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NON-SINGULAR SPIKED HARMONIC OSCILLATOR

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Abstract

A perturbative study of a class of non-singular spiked harmonic oscillators defined by the hamiltonian $H = -d^2/dr^2 + r^2 + \lambda/r^\alpha$ in the domain $[0, \infty]$ is carried out, in the two extremes of a weak coupling and a strong coupling regimes. A path has been found to connect both expansions for α near 2.

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1 The fascinating world of the spiked harmonic oscillator

The spiked harmonic oscillator system is defined by the quantum hamiltonian

$$H = -\frac{d^2}{dr^2} + r^2 + \frac{\lambda}{r^\alpha} \tag{1}$$

defined in the one-dimensional half space $[0,\infty]$, the eigenfunctions obeying Dirichlet boundary conditions. Its name comes from the graphical shape of the full potential, which shows a pronounced peak near the origin for $\lambda > 0$. We will assume this condition through the paper.

This is a two parameter problem, λ and α, with physical relevance but also with fascinating properties from the point of view of mathematical physics. First of all, there is no dominance of any of the two terms of the interaction potential, r^2 and λ/r^α, for extreme values of λ. So, in the case of $\lambda \to 0$, the $1/r^\alpha$ adds an infinite repulsive barrier near the origin. On the other hand, in the $\lambda \to \infty$ limit one cannot neglect the r^2 term, the potential being in this case like a wide valley extending from 0 to ∞.

Potentials of this kind are of relevance in a wide range of physical situations, in chemical physics, nuclear physics and particle physics. However we will pay more attention to their mathematical physics interest [1]-[5]. Detwiler and Klauder [2] realized that normal perturbation theory could not be applied for values of $\alpha \geq 5/2$, and they were able to predict the kind of dependence of the ground state energy for small values of λ. Afterwards Harrell [3] developed a special perturbation theory, called *singular perturbation theory*, and obtained the first terms of the small λ expansion, which turned out to be a non-power series expansion. Special methods have been designed to compute the eigenvalues with high precision [4,5] . Estévez and the present authors [6] managed to obtain a strong coupling expansion for the ground state energy.

Our interest in this problem was motivated by Harrell's result for the ground state energy corresponding to $\alpha = 5/2$,

$$E(\alpha = 5/2, \lambda) = 3 + \frac{2\Gamma(1/4)}{\Gamma(1/2)}\lambda + \frac{16}{\Gamma(1/2)}\lambda^2 \ln\lambda + O(\lambda^2) \tag{2}$$

and the close analogy with the low-density expansion of the energy of a many-body boson system at zero temperature [7]

$$E/N = \frac{2\pi\hbar^2}{m}\rho a[1 + C_1(\rho a^3)^{1/2} + C_2\rho a^3 \ln(\rho a^3) + C_3\rho a^3 + \ldots]. \tag{3}$$

We intended to apply extrapolant methods to eq. (2) analogous to those so successfully used in the case of the low density expansion [8] to extend the range of applicability of eq. (2) to higher values of λ. We did not find the appropriate extrapolant, even if the information coming from the strong coupling expansion was used.

So we come back again to this problem, i.e., to find a bridge connecting values of the ground state energy at small λ with values at large λ, this time focusing the simpler, non-singular cases corresponding to $\alpha < 5/2$.

The paper is organized as follows. In section 2 we obtain the weak coupling expansion using standard perturbation theory up to second order. Section 3 extends the algebraic calculations of our previous work [6] so as to compute up to the tenth order of the strong perturbation expansion. Section 4 tries to get a path from the weak coupling regime to the strong coupling one. This turns out to be quite simple for $\alpha = 2$, but again we do not find the way of dealing with the other cases. However, we have found this path for values of λ in the vicinity of 2, as we show in section 5. Finally, section 6 presents a summary of the problem, as it was treated here.

