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NON-PERTURBATIVE SUPERSYMMETRY ANOMALY IN SUPERSYMMETRIC QCD

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ABSTRACT

The zero modes of the Dirac operator in an instanton and other topologically non-trivial backgrounds are unstable in a large class of massless or partially massless supersymmetric gauge theories. We show that under a generic perturbation of the scalar fields *all* zero modes become resonances, and discuss the ensuing breakdown of conventional perturbation theory. As a result, despite of the presence of massless fermions, the field theoretic tunneling amplitude is not suppressed. In massless supersymmetric QCD with $N_c \le N_f$ the effective potential is found to be *negative* and monotonically increasing in the weak coupling regime for scalar VEVs which lie on the perturbatively flat directions. Consequently, massless supersymmetric QCD with $N_c \le N_f$ exhibits a non-perturbative supersymmetry anomaly and exists in a strongly interacting phase which closely resembles ordinary QCD. The same conclusions apply if small masses are added to the lagrangian and the massless limit is smooth.

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1. Introduction

The vacuum energy density of N=1 supersymmetric (SUSY) gauge theories vanishes to all orders in perturbation theory, if SUSY is not broken at the classical level. The exact vacuum energy may nevertheless be non-zero because of non-perturbative effects. A prominent non-perturbative effect relevant to both weakly and strongly interacting theories is quantum field tunneling as described by the BPST instanton [1]. Let us write the path integral in the one instanton sector as

$$Z_1 = \mathcal{V}\Omega\,, \tag{1.1}$$

were $\mathcal{V}$ is the euclidean space volume. As a result of tunneling the vacuum energy density $\mathcal{E}$ is lowered by $2|\Omega|$, and in SUSY theories one has (see e.g. ref. [4] for more details)

$$\mathcal{E} = -2|\Omega|\,. \tag{1.2}$$

As is well known, the SUSY algebra

$$\mathrm{tr}\{Q^\dagger, Q\} = H \tag{1.3}$$

is incompatible with a negative vacuum energy. The SUSY algebra can be reconciled with eq. (1.2) only if $\Omega = 0$ holds as an exact statement.

In the semi-classical approximation Ω indeed vanishes because of the existence of fermionic zero modes [2]. Already at this stage, however, an intriguing question arises. The zero modes exist primarily because of the topology of the instanton field and the masslessness of the fermions. They are all left-handed and naive application of the index theorem correctly predicts their number. But there are usually more zero modes than would have been anticipated on the grounds of SUSY alone.

The path integral representing Ω (to be discussed in detail below) has the general structure

$$\Omega = \int [d\beta] Det^{1/2}\{D_F(\beta)\}\, e^{-S_B}\,. \tag{1.4}$$

Here β is a generic name for all Bose fields and $[d\beta]$ is the appropriate bosonic measure. A possible way to prove that $\Omega = 0$ identically would be to show that the Dirac operator $D_F(\beta)$ has zero modes for any finite action configuration of the bosonic fields in the instanton sector. This would imply that the integrand of the bosonic functional integration vanishes identically and hence that Ω exactly vanishes.

Consider first the simplest case, namely, SUSY Yang-Mills theories. All gauge field configurations in the instanton sector are continuously connected to the instanton field. They should all have the same non-zero index and hence the Dirac operator should have *some* zero modes for all configurations of the gauge field.

We next turn to a reacher and more interesting class of theories - supersymmetric QCD. SQCD contains scalar fields and so the Dirac operator D_F no longer anticommutes with γ_5 once these fields are turned on. However, even in the presence of the scalar fields D_F anticommutes with $\gamma_5 Q_R$, were Q_R generates the anomalous R-symmetry. We refer to the two possible eigenvalues of $\gamma_5 Q_R$ as positive and negative generalized chiralities. The gaugino and quark fields have opposite charges under the action of Q_R while all zero modes are left handed. Thus, the gaugino zero modes and quark zero modes have opposite generalized chiralities. We arbitrarily asign a positive generalized chirality to the gaugino zero modes.

Having realized that the Dirac operator has a well defined index even in SQCD we may now attempt to show that $\Omega=0$ using a similar argument to the super Yang-Mills case. However, in SQCD one has zero modes of both generalized chiralities and in some cases the index is zero. One expects that when the index is zero there will exist bosonic configurations without any fermionic zero modes. Thus, we can no longer hope that the above reasoning will work in all cases.

Nevertheless, let us focus our attention on those cases were the index of D_F is non-zero on the classical solution. Is it true that D_F has the same index (and hence *some* zero modes) for any configuration of the gauge and scalar fields in the instanton sector? Surprisingly, the answer is no! In fact, as has been recently shown by Casher and the author [5], in a large class of SUSY theories (to be specified below) the index of D_F is not conserved. Moreover, for a generic configuration of the bosonic fields D_F has *no* zero modes at all.

We would like to emphasize that this result is not in conflict with any existing index theorem. Index theorems apply to compact spaces [7] as well as to massive theories in open spaces [8]. But, for reasons that will be clarified later, there is no index theorem applicable to massless or partially massless theories in euclidean space[1].

Considering the crucial role of the fermionic zero modes in protecting SUSY on the semi-classical level, one can anticipate that their instability may have far reaching consequences. Indeed, in ref. [5] a qualitative argument was given that the tunneling amplitude is positive definite in any weakly interacting[2] vector-like SUSY

[1]The APS index theorem [7] applies to compact spaces with boundaries but requires the so-called spectral boundary conditions. The infinite volume limit of spaces endowed with these non-local boundary conditions does not reproduce a physical continuum system. The subtleties involved in defining and calculating the index on compact spaces with *local* boundary conditions are discussed in ref. [9].

[2]this assumption is needed in order to suppress infra-red divergences and to allow for a reliable computation.

theory whose fermionic zero modes are unstable. However, a detailed quantitative calculation was not carried out.

In this paper we fill that gap. We show that under a generic perturbation of the scalar fields all fermionic zero modes become resonances. The dependence of the resonances' energy on the scalar perturbation is found, and is used to obtain an explicit expression for the tunneling amplitude Ω. The results confirm the qualitative arguments of ref. [5] and reveal a wealth of non-analytic features which are beyond the reach of conventional perturbation theory.

This paper is organized as follows. After introducing the necessary technical tools in sect. 2, we review in sect. 3 the instability of the fermionic zero modes [5], adding some new details needed for the discussion of the resonances. We emphasize the physical origin this instability and the ensuing breakdown of conventional perturbation theory. The conditions under which this phenomenon takes place are discussed. Fermionic zero modes are unstable against scalar perturbations whenever (a) some fermions are massless and (b) the Dirac operator has zero modes of both generalized chiralities in a rotationally invariant background. The scalar modes responsible for the instability of the fermionic zero modes (which we call destabilizing modes) are found. A particular fermionic zero mode will be destabilized by any one of the scalar modes which couple it to a zero mode of the opposite generalized chirality.

In sect. 4 we discuss the resonances in detail. Simple physical considerations allow us to predict the existence of resonances and to estimate their energy and their width. A more detailed analysis of the low energy spectrum which provides the full dependence of the resonances' energy on the amplitudes of the destabilizing modes is then carried out, while most technical details are left to an appendix.

In sect. 5 the information obtained on the properties of the resonances is used to explicitly calculate the tunneling amplitude. The result is valid for weakly coupled vector-like SUSY gauge theories whose fermionic zero modes are unstable. For the case of an SU(2) gauge group (the case $N_c > 2$ requires minor modifications) it takes the form

$$\Omega(v) = B_1 g^{-8} \int_0^\infty d\rho\, \rho^{-5+n_+ + n_-}\, e^{-\frac{8\pi^2}{g^2(\rho)} - B_2 \rho^2 v^2} \int d^{n_+ n_-}\xi\, d^{n_+ n_-}\xi^* \times$$

$$e^{-\xi^\dagger H(\rho,v)\xi - W[\rho,v,\xi]} \prod_{i=1}^{n_+} E_i^+(\rho,v,\xi) \prod_{j=1}^{n_-} E_j^-(\rho,v,\xi)\,, \qquad (1.5)$$

$$E_i^+(\rho,v,\xi) = g\left(\sum_{j=1}^{n_-} |C_{ij}\xi_{ij}|^2\right)^{\frac{1}{2}} + O(g^2)\,, \qquad (1.6a)$$

$$E_j^-(\rho, v, \xi) = g\left(\sum_{i=1}^{n_+} |C_{ij}\xi_{ij}|^2\right)^{\frac{1}{2}} + O(g^2)\,. \qquad (1.6b)$$

In this expression v is the scalar expectation value which breaks the gauge symmetry and ρ is the instanton size. B_1 and B_2 are positive constants. n_+ (n_-) is the number of *pairs* of zero modes with positive (negative) generalized chirality which exist in the unperturbed rotationally invariant classical background. The complex n_+n_--dimensional vector ξ represents the amplitudes of the destabilizing modes. Specifically, ξ_{ij} is the amplitude of the scalar mode which couples the i-th pair of zero modes with positive generalized chirality to the j-th pair of negative generalized chirality ones. $E_i^+(\rho, v, \xi)$ is the energy of the resonances that originates from the i-th pair of positive generalized chirality zero modes. Similarly, $E_j^-(\rho, v, \xi)$ is the energy of the resonances that originates from the j-th pair of negative generalized chirality zero modes. The positive numbers C_{ij} are normalization constants. The hermitian quadratic form $H(\rho, v)$ is positive definite. It arises from splitting the domain of the bosonic hamiltonian into the subspace spanned by the destabilizing modes and an orthogonal subspace. $W[\rho, v, \xi]$ is the sum of connected bubble diagrams calculated with the Feynman roles given in sect. 5. Because of the vector-like nature of the theory $W[\rho, v, \xi]$ is real.

