\begin{aligned} \rho(f,g) &= \rho(f, h \circ f \circ h^{-1}) \\ &= \sup_{x\in I} | f(x) - h \circ f \circ h^{-1}(x) | \\ &= \sup_{x\in I} | f \circ h(x) - h \circ f(x) | \\ &\le \sup_{x\in I} | f \circ h(x) - f(x) | + \sup_{x\in I} | f(x) - h \circ f(x) | \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \square \end{aligned}$$

Theorem 4.6 *Suppose $f \in C(I)$ is transitive. Then for any $\epsilon > 0$ there exists a transitive map $g \in C(I)$ satisfying the following conditions:*

(i) $\rho(f,g) < \epsilon$.

(ii) *There exists a subset C of I with $\mu(C) = 1$ such that*

$$\liminf_{n\to\infty} |g^n(x) - g^n(y)| = 0 \quad \textit{and} \quad \limsup_{n\to\infty} |g^n(x) - g^n(y)| = \lambda$$

where $1 \geq \lambda \geq 1/2$ is a constant. Especially, C is a scrambled subset.

And if f is not topologically mixing, we can choose g such that g is not topologically mixing neither

Proof. According to Theorem 4.5, it is sufficient to prove that if the transitive map f satisfies the Condition (ii) in Proposition 4.4, then we can choose a map $g \in C(I)$ satisfying the Condition (i) and (ii) of this theorem and the Condition (ii) of Proposition 4.3.

Suppose the map f satisfies the Condition (ii) of Proposition 4.3, i.e., there is a point $e \in (0,1)$ such that $f([0,e]) = [e,1]$, $f([e,1]) = [0,e]$, and both $f^2 \mid [0,e]$ and $f^2 \mid [e,1]$ are topologically mixing. Let $\nu = \min\{e, 1-e\}$.

Given $\epsilon > 0$. Choose $\delta > 0$ such that for any $x, y \in I$ we have

$$| f(x) - f(y) | < \frac{\epsilon}{2}$$

whenever $| x - y | < \delta$.

Since $f^2 \mid [0,e]$ is topologically mixing, by Theorem 2.8 there is a chaotic c-dense F_σ-subset C_1 of $[0,e]$. Let $F : C_1 \longrightarrow [0,e]$ be the map taking constant value e. By the definition of chaotic subsets there is an increasing sequence $\{p_i\}$ of positive integers such that

$$\lim_{i\to\infty} f^{2p_i}(x) = e$$

for every $x \in C_1$. Since $f^2 \mid [e,1]$ is topologically mixing, by Theorem 2.8 there is an c-dense F_σ-subset C_2 of $[e,1]$ chaotic in respect to the sequence $\{p_i\}$. Therefore, $C_3 = C_1 \cup C_2 \subset I$ is a c-dense, F_σ-subset. By Theorem 2.12 both C_1 and C_2 are of first category, and so is C_3

By Proposition 4.4 there is a homeomorphism $h \in C(I)$ such that $\mu(h(C_3)) = 1$ and $\rho(h, id) < \min\{\delta, \frac{\epsilon}{2}, \nu\}$.

Let $g = hofoh^{-1} \in C(I)$, let $C = h(C_3)$ and let $b = h(e)$. It is easy to verify that

(a) $b \in (0,1)$ is a fixed point of the map g,

(b) $g([0,b]) = [b,1]$ and $g([b,1]) = [0,b]$,

(c) both $g^2 \mid [0,b]$ and $g^2 \mid [b,1]$ are topologically mixing.

Consequently, g is a transitive map and is not a topologically mixing map. Obviously, $\mu(C) = 1$. We can check that $\rho(f,g) < \epsilon$ just like how we did this in the proof of Theorem 4.5.

It remains to verify that

$$\liminf_{n\to\infty} |g^n(x) - g^n(y)| = 0 \quad \textit{and} \quad \limsup_{n\to\infty} |g^n(x) - g^n(y)| = \lambda$$

where $\lambda = \max\{b, 1-b\}$. It is sufficient to verify these two formulas in the case $x \in h(C_1)$ and $x \in h(C_2)$. In this case the sequence $\{g^{2p_i}(x)\}$ convergences to e and the sequence $\{g^{2p_i}(y)\}$ has a subsequence which convergences to e either, so the first formula works. On the other hand, without loss of generality, suppose $b \geq 1-b$. The sequence $\{g^{2n}(x)\}$ has a subsequence which convergences to 0 and any convergent subsequence of the sequence $\{g^{2n}(y)\}$ has to convergence to a point on the right hand of b, so the second formula holds. □

There is also an interesting consequence of Theorem 4.5.

Corollary 4.7 *There is a continuous map $f \in C(I)$ satisfying the condition for any continuous map $g \in C(I)$ and for any increasing sequence $\{p_i\}$ of positive integers there exists an subsequence $\{q_i\}$ of the sequence $\{p_i\}$ such that the formula*

$$\lim_{i\to\infty} f^{q_i}(x) = g(x)$$

holds on a subset of I with Lebesgue measure 1.

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