2 The weak coupling perturbation expansion (WCP)

Let us split the hamiltonian (1) as usual in a H_0 part

$$H_0 = -d^2/dr^2 + r^2, r \geq 0 \tag{4}$$

and a perturbation

$$H_I = \lambda/r^\alpha \tag{5}$$

The eigenstates of H_0 are the odd-parity solutions of the one-dimensional harmonic oscillator satisfying the Dirichlet boundary condition $\psi(0) = 0$. We will represent these states by $|n\rangle$, $n = 0, 1, 2 \ldots$, their unperturbed energy being $E_n^{(0)} = 3 + 4n$.

The relevant matrix elements for the perturbative calculation up to second order are [6]

$$\langle 0|r^{-\alpha}|n\rangle = \frac{\Gamma[(3-\alpha)/2](-2)^n(\alpha/2)_n}{\sqrt{(2n+1)!}\Gamma(3/2)} \tag{6}$$

where $(a)_n$ is the Pochhammer symbol. The WCP expansion for the ground state energy results

$$E_0 = E_0^{(0)} + S_1\lambda + S_2\lambda^2 + \ldots \tag{7}$$

with

$$S_1 = \Gamma[(3-\alpha)/2]/\Gamma(3/2) \tag{8}$$

and

$$S_2 = -\frac{\Gamma^2[(3-\alpha)/2]}{\Gamma^2(3/2)} \sum_{n\neq 0} \frac{2^{2n}(\alpha/2)_n^2}{4n(2n+1)!} \tag{9}$$

provided the relevant integrals are defined, i.e., only valid for the non-singular cases where $\alpha < 5/2$. In the case $\alpha = 5/2$ the correction S_1 is well defined, but the sum appearing in eq. (9) is infinity [6]. An alternative way of writing S_2 is

$$S_2 = -\frac{\Gamma^2[(3-\alpha)/2]}{\Gamma^2(3/2)} \sum_{n\neq 0} \frac{(\alpha/2)_n^2}{4n(3/2)_n n!} \tag{10}$$

We have not found a simpler form for S_2, except for the case $\alpha = 2$, to be discussed later. The sum appearing in eq. (10) is very slowly convergent, so that to have several digits of precision it will be necessary to consider a huge amount of terms.

For practical use, it is convenient to use one of the standard methods for summing up convergent series. Let us make a comparison with another exactly summable series. Consider the sum

$$F = \sum_{n\neq 0} \frac{(\alpha/2)_n^2}{4(n+1)(3/2)_n n!} \tag{11}$$

obtained by replacing the quantity n of the denominator of eq. (10) by $(n+1)$. Let

$$G = \sum_{n\neq 0} \frac{(\alpha/2)_n^2}{4n(3/2)_n n!} \tag{12}$$

be the sum we are interested in. Obviously, for n large enough the terms of F and G will be very close. These series are connected by the expression

$$G = F + \sum_{n\neq 0} \frac{(\alpha/2)_n^2}{4n(n+1)(3/2)_n n!} \tag{13}$$

The correction series in this equation will require much less terms to be computed. Note finally that F may be written in terms of a special form of the hypergeometric function

$$F = \frac{1}{2(\alpha/2-1)^2}\left[{}_2F_1(\alpha/2-1,\alpha/2-1;1/2;1) - 1 - 2(\alpha/2-1)^2\right] \tag{14}$$

whereas the correcting series is evaluated by direct summation. In this way one obtains

$$\begin{aligned}
E_0(\alpha = 1/2) &= 3 + 1.022766\lambda - 0.0137842\lambda^2 + \ldots \\
E_0(\alpha = 1) &= 3 + 1.128379\lambda - 0.0778902\lambda^2 + \ldots \\
E_0(\alpha = 3/2) &= 3 + 1.382735\lambda - 0.334783\lambda^2 + \ldots \\
E_0(\alpha = 2) &= 3 + 2\lambda - 2\lambda^2 + \ldots
\end{aligned} \qquad (15)$$