Eqs. (1.5) and (1.6) and their generalization to $N_c > 2$ are the main results of this paper. The integrand of the ξ-integral is strictly positive almost everywhere and so $\Omega(v) > 0$ holds as an exact statement. (The integrand vanishes on the union of the $n_+ + n_-$ lower dimensional subspaces defined by $E_i^+(\rho, v, \xi) = 0$ or $E_j^-(\rho, v, \xi) = 0$). An approximate expression for Ω in what might be called an improved semi-classical approximation can be obtained by neglecting the $O(g^2)$ terms in eq. (1.6) as well as $W[\rho, v, \xi]$. In the SU(2) case, the result is

$$\Omega(v) \approx g^{-8+n_++n_-} v^4 \exp(-8\pi^2/g^2(v))\,. \qquad (1.7)$$

Under ordinary circumstances the destabilizing modes – like any other fluctuation mode – would be included in the bosonic propagator and their effect would be calculated using conventional perturbation theory. In the present case the bosonic propagator does not contain the destabilizing modes and their effect on the fermionic spectrum is calculated using alternative methods. This unconventional treatment is necessary because, considered as functions of the amplitudes of the destabilizing modes, the fermionic spectrum and the fermionic propagator exhibit a discontinuity when a zero mode disappears from the spectrum. The fermionic determinant is still continuous in g and ξ but non-analytic in g^2. Thus, conventional perturbation theory breaks down.

In sect. 6 we apply the general formulae of sect. 5 to SQCD. In the massless case we have to consider all possible scalar VEVs which lie on the flat directions [14]. The necessary conditions for the instability of the zero modes are always fulfilled in SQCD when the scalar VEVs vanish, as well as in SQCD with $N_c \leq N_f$ for any scalar VEV. Eq. (1.2) is a good approximation of the full scalar potential in the weak coupling regime, namely, when the scalar VEVs are much larger than Λ_{SQCD}. Consequently, the effective potential of massless SQCD with $N_c \leq N_f$ is negative and monotonically increasing in the weak coupling regime (see fig. 5). As a result, massless SQCD with $N_c \leq N_f$ has a ***negative*** vacuum energy and exists in a strongly interacting phase very similar to ordinary QCD (other properties of this phase have been recently discussed in ref. [10]). This implies a violation of the SUSY algebra (1.3) by non-perturbative effects and hence the existence of a non-perturbative SUSY anomaly [3-6]. We show that the same conclusions apply if a small mass term $m \ll \Lambda_{SQCD}$ is added to the lagrangian and that the massless limit is smooth.

In sect. 7 we give our conclusions. The question arises whether the zero modes' instability is a necessary condition for the existence of the SUSY anomaly. We mention other circumstances under which it is known that a non-perturbative SUSY anomaly exists, and discuss the conjecture that this anomaly is a very general phenomenon, possibly existing in *all* asymptotically free SUSY gauge theories.

2. Zero modes and antimodes

Before delving into a detailed technical discussion let us explain why the instability of the fermionic zero modes is not unexpected. To this end let us compare the spectra of the Dirac operator in the following situations: (a) on a compact space, (b) on an open space when all fermions are massive and (c) on an open space when some fermions are massless (see fig. 1). Notice that in all cases the spectrum is symmetric around $E = 0$ because of the assumed existence of generalized chirality. If χ is an eigenstate of energy E then $\gamma_5 Q_R\, \chi$ is an eigenstate of energy $-E$.

In the discussion of possible zero modes of the Dirac operator, one should notice the qualitative difference between cases (a) and (b) compared to case (c). In the first two cases there is an energy gap between the zero modes and the rest of the spectrum. On a compact space the spectrum is discrete and so a gap whose magnitude is bounded from below by the inverse size of the space always exists. In case (b) there is a gap because the continuous spectrum exists only for $|E| > m_0$ where m_0 is the smallest mass. The reader may easily check that this is a necessary and sufficient condition for an oscillatory behaviour of the wave function in the asymptotic region by examining the second order Schrödinger-like operator D_F^2.

When there is a gap between the zero modes and the rest of the spectrum as in cases (a) and (b), perturbation theory has a non-zero radius of convergence and so conservation of the index follows trivially. Under these circumstances the interesting question is the relation between the index and topological invariants of the background field, a question which is answered by various index theorems [7,8]. But in case (c) the continuous spectrum consists of the whole E-axis except for the single point $E = 0$. There is no gap between the zero modes and the continuous spectrum. As we will see below, any small deformation of the background field which is capable of changing the zero modes' energy will in fact destabilize them. Conventional perturbation theory breaks down and the index is not conserved.

Our goal is to find out precisely under what circumstances the zero modes disappear from the spectrum [5]. Our analysis is based on the following observation. Consider the set of *homogeneous* solutions of the equation

$$D_F \chi = 0 . \tag{2.1}$$

Since the dirac operator is first order, the number of homogeneous solution with a given total angular momentum is equal to the number of radial channels sharing this angular momentum. Furthermore, once the boundary conditions of the background fields have been specified, the asymptotic behaviour of these homogeneous solutions is completely determined by the quantum numbers of the various partial waves. This situation should be contrasted with second order Schrödinger-like operators, where two homogeneous solutions exist per each partial wave. The knowledge gained this way allows us to find all zero modes when the background field is rotationally invariant as well as to prove that, provided the conditions mentioned in the introduction are satisfied, the Dirac operator has *no* zero modes once a generic deformation of the scalar background field is made.

Let us first consider a case where the background field closely resembles the exact instanton. There is no scalar background field and the gauge field has the same spherical symmetry and asymptotic behaviour as the exact instanton. The Dirac operator is then given by (in this paper we use anti-hermitian γ-matrices)

$$D_F = D_F(A_\mu) = \gamma_\mu(\partial_\mu + igA_\mu) , \tag{2.2}$$

$$A_\mu = -\frac{2}{g}\frac{a(r)}{r^2}\eta_{a\mu\nu}x_\nu T^a , \tag{2.3}$$

where the asymptotic behaviour of $a(r)$ is

$$a(r) \sim r^2 \quad , \quad r \to 0 ; \qquad a(r) \to 1 \quad , \quad r \to \infty . \tag{2.4}$$

We recall that the exact instanton is given by $a(r) = r^2/(\rho^2 + r^2)$ where ρ is the instanton size.

It is convenient to express the Dirac operator in polar coordinates

$$D_F = \gamma_r(\partial_r - M(r)/r) , \tag{2.5}$$

$$M(r) = 4(\vec{L_1}\cdot\vec{S_1} + \vec{L_2}\cdot\vec{S_2}) - 2ia(r)\gamma_r\gamma_a T^a , \tag{2.6}$$

$$\gamma_r = \gamma_\mu x_\mu/r , \qquad \gamma_a = \eta_{a\mu\nu}\gamma_\mu x_\nu/r . \tag{2.7}$$

Here $\vec{L_1}$ and $\vec{S_1}$ ($\vec{L_2}$ and $\vec{S_2}$) correspond to a self-dual (anti-self-dual) rotation and T^a are the SU(2) generators. We also define the operators $\vec{J_1} = \vec{L_1}+\vec{S_1}$, $\vec{J_2} = \vec{L_2}+\vec{S_2}$ and $\vec{L_T} = \vec{L_1} + \vec{T}$. The conserved angular momenta in the instanton background are $\vec{J_2}$ and $\vec{K_1} = \vec{J_1} + \vec{T}$.

For left handed channels one has $s_1 = 1/2$ and $s_2 = 0$ and the angular operator $M(r)$ assumes the simple form

$$\frac{1}{2}(1-\gamma_5)M(r) = 4\left[\vec{S_1}\cdot\vec{L_1} + a(r)\vec{S_1}\cdot\vec{T}\right] . \tag{2.8}$$

It is now a simple exercise to determine the asymptotic behaviour of the left-handed homogeneous solutions. Near the origin one can neglect the gauge field. Using a basis were L_1^2 and J_1^2 are diagonal we find the asymptotic behaviour to be r^{2l} if $j_1 = l + 1/2$ and r^{-2-2l} if $j_1 = l - 1/2$. (Since $L_1^2 = L_2^2$ we denote their common eigenvalues by l. In four dimensions l takes half-integer values). For $r \to \infty$ one has to take into account the role of the gauge field which effectively replaces $\vec{L_1}$ by $\vec{L_T}$ and $\vec{J_1}$ by $\vec{K_1}$ (see eqs. (2.4) and (2.8)). Thus, for $r \to \infty$ one finds the behaviour r^{2l_T} if $k_1 = l_T + 1/2$ and r^{-2-2l_T} if $k_1 = l_T - 1/2$.