We have also included the term corresponding to $\alpha = 2$ in the above list of results, but this term cannot be obtained by the above described technique. Actually in this case eq.(14) has no sense. However, this case may be evaluated by carrying out transformations on eq(9): use the duplication formula of the gamma function [9] to obtain $(2n+1)! = (2\pi)^{-1}2^{2n+3/2}\Gamma(n+1)\Gamma(n+1/2)$ and realize that the resulting sum is equivalent to

$$S_2(\alpha = 2) = -\frac{2}{3}\,{}_2F_1(1,1;5/2;1) = -2 \qquad (16)$$

It is not a surprise to obtain simple numbers for the perturbative expansion in the case $\alpha = 2$. Actually, the Schrödinger equation may be exactly solved in this case, as found in some textbooks on Quantum Mechanics [10]. The unnormalized eigenstates of the full hamiltonian are given by

$$\psi_n(\alpha = 2) = r^p \exp(-r^2/2)M(-n, p, r^2) \qquad (17)$$

with $p = [1 + \sqrt{1 + 4\lambda}]/2$ and M is the confluent hypergeometric function which degenerates in this case to a polynomial in r^2.

The corresponding eigenvalues are

$$E_n(\alpha = 2) = 2 + 4n + \sqrt{1 + 4\lambda} \qquad (18)$$

From this closed form for the energy one can easily obtain both a weak coupling expansion and a strong coupling expansion, namely,

$$\begin{aligned}
E_0(\alpha = 2) &= 3 + 2\lambda - 2\lambda^2 + 4\lambda^3 - 10\lambda^4 + \ldots \\
E_0(\alpha = 2) &= 2 + 2\sqrt{\lambda}[1 + 1/8\lambda - 1/128\lambda^2 + 1/1024\lambda^3 - 5/32768\lambda^4 + \ldots]
\end{aligned} \qquad (19)$$

the first expansion being valid for $|\lambda| < 1/4$ whereas the second converges for $|\lambda| > 1/4$.

Before ending this section we would like to mention a question regarding these cases of non-singular perturbation theory. The problem is that whereas the resulting expansion for the energy is neatly obtained (even more, it agrees with the numerically determined eigenvalues for small λ), the perturbed wave functions cannot be always obtained in the same way. Dealing specifically with the $\alpha = 2$ case we have for the first correction to the wave function

$$\psi_0^{(1)}(\alpha = 2) = \frac{\sqrt{\Gamma(1/2)}\exp(-r^2/2)}{4\Gamma(3/2)} \sum_{j\neq 0} \frac{(j-1)!(-1)^j H_{2j+1}(r)}{(2j+1)!} \tag{20}$$

which may be rearranged in a power series in r by using the explicit form of the Hermite polynomials. It results

$$\psi_0^{(1)}(\alpha = 2) = \frac{\sqrt{\Gamma(1/2)}\exp(-r^2/2)}{4\Gamma(3/2)} \sum_{j\neq 0}(j-1)! \sum_{m=0}^{j} \frac{(-1)^m (2r)^{2m+1}}{(j-m)!(2m+1)!} \tag{21}$$

By collecting the terms of each given power of r in the double sum of eq.(21) it turns out that each coefficient is infinity. For example, the r term of this double sum has the coefficient $2\sum_{j=1}^{\infty} 1/j$, i.e., the harmonic series. This is a quite surprising result, having a properly defined expansion for the energy eigenvalues but an inappropriate expansion for the wave function. In other words, at least for $\alpha = 2$ the *standard* perturbation theory is in fact a *singular* perturbation theory in Harrell's sense (2).