Most homogeneous solutions are normalizable at one end of the positive r-axis (say, at the origin) and non-normalizable at the other end (at infinity in this example). However, there are certain partial waves which satisfy the conditions $j_1 = l + 1/2$ and $k_1 = l_T - 1/2$ simultaneously in the instanton background. the corresponding homogeneous solutions are normalizable both at the origin and at infinity and hence are zero modes. The above conditions are satisfied simultaneously if and only if the vectors $\vec{S_1}$, $\vec{L_1}$, $\vec{K_1}$ and $\vec{T}$ are colinear and $\vec{T}$ is the sum of the other three, i.e. $t = l + k_1 + 1/2$ (t is isospin). We therefore have a multiplet of zero modes for every l and t such that $t - l \geq 1/2$. In a channel supporting a zero mode eq. (2.1) takes the explicit form

$$\left[\partial_r - \frac{2(l-(t+1)a(r))}{r}\right] \mathcal{U}(r) = 0 , \tag{2.9}$$

were $\mathcal{U}(r)$ denotes the radial dependence of the zero mode.

We now turn our attention to the right-handed channels. Those channels will become relevant in the study of the zero modes once the scalar fields are turned on. Instead of repeating the whole analysis we make a short cut by using a natural mapping between the left and right handed channels. To every left-handed eigenspinor $Y_L = Y_L(x_\mu/r)$ we associate the right-handed eigenspinor obtained by acting on it with γ_r, namely, $Y_R(x_\mu/r) = \gamma_r Y_L(x_\mu/r)$.

The importance of the above mapping stems from the anticommutation relation

$$\{\gamma_r, M(r)\} = -3\gamma_r . \tag{2.10}$$

Eq. (2.10) provides a simple relation between the asymptotic behaviour of homogeneous solutions in corresponding left-handed and right-handed channels, which applies both at the origin and at infinity. If the left-handed solution behaves like r^α for some α, the corresponding right-handed solution behaves like $r^{-3-\alpha}$. This relation implies that if a left-handed homogeneous solution is normalizable, say, at the origin, the corresponding right-handed solution is non-normalizable (and vice versa). The same conclusion applies to the behaviour at infinity.

Let us now concentrate on the right-handed channels that correspond to the left-handed zero modes. We denote these channels, which will play an important role below, as antimode channels. Using eqs. (2.9) and (2.10) we find that the radial equation in an antimode channel is

$$\left[\partial_r + \frac{3 + 2(l - (t+1)a(r))}{r}\right] h(r) = 0 , \tag{2.11}$$

The homogeneous solution $h(r)$ can be expressed in terms of the zero mode $\mathcal{U}(r)$ as follows

$$h(r) = \left(r^3 \mathcal{U}(r)\right)^{-1} . \tag{2.12}$$

The homogeneous solution $h(r)$ is non-normalizable *both* at the origin and at infinity (see fig. 2). We point out that homogeneous solutions with this property are just as rare as zero modes. As we will see in the next section, this property plays a crucial role in the destabilization of the zero modes once the scalar background field is turned on.

3. Instability of the fermionic zero modes

The simple tools developed in the previous section allow us to study what happens to the zero modes when we deform the background field. We will be specifically interested in the Dirac operator $D_F(A_\mu, \phi)$ of SQCD, where ϕ collectively

denotes all scalar fields. The precise form of $D_F(A_\mu, \phi)$ is given in the appendix. Here it will only be important to remember that $D_F(A_\mu, \phi)$ depends linearly on the scalar fields and that the scalar fields couple the matter fermions to the gauginos.

In order to avoid excessive notation we first discuss the case that all scalar VEVs are zero. Thus, at the starting point we have a classical background field which consists of a gauge field only (see eqs. (2.3) and (2.4)). The Dirac operator of eq. (2.2) admits both gaugino and matter left-handed zero modes. We therefore have zero modes of both generalized chiralities. We will study the effect of small scalar perturbations $\delta\phi(x)$ on the fermionic zero modes. The perturbation should not change the boundary conditions of the scalar fields and so we require that the c-number function $\delta\phi(x)$ be normalizable. The size of the perturbation is generically $O(1)$ and it is small compared to the classical solution which is $O(1/g)$. We will show that when a generic scalar perturbation is turned on all zero modes disappear from the spectrum. Later we will describe the most general conditions under which this phenomenon takes place. In the next section we will show that the zero modes become reso nces and calculate the resonances' energy.

What w have in mind is to explicitly calculate the effect of the scalar perturbation on the fermionic spectrum. Under ordinary circumstances this goal would have been achieved to any required accuracy by simply computing the relevant Feynman diagrams. However, in the case at hand the naive perturbative expansion of the path integral in terms of Feynman diagrams breaks down. This property is manifest in the integrand of eq. (1.5) which is non-analytic in g^2.

The breakdown of the naive diagrammatic expansion is a direct consequence of a similar phenomenon which occurs already at the level of first quantized systems. When, as a result of a continuous change in the potential, a bound state moves into the continuous spectrum it ceases to exist as a bound state – it becomes a resonance. The disappearance of the bound state from the spectrum implies a discontinuity in the dependence of the Hilbert space of the quantum mechanical problem on the potential. The Born series diverge precisely when a bound state disappears from the spectrum (see e.g. ref. [11]).

In fact, the breakdown of the naive perturbative expansion in the instanton sector is not an unfamiliar phenomenon. Already in purely bosonic theories there are certain directions in which the functional integration cannot be approximated by a gaussian integral. These directions correspond to symmetries of the action which are explicitly broken by the classical instanton solution. Such symmetries manifest themselves through the existence of bosonic zero modes, which must be eliminated before a bosonic propagator can be defined. The resolution of this problem is to replace the bosonic zero modes by collective coordinates and to perform the integration over the collective coordinates exactly. For instance, the euclidean space volume $\mathcal{V}$ in eq. (1.1) arises by applying this procedure to the translational zero modes.

The above well-known non-gaussian behaviour arises from the existence of zero curvature directions in the bosonic action. In the present context there is another source of non-gaussian behaviour, namely, the non-analytic dependence of the fermionic spectrum on g^2 and on the amplitudes of the destabilizing modes. Now, the scalar modes which destabilize the fermionic zero modes – the destabilizing modes – correspond to positive curvature directions of the bosonic action. Thus, there is no need to to replace the destabilizing modes by collective coordinates. But the integration over the amplitudes of the destabilizing modes must be carried out carefully, taking into account correctly their effect on the fermionic spectrum. The rest of the bosonic modes can be handled using a diagrammatic expansion but with a modified set of Feynman rules, to be described in sect. 5.

Fortunately, even in studying the instability of the fermionic zero modes some form of perturbation theory is still valid. This stems from the fact that the solution of an ordinary linear differential equation over a bounded interval with local boundary conditions depends analytically on the coefficient functions (so long as the latter are non-singular in the interval). As a result, the fermionic determinant, while not analytic in g^2, is still a continuous function of g and of the amplitudes of the destabilizing modes. The fermionic propagator, however, exhibits a discontinuity when a zero mode disappears from the spectrum. In this section we will prove that a generic scalar perturbation destabilizes *all* zero modes. This statement is exact. The scalar modes which destabilize the different fermionic zero modes will be found to leading order in g. This information will be used in sect. 4 to calculate the resonances' energy to $O(g)$, while in sect. 5 we will perform the functional integration and derive eq. (1.5) for the tunneling amplitude.

Let us now proceed with the actual demonstration that a generic scalar perturbation destabilizes all zero modes. The logic of the proof is as follows. We first turn on a perturbation containing only the scalar mode which mixes a particular gaugino zero mode with a particular matter zero mode. (Because of charge conjugation symmetry all fermionic zero modes occur in pairs. The above perturbation will also mix the charge conjugate gaugino zero mode with the charge conjugate matter zero mode). We show that as a result of the perturbation both pairs of zero modes disappear from the spectrum. Notice that at this stage the index is still conserved – we have lost two zero mode of each generalized chirality.

We then make a second perturbation, chosen such that in the absence of the first perturbation it mixes the *same* pair of gaugino zero modes with another pair of matter zero modes. we will show that *even in the presence of the first perturbation* this second perturbation destabilizes the second pair of matter zero modes. This, of course, implies non-conservation of the index. Physically, what happens is that after the first perturbation the pair of gaugino zero modes and the first pair of matter zero modes become resonances. After the second perturbation, the second pair of matter zero modes also becomes a pair of resonances, while the energy of the already existing gaugino resonances increases. In general, regardless of the presence or absence of other perturbations, turning on the destabilizing mode $\delta\phi_{ij}$ that mixes the i-th pair of gaugino zero modes with the j-th pair of matter zero modes is a *sufficient* condition that both pairs of zero modes will no longer exist. The announced result follows immediately once we recognize that a generic scalar perturbation contains all destabilizing modes.

Consider then a scalar perturbation that mixes a given gaugino zero mode with a given matter zero mode. In addition to the left-handed channel in which it originally existed, each zero mode must now have a non-vanishing component in the right-handed antimode channel that corresponds to the *other* zero mode. Explicitly, the radial equations are (see eqs. (2.9), (2.11) and the appendix)

$$\partial_r u_\lambda(r) - 2r^{-1}(l_\lambda - a(r)(t_\lambda+1))\, u_\lambda(r) - gC_1\delta\phi^*(r)v_\psi(r) = 0\,, \qquad (3.1a)$$

$$\partial_r v_\psi(r) + r^{-1}[3+2(l_\psi - a(r)(t_\psi+1))]\, v_\psi(r) - gC_1\delta\phi(r)u_\lambda(r) = 0\,. \qquad (3.1b)$$

Here l_λ and t_λ (l_ψ and t_ψ) are the quantum numbers of the gaugino (matter) zero mode, $\delta\phi(r)$ is the radial dependence of the scalar perturbation and C_1 is a non-zero constant which arises upon performing the angular integration and contracting all indices (see eq. (A.14)). A similar equation holds where the roles of λ and ψ as well as $\delta\phi(r)$ and $\delta\phi^*(r)$ are interchanged.

The question is whether or not the coupled eqs. (3.1) still have a normalizable solution. To first order is g eq. (3.1b) reads

$$\partial_r v_\psi(r) + r^{-1}[3+2(l_\psi - a(r)(t_\psi+1))]\, v_\psi(r) = gC_1\delta\phi(r)\mathcal{U}_\lambda(r) + O(g^3)\,, \qquad (3.2)$$

where $\mathcal{U}_\lambda(r)$ denotes the radial dependence of the unperturbed gaugino zero mode. using eq. (2.12) and neglecting the $O(g^3)$ terms the solution is

$$v_\psi(r) = gC_1h_\psi(r)\left(c + \int_{r_0}^{r} dr'\,(r')^3\mathcal{U}_\lambda(r')\mathcal{U}_\psi(r')\delta\phi(r')\right)\,. \qquad (3.3)$$

Here $0 < r_0 < \infty$ is an arbitrary reference point, c is an integration constant and $\mathcal{U}_\psi(r)$ denotes the radial dependence of the unperturbed matter zero mode. Since

$h_\psi(r)$ is non-normalizable both at the origin and at infinity, $v_\psi(r)$ of eq. (3.3) will be normalizable *iff* the following two conditions are satisfied

$$c + \int_{r_0}^{\infty} dr\, r^3 \mathcal{U}_\lambda(r)\mathcal{U}_\psi(r)\delta\phi(r) = 0\,, \tag{3.4a}$$

$$c - \int_{0}^{r_0} dr\, r^3 \mathcal{U}_\lambda(r)\mathcal{U}_\psi(r)\delta\phi(r) = 0\,. \tag{3.4b}$$

Notice that we have two conditions but only one integration constant whose value can be adjusted. Eliminating c between the two equations we find that a normalizable solution exists *iff* $\delta\phi(r)$ satisfies the constraint

$$\int_{0}^{\infty} dr\, r^3 \mathcal{U}_\lambda(r)\mathcal{U}_\psi(r)\delta\phi(r) = 0\,. \tag{3.5}$$

Clearly, a generic $\delta\phi(r)$ will not satisfy eq. (3.5) and will therefore lead to the disappearance of both (pairs of) zero modes. Eq. (3.5) implies that to leading order in g the scalar mode which destabilize the above fermionic zero modes is

$$\delta\phi(r) \sim \mathcal{U}_\lambda(r)\mathcal{U}_\psi(r)\,. \tag{3.6}$$

For the precise form of the destabilizing mode including angular dependence and colour and flavour indices see eq. (A.10).

What changes if the higher order terms in eq. (3.2) are kept? The integrand on the r.h.s. of eq. (3.3) will now contain higher order terms in g and $\delta\phi(r)$. Likewise, the constraint (3.5) will become non-linear in $\delta\phi(r)$. But the fact remains that the scalar perturbation must satisfy a constraint in order that the fermionic zero modes will survive.

As can be seen from eq. (3.3), the necessity to satisfy a constraint arises because of the peculiar asymptotic behaviour of $h_\psi(r)$ which, in turn, is fixed by the asymptotic behaviour of the l.h.s. of eq. (3.2). The characteristic behaviour of the homogeneous solution in the antimode channel will not change if we deform the background field. In particular, if the background field has already been deformed previously, when we turn on a new destabilizing mode the integral on the r.h.s. of eq. (3.3) will depend on the previous deformations as well as on the new one. But this integral will still be multiflied by a homogeneous solution which is non-normalizable both at the origin and at infinity, and so the necessity to impose a constraint in order to maintain the zero mode (which, to leading order, simply requires that the amplitude of the new destabilizing mode be zero) will persist.

In summary, the existence of a single antimode implies that under a generic deformation of the background field no zero modes with the same generalized chirality as the antimode will survive. Furthermore, this antimode channel exists provided

the original rotationally symmetric background field had a corresponding zero mode with the opposite generalized chirality. Thus, the existence of a single zero mode in a rotationally invariant background is a sufficient condition that, under a generic perturbation there will remain no zero modes of the *opposite* generalized chirality. The number of constraints that must be imposed on the background field in order to maintain a particular zero mode is equal to the original number of zero modes with the opposite generalized chirality. This statement is trivially true if in the original rotationally invariant background field the Dirac operator has zero modes of only one generalized chirality (we will encounter examples of this situation below). But if initially the Dirac operator has zero modes of both generalized chiralities, *no* zero mode will survive after a generic deformation of the background field is made.

Before we can apply our conclusions to instantons of weakly interacting theories we must generalize our discussion to include the case that some scalar fields have non-zero expectation values. At the starting point we now have both gauge and scalar (rotationally invariant) background fields, and some (but not all) of the fermions are massive. In this more general situation, zero modes may consist of more than one radial channel. The antimode is defined as the set of channels obtained by acting on all the zero mode's channels with γ_r. The linear space of homogeneous solutions of the antimode has the property that it is not spanned by those homogeneous solutions which are normalizable *either* at the origin *or* at infinity. The destabilizing modes are still given by eq. (A.10). Once again, one finds that the number of conditions needed to maintain a normalizable solution is larger than the number of integration constants if the zero mode mixes with an antimode, and so a zero mode will exist only if the deformed background field satisfies certain constraints.

Notice that, to be precise, what has been shown above is that, if the Dirac operator has zero modes of both generalized chiralities in a rotationally invariant background, then after a generic deformation is made there remain no bound states with *zero* energy. Is it possible that the bound states are still there but they have simply acquired a small non-zero energy? The answer is no, but the reasons are different depending on whether or not some of the fermions are massless.

If all fermions are massive the index must be conserved. The only way this can be reconciled with our result is that the conditions necessary for non-conservation of the index never arise. We therefore have the following corollary. All fermionic zero modes must have the same generalized chirality in a completely massive theory whose Dirac operator anticommutes with the generator of generalized chirality transformations.

If some fermions are massless then, as explained earlier, zero is the *only* possible energy for a bound state. If a fermionic bound state ceases to have zero energy it ceases to exist as such – it becomes a resonance. The study of the resonances will be the subject of the next section. We conclude this section by recapitulating the conditions under which fermionic zero modes are unstable. If (a) some fermions are massless and (b) the Dirac operator has zero modes of both generalized chiralities in a rotationally invariant background, then no zero modes survive under a generic deformation of the background field.

4. The resonances

We have seen in the previous section that under certain circumstances the zero modes of the Dirac operator are unstable, and disappear from the spectrum under a generic scalar perturbation. On rather general grounds we expect that, when a bound state disappears from the spectrum, there should appear a low energy resonance. The destabilization of a zero mode and the properties of the resulting resonance are both governed by the massless fermionic channels. We will discuss below the resonances in the simplest case, namely, when initially the scalar background field vanishes and the Dirac operator is given by eqs. (2.2)-(2.4). Except for some technical details, the whole analysis is carried over to the more complicated case that the initial background field contains scalar fields with non-zero expectation values.

Since $D_F(A_\mu,\phi)$ anticommutes with $\gamma_5 Q_R$ it has the structure

$$D_F = \begin{pmatrix} 0 & \mathcal{D}^\dagger \\ \mathcal{D} & 0 \end{pmatrix} . \tag{4.1}$$

Here $\mathcal{D}$ takes left-handed gaugino channels and right-handed matter channels to right-handed gaugino channels and left-handed matter channels. We also introduce the second order operator

$$D_F^2 = \begin{pmatrix} \mathcal{D}^\dagger\mathcal{D} & 0 \\ 0 & \mathcal{D}\mathcal{D}^\dagger \end{pmatrix} . \tag{4.2}$$

Notice that the gaugino zero modes are zero modes of $\mathcal{D}^\dagger\mathcal{D}$ while the matter zero modes are zero modes of $\mathcal{D}\mathcal{D}^\dagger$. All non-zero eigenstates of D_F can be constructed from the eigenstates of either $\mathcal{D}^\dagger\mathcal{D}$ or $\mathcal{D}\mathcal{D}^\dagger$. For example, starting from an eigenstate χ_+ of $\mathcal{D}^\dagger\mathcal{D}$,

$$\mathcal{D}^\dagger\mathcal{D}\chi_+ = E^2\chi_+ , \tag{4.3}$$

one has

$$D_F\chi = E\chi , \tag{4.4}$$

where

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_+ \\ E^{-1}\mathcal{D}\chi_+ \end{pmatrix} . \tag{4.5}$$

In order to understand the existence of resonances let us examine the structure of the Schrödinger-like operator $\mathcal{D}^\dagger\mathcal{D}$ in some detail. $\mathcal{D}^\dagger\mathcal{D}$ has a centrifugal term which takes the asymptotic form $4l_T(l_T+1)/r^2$ in a left-handed channel supporting a gaugino zero mode. All zero modes have $l_T = t-l \geq 1/2$. One can think of $\mathcal{D}^\dagger\mathcal{D}$ as describing a non-relativistic scattering problem in four *space* dimensions. Imagine a particle living in five dimensional space-time and having the same quantum numbers as a zero mode. This particle will encounter a centrifugal barrier while approaching the instanton center from infinity. Near the origin, the non-trivial structure of the instanton creates a local well in which the particle can be trapped in the absence of scalar perturbations (see fig. 3).