This result for the wave function is easily understood if one expands the exact wave function in powers of λ. One has to use the wave function (17) normalized, because the norm depends also on λ. For n=0 we have

$$\psi_0(\alpha = 2) = \left(\frac{2}{\Gamma\left(\frac{2+\sqrt{1+4\lambda}}{2}\right)}\right)^{1/2} r^{(1+\sqrt{1+4\lambda})/2}\exp(-r^2/2) \tag{22}$$

and the first order perturbative correction is

$$\psi_0^{(1)} = 2\pi^{-1/4}\lambda r\left(\ln r - \frac{1}{2}\psi(3/2)\right)\exp(-r^2/2) \tag{23}$$

the coefficient of the exp term is no longer a polynomial in r. This explains the strange result of the perturbation expansion for the wave function, but nevertheless does not clarify why the energy has a regular expansion. It would be interesting to find a special summation formula for eq.(21) which could produce the non-power series expansion of eq.(23)

3 The strong coupling perturbation expansion (SC

Probably the most popular case of a SCP expansion is the anharmonic oscillator $x^2 + \lambda x^4$, which is also described in quantum mechanics textbooks. When λ is very large, the x^2 term can be neglected with respect to the perturbation λx^4, and a scaling in the coordinate leads to the conclusion that the energy in the $\lambda \to \infty$ limit is given by $\epsilon\lambda^{1/3}$, where ϵ is the eigenvalue of the operator $-d^2/dx^2 + x^4$ [11].

In our case, however, there is not such a simple transition. Even for large values of the coupling constant both terms of the interaction, x^2 and λ/x^α, are important. A special way of dealing with the strong coupling limit of the spiked oscillators was presented in full detail in our previous work [6]. The procedure is based in an idea borrowed from Witten's study of hydrogenic atoms in the so called $1/N$ expansion [12], where N is the number of dimensions of the space. The system is described starting from the classical equilibrium point, i.e., the minimum of the full potential, and afterwards the quantum corrections and the residual interaction are added up. In practice this means that there are a base or zero energy given by the value of the potential at the minimum $V(r_{min})$, a zero order hamiltonian $H_0 = -d^2/dx^2 + \omega^2 x^2$ of harmonic type, ω being related to the second derivative of the potential at the minimum, and finally anharmonic corrections x^3, x^4, ... In this description x is the distance measured from the classical equilibrium point, $x = r - r_{min}$.

In this way there results an expansion of the energy in powers of the quantity

$$\mu = (2/\lambda\alpha)^{1/(\alpha+2)} \tag{24}$$

of the form

$$E(\mu) = E_{-2}\mu^{-2} + E_0 + E_2\mu^2 + E_4\mu^4 + \dots \tag{25}$$

i.e., involving non-integral powers of the coupling constant λ. The way of computing the coefficients E_n as well as some of them (up to E_4) was presented in Ref. [6]. The calculation was greatly reduced by using computer algebraic manipulation codes, but, perhaps due to our lack of experience with this technology, we were only able to compute up to E_4 before running out of memory. Some improvements have permitted us to arrive up to E_{10}.

The results are simpler to display (and compute) in terms of the parameter

$$\omega = \sqrt{\alpha + 2} \tag{26}$$

and are listed below

$$
\begin{aligned}
E_{-2} &= \omega^2/(\omega^2 - 2) \\
E_0 &= \omega \\
E_2 &= (-\omega^4 + 11\omega^2 - 10)/72 \\
E_4 &= (\omega^8 - 10\omega^6 - 39\omega^4 + 320\omega^2 - 272)/1728\omega \\
E_6 &= (-317\omega^{12} + 2847\omega^{10} + 29661\omega^8 - 162779\omega^6 - 793236\omega^4 \\
&\quad +4130832\omega^2 - 3207008)/9331200\omega^2 \\
E_8 &= (2177\omega^{16} - 8692\omega^{14} - 551578\omega^{12} + 2902556\omega^{10} + 14552897\omega^8 \\
&\quad -566299936\omega^6 - 324369568\omega^4 + 1337123072\omega^2 \\
&\quad -973020928)/895795200\omega^3 \\
E_{10} &= (-624689\omega^{20} - 5466679\omega^{18} + 363722706\omega^{16} - 1117832430\omega^{14} \\
&\quad -22741915965\omega^{12} + 104066243541\omega^{10} + 265411385292\omega^8 - \\
&\quad 1017106275936\omega^6 - 5381642164608\omega^4 + 20151510274304\omega^2 \\
&\quad -14098737345536)/3160365465600\omega^4
\end{aligned}
\qquad (27)
$$