All eigenvalues of the operator $\mathcal{D}^\dagger\mathcal{D}$ are non-negative. Since the potential tends to zero at infinity, positive eigenvalues correspond to scattering states and a bound state may exist only if its eigenvalue coincides with the infimum of the spectrum, i.e. for $E=0$. The particle is only barely trapped. Notice that, by abuse of language, we use E to denote the square root of the eigenvalue of $\mathcal{D}^\dagger\mathcal{D}$ (see eq. (4.3)). The true energy of the five-dimensional particle would be $E^2/2m$.

Consider now the effect of a scalar perturbation that mixes two zero modes $\mathcal{U}_\lambda$ and $\mathcal{U}_\psi$ of opposite generalized chiralities. One has

$$\langle \mathcal{U}_\lambda | D_F(A_\mu,\delta\phi) | \mathcal{U}_\psi \rangle = \langle \mathcal{U}_\lambda | gC_1\delta\phi | \mathcal{U}_\psi \rangle , \tag{4.6}$$

where we have used the fact that the unperturbd Dirac operator annihilates the zero modes. Let us assume that at $t=0$ the wave function of our particle *coincides* with one of the zero modes of the unperturbed Dirac operator. In the perturbed "hamiltonian" $\mathcal{D}^\dagger\mathcal{D}(A_\mu,\delta\phi)$ the particle's "energy" is $E_{rs} = O(g\delta\phi)$. Since $|E_{rs}| > 0$, the particle is no longer trapped in the well and will eventually escape to infinity. However, before it can propagate as a free particle it has to tunnel though the centrifugal barrier. Because the tunneling probability is small this will take a very long time and so we will have a resonance.

The tunneling probability is just the probability to find the particle at a distance r where the height of the centrifugal barrier is equal to E_{rs}^2, i.e. at $r \approx E_{rs}^{-1}$. For $r \gg 1$ (we henceforth set $\rho = 1$) the zero mode's wave function is

$$\mathcal{U}(r) = b\, r^{-2-2l_T} + O(r^{-4-2l_T}) , \tag{4.7}$$

where b is an $O(1)$ constant. The probability to find the particle at this distance is therefore

$$\Gamma \approx r^3 \mathcal{U}^2(r)\Big|_{r\approx E_{rs}^{-1}} \approx E_{rs}^{1+4l_T}. \tag{4.8}$$

Evidently, Γ is nothing but the width of the resonance.

What we have found so far is that under a generic scalar perturbation all zero modes become resonances with $E_{rs} = O(g\delta\phi)$. In order to calculate $\Omega(v)$ in the next section we need a more detailed expression for the dependence of the resonances' energy on the scalar perturbation. This information can be extracted by examining the structure of the low energy scattering states at the intermediate range $1 \ll r \ll E^{-1}$ and the asymptotic range $r \gg E^{-1}$. The analysis is straightforward but rather technical and so we leave most details to the appendix.

Consider first the behaviour of the low energy continuum states of the unperturbed operators $\mathcal{D}^\dagger\mathcal{D}(A_\mu)$ and $\mathcal{D}\mathcal{D}^\dagger(A_\mu)$. We are interested in particular in the behaviour of the scattering states $u(E,r)$ in left-handed channels supporting a gaugino or matter zero mode. Explicitly (see the appendix for more details),

$$u(E,r) \approx \mathcal{F}(E)\left(r^{-2-2l_T} - b^{-2}E^2\frac{r^{2l_T}}{4l_T+2}\right) \quad , \quad 1 \ll r \ll E^{-1}. \tag{4.9}$$

The normalization constant $\mathcal{F}(E)$ is the Jost function. $\mathcal{F}^2(E)$ measures the probability to find the scattering particle near the origin. Notice that, up to a normalization factor, $u(E,r)$ behaves like the zero mode $\mathcal{U}(r)$ for $r \sim 1$, while at some distance which is still much smaller than E^{-1} the increasing term in $u(E,r)$ dominates over the decreasing one. Eq. (4.9) shows the highest powers of r that occur at zeroth and second order in E. Corrections to it involve either higher powers of E, or lower powers of r compared to the ones shown for zeroth and second order in E.

For $r \gtrsim E^{-1}$ the $O(E^4)$ corrections to eq. (4.9) cannot be neglected any more and eventually one has

$$u(E,r) \approx r^{-3/2}\sin(Er - \pi(l_T + 1/4) + \eta) \quad , \quad r \gg E^{-1}, \tag{4.10}$$

where η is a phase shift. The r.h.s. of eq. (4.10) has the correct continuum normalization, and $\mathcal{F}(E)$ in eq. (4.9) can be determined by matching the intermediate and large distance behaviour at $r \approx E^{-1}$. One finds that $\mathcal{F}(E) \sim E^{2l_T-1/2}$.

We now turn on one of the destabilizing modes

$$\delta\phi(r) = \overline{\delta\phi}(r)\xi. \tag{4.11}$$

Here $\overline{\delta\phi}(r)$ is normalized and ξ is the amplitude of the destabilizing mode. The perturbed scattering state of e.g. $\mathcal{D}^\dagger\mathcal{D}$ has non-vanishing components both in the

original gaugino zero mode channel and in the matter antimode channel. The new component $v_\psi(\xi,E,r)$ is non-zero for $r \lesssim 1$ and it decreases rapidly for $r \gg 1$. In the intermediate range $1 \ll r \ll E^{-1}$ the perturbed component $u_\lambda(\xi,E,r)$ is given by

$$u_\lambda(\xi,E,r) \approx \mathcal{F}(\xi,E)\left(r^{-2-2l_T} + b^{-2}\left(|gC\xi|^2 - E^2\right)\frac{r^{2l_T}}{4l_T+2}\right) \quad , \quad 1 \ll r \ll E^{-1}. \tag{4.12}$$

For the derivation of eq. (4.12) and the value of the non-zero constant C see the appendix. Similarly to eq. (4.9), corrections to eq. (4.12) are either of higher order in E and ξ, or involve lower powers of r compared to the ones shown for zeroth and second order in E and ξ. The asymptotic large r behaviour of $u_\lambda(\xi,E,r)$ is again given by eq. (4.10) but with a different phase shift.

The resonance occurs at

$$E_{rs} = |gC\xi| + O(g^2\xi^2). \tag{4.13}$$

For $E = E_{rs}$, the E^2 and ξ^2 terms shown in eq. (4.12) as well as non-leading terms cancel each other, and the scattering state resembles the zero mode all the way to $r \approx E_{rs}^{-1}$. The phase shift is $\eta_{rs} = (\pi/2 \bmod \pi)$, and $u_\lambda(\xi,E_{rs},r)$ behaves like the "wrong" solution of Bessel's equation for $r \gg 1$. The existence of an $O(g\xi)$ energy for which the phase shift is $(\pi/2 \bmod \pi)$ is guaranteed, because $\eta = (0 \bmod \pi)$ for $E = 0$ and the phase shift increases by almost π over the range $0 < E \sim g\xi$. Consequently, $\mathcal{F}(\xi,E_{rs}) \approx E_{rs}^{-1/2-2l_T}$. The probability to find a particle with $E = E_{rs}$ near the origin is greatly enhanced. As expected, it is equal to Γ^{-1}. For E outside the resonance's peak one still has $\mathcal{F}(\xi,E) \sim E^{2l_T-1/2}$.

The generalization of eq. (4.12) to the case that more than one destabilizing mode is turned on is straightforward. For instance, if the resonance corresponds to the i-th positive generalized chirality zero mode, one replaces $|gC\xi|^2$ on the r.h.s. of eq. (4.12) by $\sum_{j=1}^{n_-}|gC_{ij}\xi_{ij}|^2$. The resulting expression for the energy of the resonance is eq. (1.6).

5. The tunneling amplitude

We now proceed to calculate the tunneling amplitude $\Omega(v)$ in vector-like N=1 SUSY gauge theories. We make the following assumptions. (a) The scalar expectation values break the non-abelian gauge symmetry completely or, at most, the unbroken symmetry is abelian. This assumption implies that a weak coupling calculation is possible and in particular that the dilute gas approximation leading to eq. (1.2) is reliable. (b) Some fermions (or, equivalently, some supermultiplets) are classically massless. (c) The Dirac operator $D_F(A_\mu, \phi)$ anticommutes with the generator of generalized chirality. As we will see below, this assumption can be somewhat relaxed. (d) The Dirac operator has zero modes of both generalized chiralities in the rotationally invariant classical field of the instanton.

Recall that, in the presence of scalar fields whose expectation values break the gauge symmetry, the euclidean field equations do not admit an exact instanton solution. The scalar VEVs explicitly break the classical scale symmetry of the Yang-Mills action, and the action of any bosonic configuration can be reduced by simply shrinking its size. This problem is handled by the so-called constrained instanton technique [12,19]. One first imposes a constraint on the bosonic fields which fixes the size of the instanton. The constrained field equations are then solved by introducing a Lagrange multiplier, and a diagrammatic expansion is set up for each value of the constraint. Finally, the integration over the constraint parameter is carried out. Technically, the ρ-integral in eq. (1.5) is an integral over the constraint parameter. Physically, it has the same significance as the scale collective coordinate of the massless theory.

We will now obtain an explicit expression for the tunneling amplitude in the case of an SU(2) gauge group. The generalization to $N_c > 2$ requires the introduction of additional collective coordinates which take care of the possible orientations of the instanton in the broken gauge group. Also, the inverse power of g in front of the integral on the r.h.s. of eq. (1.5) changes from 8 to α where $\alpha - 8$ is number of additional collective coordinates. Aside from these minor technical modifications, everything that will be said below about the SU(2) case applies also in the more general case of $N_c > 2$.

We begin by writing $\Omega(v)$ as

$$\Omega(v) = \int_0^\infty d\rho\, \rho^{-5} \int d^{n_+ n_-}\xi\, d^{n_+ n_-}\xi^* \times$$
$$\int [d\beta^\perp][d\chi]\, J\, Det\,\{\mathcal{M}_{gh}\}\, e^{-S_B - S_F}\,. \tag{5.1}$$

Here $[d\beta^\perp]$ denotes the functional integration over the bosonic subspace $\mathcal{B}^\perp$ orthogonal to the destabilizing modes as well as to the infinitesimal translations and dilatation modes, which have been eliminated in the process of introducing the translation and scale collective coordinates. This change of variables also leads to the jacobian J. $Det\,\{\mathcal{M}_{gh}\}$ is the Faddeev-Popov determinant. In eq. (5.1) χ is a generic name for all fermion fields. In euclidean space left-handed and right-handed fermions must be treated as independent variables. Because of the vector-like nature of the theory it is convenient to use Majorana notation for both gaugino and matter

fermions. For every Dirac field we introduce two Majorana fields, one transforming under the same representation as the original Dirac field and the other transforming under the conjugate representation (see the appendix for more details).

We now have to set up a diagrammatic expansion for each value of ρ and ξ_{ij}. Notice that if the amplitudes of too many destabilizing modes vanish, some zero modes will still survive. We refer to the ξ-values for which $D_F(\rho, v, \xi)$ admits *some* zero modes as exceptional values. However, the set of all exceptional values has zero measure under the ξ-integration and can therefore be neglected. We have to derive the Feynman rules only for non-exceptional values of ξ.

Let $(A_\mu(\rho, v, x), \varphi(\rho, v, x))$ be the rotationally invariant classical field that solves the constrained field equations for some values of ρ and v. The background field around which the perturbative expansion will be set up is $(A_\mu(\rho, v, x), \varphi(\rho, v, \xi_{ij}, x))$ where

$$\varphi(\rho, v, \xi_{ij}, x) = \varphi(\rho, v, x) + \sum_{i=1}^{n_+}\sum_{j=1}^{n_-} \overline{\delta\phi}_{ij}(\rho, v, x)\xi_{ij}\,. \tag{5.2}$$

The background field $\varphi(\rho, v, \xi_{ij}, x)$ satisfies a certain reality condition (see eq. (A.7)). By performing standard manipulations we arrive at

$$\Omega(v) = B_1' g^{-8}(\mu_0\rho)^8 \int_0^\infty d\rho\, \rho^{-5} \int d^{n_+ n_-}\xi\, d^{n_+ n_-}\xi^*\, e^{-S_B(\rho,v,\xi)} \times$$
$$Det^{-1/2}\,\{\mathcal{H}_B\}\, Det^{1/2}\,\{D_F\}\, Det\,\{\mathcal{M}_{gh}\}\, e^{-W'[\rho,v,\xi]}\,, \tag{5.3}$$

where

$$S_B(\rho, v, \xi) = \frac{8\pi^2}{g^2(\mu_0)} + B_2\rho^2 v^2 + \xi\text{-dependent terms}\,, \tag{5.4}$$

and μ_0 is an arbitrary renormalization point. The differential operators $\mathcal{H}_B$, D_F and $\mathcal{M}_{gh}$ define the fluctuations of boson, fermion and ghost fields respectively. They all depend on the background field $(A_\mu(\rho, v, x), \varphi(\rho, v, \xi_{ij}, x))$. Their determinants depend on the renormalization point μ_0. The fermionic determinant is raised to the power 1/2 because of our use of Majorana notation. $W'[\rho, v, \xi]$ is the sum of connected bubble diagrams calculated with a set of Feynman rules that we will not bother to specify, because we are shortly going to replace them by more convenient ones. The numerical factor B_1' as well as $g^{-8}(\mu_0\rho)^8$ arise from the semi-classical value of the jacobian J in eq. (5.1). Notice that the jacobian can be treated using a trick similar to the representation of the Faddeev-Popov determinant in terms of a ghost field. However, because J is the determinant of a finite dimensional matrix there is no need to introduce a whole new ghost field. A finite number of ghost coordinates whose "propagator" is the unit matrix will suffice.

The reason why the destabilizing modes must be given a special treatment is the discontinuity that the fermionic spectrum exhibits at exceptional values of their amplitudes. As far as the bosonic sector of the theory is concerned there is nothing special about the destabilizing modes. We may therefore try to simplify the cumbersome set of Feynman rules implicit in eq. (5.3) by reexpanding the background field (5.2) around $\varphi(\rho, v, x)$ in the bosonic sector. We also recall that when the bosonic, fermionic and ghost determinants are evaluated on the classical background field $(A_\mu(\rho, v, x), \varphi(\rho, v, x))$, their effect is to replaced $g^2(\mu)$ and $(\mu_0\rho)^8$ in eq. (5.4) by the running coupling $g^2(\rho)$. These considerations lead to the following expression

$$\Omega(v) = B_1 g^{-8} \int_0^\infty d\rho\, \rho^{-5}\, e^{-\frac{8\pi^2}{g^2(\rho)} - B_2\rho^2 v^2} \int d^{n_+ + n_-}\xi\, d^{n_+ + n_-}\xi^* \times$$
$$e^{-\xi^\dagger H(\rho,v)\xi - W[\rho,v,\xi]}\, \underline{Det}^{1/2}\left\{D_F(\rho, v, \xi)/D_F(\rho, v, 0)\right\} . \qquad (5.5)$$

The ratio of the determinants of $D_F(\rho, v, \xi)$ and $D_F(\rho, v, 0)$ is finite. The underline indicates that we are evaluating only the contribution of the continuous spectrum to the determinant. Notice that for non-exceptional ξ-values there are no zero modes and so $\underline{Det}\{D_F(\rho, v, \xi)\} = Det\{D_F(\rho, v, \xi)\}$. Because of the vector-like nature of the theory and the reality condition (A.7) satisfied by the background field, the sum of bubble diagrams $W[\rho, v, \xi]$ is real.

The fermionic propagator $G_F(\rho, v, \xi)$ to be used in the calculation of $W[\rho, v, \xi]$ is the inverse of $D_F(\rho, v, \xi)$. The bosonic propagator $G_B^\perp(\rho, v)$ is the inverse of the unperturbed bosonic hamiltonian $\mathcal{H}_B(\rho, v)$ on the subspace $\mathcal{B}^\perp$. The ghost propagator is the inverse of $\mathcal{M}_{gh}(\rho, v)$. The positive quadratic form $H(\rho, v)$ is

$$H(\rho, v)_{ijkl} = \left\langle \overline{\delta\phi}_{ij} \left| \mathcal{H}_B(\rho, v) - \mathcal{H}_B(\rho, v) G_B^\perp(\rho, v) \mathcal{H}_B(\rho, v) \right| \overline{\delta\phi}_{kl} \right\rangle . \qquad (5.6)$$

In addition to the usual vertices, the bosonic sector contains new linear, bilinear and trilinear vertices which arise from the separation of the background field (5.2) into its classical and perturbative part. Notice in particular the presence of linear vertices (see fig. 4a) which occur because the destabilizing modes are not eigenstates of $\mathcal{H}_B(\rho, v)$. The diagram shown in fig. 4b is excluded from $W[\rho, v, \xi]$ because it has already been taken into account in $H(\rho, v)$.

Let us now obtain a more explicit expression for the ratio of fermionic determinants in eq. (5.5). Using eqs. (4.2) and (4.3) one has

$$\begin{aligned} \frac{1}{2}\log \underline{Det}\{D_F\} &= \frac{1}{4}\left(\log \underline{Det}\{\mathcal{D}^\dagger\mathcal{D}\} + \log \underline{Det}\{\mathcal{D}\mathcal{D}^\dagger\}\right) \\ &= \frac{1}{2}\int_0^\infty dE\, \left(\rho_{\mathcal{D}^\dagger\mathcal{D}}(E) + \rho_{\mathcal{D}\mathcal{D}^\dagger}(E)\right)\log(\rho E) . \qquad (5.7) \end{aligned}$$

Here $\rho_{\mathcal{D}^\dagger\mathcal{D}}(E)$ and $\rho_{\mathcal{D}\mathcal{D}^\dagger}(E)$ are the spectral densities of the continuous spectrum of the operators $\mathcal{D}^\dagger\mathcal{D}$ and $\mathcal{D}\mathcal{D}^\dagger$ respectively. The scale parameter ρ is needed in order to

make the argument of the logarithm dimensionless and with the same normalization as in ref. [2].