In Table 1 we show the values of these coefficients for some particular values of α which correspond to non-singular perturbation cases. These values are of interest for further analytic study of the series, but it does not help very much taking a quick look at the tendency of these coefficients. So in Table 2 we have included the values of the same set of coefficients, this time computed in FORTRAN with a floating-point representation. The table extends up to E_{20}. Note that with the exception of the case $\alpha = 2$, the coefficients behave rather erratically with the order.

It turns out that this perturbation expansion is abnormal. The eigenvalues obtained with this method, as we have shown in [6], do actually give a very good value of the ground state energies for very large λ, but the radius of convergence of the SCP expansions is not known (obviously, with the exception of the case $\alpha = 2$). On the other hand, we know that the expansion for the wave function is not convergent because the successive corrections extend also into the interval $[-\infty, 0]$. Note that we are dealing in this SCP expansion with a one-dimensional harmonic oscillator problem extending in all the space. So we are again facing an expansion reasonable for the energy but inappropriate for the wave function.

4 Is there a path from $\lambda \to \infty$ to $\lambda \to 0$?

The content of this section is quite speculative. We know few terms of the WCP expansion and also several terms of the SCP expansion. On the other

α	1/2	1	3/2	2	5/2
E_{-2}	5	3	$\frac{7}{3}$	2	$\frac{9}{5}$
E_0	$\sqrt{\frac{5}{2}}$	$\sqrt{3}$	$\sqrt{\frac{7}{2}}$	2	$\frac{3}{\sqrt{2}}$
E_2	$\frac{5}{32}$	$\frac{7}{36}$	$\frac{65}{288}$	$\frac{1}{4}$	$\frac{77}{288}$
E_4	$\frac{99}{512\sqrt{10}}$	$\frac{37}{432\sqrt{3}}$	$\frac{1465}{13824\sqrt{14}}$	0	$\frac{-1967}{41472\sqrt{2}}$
E_6	$\frac{85877}{2048000}$	$\frac{2573}{139968}$	$\frac{-29921}{11943936}$	$\frac{-1}{64}$	$\frac{-46748387}{2687385600}$
E_8	$\frac{20240451}{131072000\sqrt{10}}$	$\frac{168233}{6718464\sqrt{3}}$	$\frac{-73238695}{16052649984\sqrt{14}}$	0	$\frac{25567506209}{1547934106600\sqrt{2}}$
E_{10}	$\frac{1673843203}{20971520000}$	$\frac{11834297}{725594112}$	$\frac{-1085434210835}{1585744976019456}$	$\frac{1}{512}$	$\frac{1483965461593}{1337415067238400}$

Table 1: The coefficients of the SCP expansion up to order ten in rational (or irrational) form for several values of the exponent α from 1/2 up to 5/2.

N	α = 1/2	α = 1	α = 3/2	α = 2	α = 5/2
-2	5.00000000	3.00000000	2.33333333	2.00000000	1.8000000000
0	1.58113883	1.73205081	1.87082869	2.00000000	2.12132034
2	0.15625000	0.19444444	0.22569444	0.25000000	0.26736111
4	0.06114560	0.04944898	0.02832304	0	-0.03353779
6	0.04193213	0.01838277	-0.00250512	-0.01562500	-0.01739549
8	0.04883265	0.01445708	-0.00121935	0	0.01167941
10	0.07981506	0.01630980	-0.00068449	0.00195312	0.00110958
12	0.16418623	0.02190383	-0.00131199	0	-0.00682551
14	0.40470360	0.03492499	-0.00116044	-0.00030518	0.00254260
16	1.16029548	0.06508549	-0.00044473	0	0.00817874
18	3.78818681	0.13837039	0.00005696	0.00005341	-0.01109314
20	13.86291828	0.32974546	-0.00002870	0	-0.01442769