The last row on the r.h.s. of eq. (5.7) should be properly considered as a *definition* of $\log \underline{Det}^{1/2}\{D_F\}$. Notice that it is free of any global sign ambiguities that may occur for a general first order operator lacking generalized chirality symmetry because of the necessity to continue the argument of the logarithm to negative energies. Using eq. (5.7) we find

$$\underline{Det}^{1/2}\left\{\frac{D_F(\rho, v, \xi)}{D_F(\rho, v, 0)}\right\} = \exp\left[\frac{1}{2}\int_0^\infty dE\, \left(\delta\rho_{\mathcal{D}^\dagger\mathcal{D}}(E) + \delta\rho_{\mathcal{D}\mathcal{D}^\dagger}(E)\right)\log(\rho E)\right] , \qquad (5.8)$$

where

$$\delta\rho_{\mathcal{D}^\dagger\mathcal{D}}(E) = \rho_{\mathcal{D}^\dagger\mathcal{D}(\rho,v,\xi)}(E) - \rho_{\mathcal{D}^\dagger\mathcal{D}(\rho,v,0)}(E) , \qquad (5.9a)$$
$$\delta\rho_{\mathcal{D}\mathcal{D}^\dagger}(E) = \rho_{\mathcal{D}\mathcal{D}^\dagger(\rho,v,\xi)}(E) - \rho_{\mathcal{D}\mathcal{D}^\dagger(\rho,v,0)}(E) . \qquad (5.9a)$$

Eqs. (5.5) and (5.8) provide an exact expression for the tunneling amplitude from which it follows that $\Omega(v) > 0$. The reason is simply that the integrand of the ξ-integral is always non-negative and it is almost everywhere positive.

Notice that eq. (5.7) applies provided the Dirac operator anticommutes with generalized chirality on the background field (5.2). We may therefore somewhat relax the assumption that that $D_F(A_\mu, \phi)$ anticommutes with generalized chirality for every scalar configuration. In order to prove that $\Omega(v) > 0$ it suffices that $D_F(A_\mu, \phi)$ anticommutes with generalized chirality on the background field (5.2). A model which satisfies this milder assumption is described in Appendix C of ref. [5].

The physical content of eq. (5.8) can be understood using the results of the previous section. We first observe that, for non-exceptional values of ξ, a suitable generalization of Levinson's theorem [11,13] gives rise to

$$\int_0^\infty dE\, \delta\rho_{\mathcal{D}^\dagger\mathcal{D}}(E) = 2n_+ , \qquad (5.10a)$$

$$\int_0^\infty dE\, \delta\rho_{\mathcal{D}\mathcal{D}^\dagger}(E) = 2n_- . \qquad (5.10b)$$

Levinson's theorem provides a quantitative relation which ensures that the "total number of states" is conserved in the appropriate sense. As we have seen in the previous section, the zero modes become narrow resonances for $O(1)$ values of ξ. Under these circumstances practically the entire change in the spectral density $\delta\rho(E)$ arises from the resonances and so

$$\delta\rho_{\mathcal{D}^\dagger\mathcal{D}}(E) = 2\sum_{i=1}^{n_+}\delta(E - E_i^{res}) + O(g^2) , \qquad (5.11a)$$

$$\delta\rho_{DDt}(E) = 2\sum_{j=1}^{n_-}\delta(E - E_j^{r\epsilon}) + O(g^2)\,. \tag{5.11b}$$

Substituting this in eq. (5.8) we obtain

$$\underline{Det}^{1/2}\left\{\frac{D_F(\rho,v,\xi)}{D_F(\rho,v,0)}\right\} = \prod_{i=1}^{n_+} E_i^+(\rho,v,\xi)\prod_{j=1}^{n_-} E_j^-(\rho,v,\xi) + O(g^{n_++n_-+1})\,, \tag{5.12}$$

where the resonances' energies are given by eq. (1.6). Eq. (1.5) is obtained by substituting eq. (5.12) in eq. (5.5).

Eq. (5.12) implies that the fermionic determinant is a continuous function of g and ξ. The same is not true for the fermionic propagator. The easiest way to see this is to observe that the propagator of the unperturbed Dirac operator satisfies the relation

$$D_F(\rho,v,0)G_F(\rho,v,0;x,y) = \delta^4(x-y) - \sum_{i=1}^{n_+}|i\rangle\langle i| - \sum_{j=1}^{n_-}|j\rangle\langle j|\,, \tag{5.13}$$

where $|i\rangle$ and $|j\rangle$ denote positive and negative generalized chirality zero modes respectively. On the other hand, for non-exceptional values of ξ the propagator of the perturbed Dirac operator satisfies the relation

$$D_F(\rho,v,\xi)G_F(\rho,v,\xi;x,y) = \delta^4(x-y)\,. \tag{5.14}$$

The difference between the r.h.s. of eq. (5.13) and the r.h.s. of eq. (5.14) is $O(1)$. Consequently, the fermionic propagator must exhibit a discontinuity in ξ when the vector ξ approaches one of the subspaces on which $D_F(\rho,v,\xi)$ has zero modes. By calculating the resonance's contribution to $G_F(\rho,v,\xi)$ one can show that for non-exceptional values of ξ

$$G_F(\rho,v,\xi) - G_F(\rho,v,0) \approx \sum_{i=1}^{n_+}(|i'\rangle\langle i| + |i\rangle\langle i'|) + \sum_{j=1}^{n_-}(|j'\rangle\langle j| + |j\rangle\langle j'|)\,, \tag{5.15}$$

where the normalizable states $|i'\rangle$ and $|j'\rangle$ satisfy

$$D_F(\rho,v,0)\,|i'\rangle \approx |i\rangle\,, \tag{5.16a}$$

$$D_F(\rho,v,0)\,|j'\rangle \approx |j\rangle\,. \tag{5.16b}$$

Corrections to eqs. (5.15) and (5.16) are $O(g\xi)$. These corrections, while small, are essential. $|i'\rangle$ and $|j'\rangle$ live in the antimode channels of the corresponding zero modes, and using the same arguments as in sect. 3 one can show that eq. (5.16) cannot be satisfied exactly for normalizable $|i'\rangle$ and $|j'\rangle$. Needless to say, the fermionic propagator $G_F(\rho,v,\xi)$ cannot be obtained from the propagator $G_F(\rho,v,0)$ of the unperturbed Dirac operator by summing the formal series

$$G_F(\rho,v,0)\sum_{p=0}^{\infty}(-gC_1\delta\phi G_F(\rho,v,0))^p\,. \tag{5.17}$$

Conventional perturbation theory has broken down.

Among the zero mode, of particular importance are those which have $l_T = 1/2$ in their massless channels. (In SQCD with matter fields in the fundamental representation all zero modes are of this kind for non-zero scalar expectation values). For $l_T = 1/2$, the wave function of the zero mode decays like r^{-3} in the massless channels. Going to a singular gauge we realize that this behaviour is the same as the large r behaviour of the free fermionic propagator. The propagator $G_F(\rho,v,\xi)$ also exhibits the same asymptotic behaviour for non-exceptional values of ξ. On the other hand, because of the presence of the extra terms on the r.h.s. of eq. (5.13) in the unperturbed instanton background, the propagator $D_F(\rho,v,0)$ decays only like r^{-2}. The role of the r.h.s. of eq. (5.15) is to cancel out this slower decaying tail of $D_F(\rho,v,0)$.

6. Supersymmetric QCD with $N_c \le N_f$

The non-perturbative properties of Supersymmetric QCD have been extensively discussed in the literature. Instanton induced effects in SQCD have been discussed within the semi-classical approximation in refs. [14-16]. More heuristic discussions are based on the effective lagrangian approach [17] and Witten's index [18] arguments.

SQCD is the supersymmetric SU(N_c) gauge theory whose matter sector consists of the chiral supermultiplets (ϕ_p^N,ψ_p^N) and $(\phi_p^{\bar N},\psi_p^{\bar N})$, $p = 1,..,N_f$. (ϕ_p^N,ψ_p^N) belongs to the fundamental representation and $(\phi_p^{\bar N},\psi_p^{\bar N})$ to the complex conjugate one. To all orders in perturbation theory, the scalar potential retains its classical value

$$V = \sum_{p=1}^{N_f} m_p^2\left(|\phi_p^N|^2 + |\phi_p^{\bar N}|^2\right) + \frac{g^2}{2}D^aD^a\,, \tag{6.1}$$

$$D^a = \sum_{p=1}^{N_f}\left((\phi_p^N)^\dagger T^a\phi_p^N - (\phi_p^{\bar N})^T T^a(\phi_p^{\bar N})^*\right)\,. \tag{6.2}$$

In the massive case ($m_p \ne 0$, all p) the scalar potential vanishes only at the origin. Massive SQCD has a unique supersymmetric classical vacuum. But if some of the matter fields are massless there are flat directions [14] along which the potential remains zero, given by

$$\left\langle\phi_p^N\right\rangle=\left\langle\phi_p^{\bar N}\right\rangle^*=v_p\,, \tag{6.3}$$

where v_p, $p=1,..,N_f$ are arbitrary vectors in the fundamental representation. For $m_p=0$ perturbation theory allows the scalar expectation values to lie anywhere along the flat directions, and so they can be determined only non-perturbatively. The scalar fluctuations along the flat directions may be very large also in the massive case provided $m \ll \Lambda_{SQCD}$ (for simplicity we take $m_p=m$ for all p).

The main conclusions of Affleck, Dine and Seiberg [14] are as follows. For $N_c > N_f$ a non-perturbative superpotential is generated. The resulting scalar potential is essentially an instanton-antiinstanton effect [14,19], with one instanton corrections proportional to m. For $0<m\ll\Lambda_{SQCD}$ the full (classical plus non-perturbative) scalar potential admits N_c supersymmetric minima, in agreement with Witten's index calculations. However, when $m\to 0$ the scalar expectation values tend to infinity, a phenomenon known as "runaway" behaviour. Eventually, for $m=0$ the scalar potential has no minimum for finite scalar VEVs. The conclusion reached in ref. [14] is that these theories have *no* ground state. Other authors [15-17] have reached similar conclusions regarding the pathological behaviour of the massless limit.