Table 2: The coefficients of the SCP expansion up to order twenty in floating form for several values of the exponent α from 1/2 up to 5/2 . All digits are exact but the last, which may have been rounded.

hand, except for the case $\alpha = 2$, we do not know the radii of convergence of these expansions. The question is if it is possible to find a constructive way of connecting both expansions and, still better, to get an estimate (even if approximate) of the ground state energy valid for all regimes of the coupling constant.

The case of $\alpha = 2$ is quite illuminating because in this case there is a simple and obvious way of carrying out this connection. Going back to the second of eqs. (19), which is a particular way of writing down the SCP expansion, we square the series in brackets and take the square root of the result

$$E_0(\alpha = 2) = 2+2\sqrt{\lambda}\left[\left[1 + 1/8\lambda - 1/128\lambda^2 + 1/1024\lambda^3 - 5/32768\lambda^4 + \ldots\right]^2\right]^{1/2} \tag{28}$$

Computing the square of the inner bracket produces, within the known coefficients, the very simple result of $1 + 1/4\lambda$ so that one ends up with

$$E_0(\alpha = 2) = 2 + 2\sqrt{\lambda}\left(1 + \frac{1}{4\lambda}\right)^{1/2} \tag{29}$$

which is the exact result quoted in (18). Of course, the manipulation of this series may seem speculative, but it is a common useful practice in the field of series extrapolation. Actually there are two steps involved in this procedure. First, the explicit presence of non-integral powers is removed by using the trick of rising the series to some adequate power, and maintaining frozen (i.e. not expanding) the compensating inverse power. The second step, which was not used here, is to construct Padé or similar extrapolants to the transformed series, so as to have still control of the (known) dominating power in the limit of interest. This technique has been largely applied in different physical systems as 2-species-fermion hard-spheres [13] , fully interacting fermion gases [14] , and Lennard-Jones boson fluids [15] .

We now try to apply the same procedure to other cases. The SCP series has only even powers of the effective coupling constant $\mu = (2/\lambda\alpha)^{1/(\alpha+2)}$ so the simpler case will correspond to $\alpha = 1$. The actual series for the energy has the form

$$\begin{aligned} E(\alpha = 1) \;=\; & E_{-2}(2/\lambda)^{-2/3} + E_0 + E_2(2/\lambda)^{2/3} + E_4(2/\lambda)^{4/3} + \\ & E_6(2/\lambda)^{6/3} + E_8(2/\lambda)^{8/3} + E_{10}(2/\lambda)^{10/3} + \ldots \end{aligned} \tag{30}$$

with E_n given in Tables 1 and 2. The most immediate way of getting rid of the non integral powers parallels the $\alpha = 2$ case. Consider the dominant term with coefficient E_{-2}. The next exponent differing from 2/3 in an integer is the term E_4, the next one is E_{10}, and so on. So we may extract a sub-series from eq.(30)

$$\left(\frac{\lambda}{2}\right)^{2/3}\left[E_{-2}+E_4(2/\lambda)^2+E_{10}(2/\lambda)^4+\ldots\right] \tag{31}$$

and the way of having only integral powers of λ is to raise the square bracket to the power 3 and later take the cubic root, or some other combination like 2/3 and 3/2.

Something similar may be done with the other two subseries contained in eq.(30), namely the series containing E_0, E_6, ... and the series E_2, E_8, ... Unfortunately the method does not work. In fact, at the end there will result expressions depending upon λ^2 that will not be able to generate the WCP expansion which contains also odd powers of λ. In addition, this mechanism of extracting subseries will depend critically on the value of α and on the congruencies of the exponents $2n/(\alpha+2)$.

In conclusion, some new kind of extrapolation mechanism should be devised to deal with this analytic continuation. Certainly, there remains still the question of the existence of such a path connecting the SCP and the WCP expansion. This question is analyzed to some extent in the next section.

5 A two-parameter perturbation expansion

The key point of this section is that because we do not know what to do with one-parameter expansion, let us jump forward and add an extra parameter. Again we are inspired by the many-body problem, specifically in the so called van der Waals or Quantum Thermodynamic Perturbation Theory (also applied in the study of classical fluids), in which the low-density expansion of the energy per particle is expanded again by splitting the interaction into a strongly repulsive part and an attractive part switched on by means of the new expansion parameter [16,17,18,19] . The resulting double series is afterwards analyzed by constructive methods.

Our idea is to expand the perturbative series around the values $\alpha = 2$, where we know the way of connecting the two perturbation regimes. So consider $\alpha = 2 + \eta$ and expand both the SCP and the WCP expansion up to first order in η. From eq.(24) we readily get

$$\mu = \lambda^{-1/4}\left[1 + \left(\frac{\ln\lambda}{16} - \frac{1}{8}\right)\eta + \ldots\right] \tag{32}$$

and the expansions (27) of the E_n in terms of η turn out to be

$$\begin{aligned} E_{-2} &= 2 - \eta/2 + \ldots \\ E_0 &= 2 + \eta/4 + \ldots \\ E_2 &= 1/4 + \eta/24 + \ldots \\ E_4 &= -\eta/16 + \ldots \\ E_6 &= -1/64 - 61\eta/3840 + \ldots \\ E_8 &= \eta/64 + \ldots \\ E_{10} &= 1/512 + 527\eta/64512 + \ldots \end{aligned} \tag{33}$$

From here we may obtain the new SCP expansion, which with some special ordering of its terms turns out to be

$$\begin{aligned} E = {} & 2 + 2\sqrt{\lambda}\,[1 + 1/8\lambda - 1/128\lambda^2 + 1/1024\lambda^3 + \ldots] + \\ & \eta\sqrt{\lambda}\ln\lambda\,[-1/4 + 1/32\lambda - 3/512\lambda^2 + 5/4096\lambda^3 + \ldots] + \\ & \eta\,[1/4 - 1/16\lambda + 1/64\lambda^2 + \ldots] + \\ & \frac{\eta}{\sqrt{\lambda}}\,[-1/48 - 1/240\lambda + 739/129024\lambda^2 + \ldots] + O(\eta^2) \end{aligned} \tag{34}$$

valid for small η and large λ. The second term appearing in the r.h.s. of eq.(34) is our previous series (28), so that, for $\lambda > -1/4$, its value is $\sqrt{1+4\lambda}$. The series of the third term can be recognized as $-1/4\sqrt{1+1/4\lambda}$, so that the third term becomes

$$-\frac{1}{2}\eta\ln\lambda\frac{\lambda}{\sqrt{1+4\lambda}} \tag{35}$$

which we guess to be valid for $\lambda > 0$. Something analogous happens with the fourth term, its value being $\eta\lambda/(1+4\lambda)$. Unfortunately neither chance nor intuition suggest any closed form for the last series.

There is however an alternate approach to this two-parameter expansion problem, which consists in using standard perturbation theory to compute $E(2+\eta)$, at least up to first order in η and *valid for all physical values of* λ. Take now as the unperturbed hamiltonian

$$H_0 = -d^2/dx^2 + r^2 + \lambda/r^2 \tag{36}$$

and use as perturbation

$$H_I = -\lambda\eta r^{-2}\ln r \tag{37}$$

which is the $O(\eta)$ term of $\lambda(1/r^{2+\eta} - 1/r^2)$. Note that due to the particular dependence on λ of the perturbation (37), which is linked to η in the form $\lambda\eta$, a perturbative calculation at first order in η will be valid for all values of λ, provided the integrals are well defined. The final result will be valid for $\lambda > -1/4$, as it will be shown.

One can compute the $O(\eta)$ correction by direct integration of H_I times the square of the ground state wave function mentioned in section 2, eq.(22). However, the presence of the logarithm in H_I invites to search for an alternate procedure. It turns out to be much simpler to get the first order correction by expanding in powers of η the expectation value of the full perturbation $(1/r^{2+\eta} - 1/r^2)$. This is a valid procedure for the first order correction, but it will not be appropriate for higher orders. One obtains in this way

$$\langle 1/r^{2+\eta}\rangle - \langle 1/r^2\rangle = \frac{2}{\sqrt{1+4\lambda}}\left[\frac{\Gamma([(\sqrt{1+4\lambda}-\eta)/2]}{\Gamma([\sqrt{1+4\lambda}/2]} - 1\right] \tag{38}$$

and the energy is given by

$$E = 2 + \sqrt{1+4\lambda} - \frac{\eta\lambda}{\sqrt{1+4\lambda}}\psi\left(\sqrt{1+4\lambda}/2\right) + O(\eta^2) \tag{39}$$

where ψ is the digamma function [9] .

The importance of the above equation is that it is the link between the SCP regime and the WCP regime, at least in some region around $\alpha = 2$. From equation (39) one can obtain the new form of the WCP expansion corresponding to eqs. (15) , with the result

$$E(2+\eta) = 3 + \lambda(2 - \eta\psi(1/2)) + \lambda^2(-2 + 2\eta\psi(1/2) - \eta\psi'(1/2)) + \dots \tag{40}$$

It is quite easy to check that the coefficient of λ coincides with the expansion of S_1, eq.(8). However, it is quite cumbersome to obtain directly the coefficient of λ^2 from the value of S_2, eq.(9).

More interesting is the case of large λ, i.e., the SCP regime. It is not easy to obtain the expansion of eq.(39) for large λ. Terms like $\sqrt{1+4\lambda}$ are transformed as

$$\begin{aligned}\sqrt{1+4\lambda} &= 2\sqrt{\lambda}\left(1+\frac{1}{4\lambda}\right)^{1/2} \\ &= 2\sqrt{\lambda}\,(1 + 1/8\lambda - 1/128\lambda^2 + 1/1024\lambda^3 + \dots)\end{aligned} \tag{41}$$

resulting in an expansion in non-integral powers of λ. The trouble comes with the digamma function. It is convenient to use the asymptotic expansion [9]

$$\psi(z) = \ln z - 1/2z - 1/12z^2 + 1/120z^4 - 1/252z^6 + \dots \qquad (42)$$

with

$$z = \sqrt{\lambda}\left(1 + \frac{1}{4\lambda}\right)^{1/2} \qquad (43)$$

expanded as above, and reexpand each of the terms of the digamma asymptotic expansion. The final result of all this procedure is, as expected, exactly eq.(34).

So, to conclude this section, we have at least found an expansion of the ground state energy, given in eq(39), which serves as a bridge between the $\lambda \to 0$ and the $\lambda \to \infty$ regimes. This representation, however, is only valid in some region around $\alpha = 2$. One then may guess that there must exist a constructive method to go from the SCP to the WCP regimes. To find it makes up an interesting challenge.

6 Summary

In this work we have obtained several results concerning the family of non-singular spiked harmonic oscillators, as described below.

1. The weak coupling expansions for several values of the exponent α of the repulsive barrier, eqs.(15).

2. The strong coupling expansion up to tenth order, in algebraic form.

3. A constructive method to connect both expansions for arbitrary coupling constant λ in a region around $\alpha = 2$.

However some new questions have arisen regarding this family of potentials, and we will single out two of them.

1. The abnormal behavior of the standard, weak coupling perturbation theory, when trying to get the first correction to the wave function, and

2. Our failure in obtaining a general constructive method to connect the two perturbative regimes for general values of the exponent α of the perturbation.

We hope our work will encourage further research, not only in the simpler case of non-singular interactions, but also in the more challenging situation of singular perturbative cases.

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