For $N_c\le N_f$ the authors of ref. [14] found that no instanton induced scalar potential is generated non-perturbatively. In ref. [15], which uses slightly different methods, the conclusions for $N_c\le N_f$ are as follows. For $0<m\ll\Lambda_{SQCD}$ there are N_c supersymmetric minima whereas for $m=0$ the infinite degeneracy of the classical vacua remains intact. As in the $N_c>N_f$ case, the vacua of the massive theory exhibit a runaway behaviour as $m\to 0$.

We will now show that the results of refs. [14-18] are incorrect for $N_c\le N_f$. In this case SUSY is *explicitly* broken by non-perturbative effects and SQCD exists in a strongly interacting phase similarly to ordinary QCD. For $m=0$ all we have to do is to verify that the conditions needed for the applicability of the results of sect. 5 are satisfied for generic scalar VEVs $v_p\gg\Lambda_{SQCD}$ which lie on the flat directions. We will also be able to show that the same conclusions hold for $0<m\ll\Lambda_{SQCD}$ and that the massless limit is smooth. The erroneous results of refs. [14-18] arise from a semi-classical approximation or rely on the unjustified assumption that a non-perturbative SUSY anomaly does not exist. For the case $N_c>N_f$ we have no new results to report, but we will argue that the conclusions of refs. [14-18] should be reexamined, considering that the SUSY anomaly may exist for $N_c>N_f$ as well.

We now proceed to show that the conditions mentioned in the beginning of sect. 5 are satisfied in SQCD with $m=0$ and $N_c\le N_f$. The classical vacuum

structure as well as the zero modes of massless SQCD have beed discussed in detail in ref. [14]. A generic configuration of the scalar VEVs breaks the gauge symmetry completely. (In fact, this statement is true already for $N_c=N_f+1$). For instance, in the special case $N_c=N_f$, up to a symmetry transformation the scalars VEVs can be taken to be $v_p^A=v\delta_p^A$ where A is a colour index. Consider next the fermion masses. For generic scalar VEVs the N_c^2-1 Majorana fields of the gaugino couple to an equal number of matter fields to make N_c^2-1 massive Dirac fermions. The rest of the matter fermions remain massless.

We next discuss the fermionic zero modes of SQCD for $m=0$. Consider first the case that all scalar VEVs are zero. The Dirac operator is then given by eqs. (2.2)-(2.4). There are two pairs of gaugino zero modes in the SU(2) subgroup which supports the instanton. They are conventionally denoted λ_{SS} and λ_{SC}. Their quantum numbers are $t=1$, $l=0$ and $k_1=1/2$ for λ_{SS} and $t=1$, $l=1/2$ and $k_1=0$ for λ_{SC}. In addition, the gaugino multiplet contains $2N_c-4$ fields transforming as doublets of SU(2). Correspondingly, there are N_c-2 additional pairs of zero modes with the quantum numbers $t=1/2$, $l=0$ and $k_1=0$. In the matter sector there is a pair of zero modes for each flavour whose quantum numbers are the same as those of the gaugino doublets i.e. $t=1/2$, $l=0$ and $k_1=0$.

For generic scalar VEVs which lie on the flat directions the classical background field contains both gauge and scalar fields but is still rotationally invariant. Because of their rotational symmetry, the scalar fields mix an *equal* number of gaugino and matter zero modes with the same total angular momentum i.e. the same k_1, and lead to their disappearance from the spectrum. Since matter and gaugino zero modes have opposite generalized chiralities, the index of the Dirac operator is conserved in this process. (Recall that the index is defined relative to generalized chirality). The above picture was derived in ref. [14] using first order perturbation theory, and it can be made rigorous using the tools developed in sect. 2.

For $N_c\le N_f$, all N_c-1 pairs of gaugino zero modes with $k_1=0$ mix with matter zero modes. The remaining zero modes are as follows. there are $n_-=N_f-N_c+1$ pairs of matter zero modes with negative generalized chirality and one pair ($n_+=1$) of positive generalized chirality ones. The zero modes with positive generalized chirality correspond to λ_{SS}, which now have additional components in right-handed matter channels.

We have seen that the assumptions stated in the beginning of sect. 5 are satisfied in SQCD with $N_c\le N_f$ and $m=0$. Using the general formulae of sect. (5) as well as eq. (1.7) we find that the resulting scalar potential is (see fig. 5)

$$\mathcal{E}(v)=-2|\Omega(v)|\approx -g^{-\alpha+N_f-N_c+2}v^4\exp(-8\pi^2/g^2(v))\quad,\quad v\gg\Lambda_{SQCD}\,. \tag{6.4}$$

This scalar potential is *negative* and monotonically increasing. Its minimum occurs for $v \sim \Lambda_{SQCD}$ where the dilute gas approximation leading to eq. (1.2) breaks down. Thus, massless SQCD with $N_c \leq N_f$ exists in a strongly interacting phase similarly to ordinary QCD. It has negative vacuum energy density, which implies the existence of a non-perturbative SUSY anomaly. It is expected that other properties of this phase such as the dynamical breaking of chiral symmetry will also be similar to ordinary QCD [10].

The next question we want to examine is the behaviour of SQCD with $N_c \leq N_f$ when a small mass term $0 < m \ll \Lambda_{SQCD}$ is included. For $m \neq 0$ the classical potential becomes $V(v) = 2N_f m^2 v^2$ in the previously flat directions. We must still look for possible non-perturbative contributions to the scalar potential which may be more important than the classical term if m is small enough. Notice that for $m \neq 0$ the Dirac operator no longer anticommutes with $\gamma_5 Q_R$.

For generic scalar VEVs and $m \neq 0$, the remaining $2(N_f - N_c + 1)$ matter zero modes of the $m = 0$ theory acquire an energy $E = m$ in the background field of the constrained instanton. This conclusion follows trivially once we recognize that the new term in the Dirac operator is equal to $m\gamma_5$ times a projection operator onto the matter sector. We point out that the matter bound states still remain as threshold modes. A rigorous analysis of the fate of λ_{RR} is more complicated. It is not clear whether these zero modes remain as bound states or become resonances with $E \approx m$. However, as can be seen from the analysis of sect. 5, in both cases one has $Det\{D_F(\rho, v, 0)\} = O(m)$ in the classical background field of the constrained instanton.

Let us now ask ourselves what happens when we turn on the destabilizing modes. The analysis of the low energy spectrum carried out in sect. 4 for $m = 0$ can now be repeated with $m \neq 0$. To be specific, we require that $m \lesssim v_0 \exp(-8\pi^2/g^2(v_0))$ for some $v_0 \gg \lambda_{SQCD}$. The effect of m will be to modify eqs. (4.9) and (4.12) by $O(m^2)$ terms. By our assumption on the magnitude of m these terms are negligible. We thus conclude that, when the destabilizing modes are turned on, we again have resonances whose energies are given by eq. (1.6) up to $O(m)$ corrections. Furthermore, the r.h.s. of eq. (5.11) still gives the contribution of these resonances to the spectral densities of $D^\dagger D$ and $DD^\dagger$ (again, up to $O(m)$ corrections). This implies that the resonances of eq. (1.6) *saturate* the generalized Levinson's theorem (5.10). There can be no bound states nor additional resonances whose energy is $O(m)$ for non-exceptional values of the amplitudes of the destabilizing modes. Thus, eq. (5.12) for the fermionic determinant holds up to additive $O(m)$ corrections. We finally conclude that for $N_c \leq N_f$ and $0 < m \lesssim v_0 \exp(-8\pi^2/g^2(v_0))$ the full scalar potential is still given by eq. (6.4) up to $O(\exp(-16\pi^2/g^2(v)))$ corrections for $\Lambda_{SQCD} \ll v \ll v_0$. This proves that all the qualitative properties of the massless case persist if small enough mass terms are added to the lagrangian and that the massless limit is smooth.

We now wish to comment on the $N_c > N_f$ case. For $N_c = N_f + 1$ the gauge symmetry is still broken by generic scalar VEVs. In this case no matter zero modes survive and we have only the λ_{SS} pair. Thus, there are zero modes of only one generalized chirality. As discussed earlier, under these circumstances the zero modes are absolutely stable and the index is conserved. The results of ref. [14] for $N_c = N_f + 1$ and $v \gg \Lambda_{SQCD}$ are therefore correct. Notice that this still leaves open the question of what is the scalar potential in the strong coupling regime $v \sim \Lambda_{SQCD}$. For $N_c > N_f + 1$ an unbroken non-abelian subgroup remains and so the heuristic discussion of this case in refs. [14-18] totally hinges on the assumption that a non-perturbative anomaly does not exist.

It is clear that the conditions mentioned in the beginning of sect. 5 *except* for the requirement of weak coupling situation are satisfied in all massless SQCD theories for vanishing scalar VEVs. Thus ,the integrand of the ρ-integral in eq. (1.5) is positive definite but we are faced with an infrared divergence. The same statement holds if small mass terms are added to the lagrangian. In our view this is a strong indication that the effective potential is negative in all SQCD theories for small scalar VEVs provided m is small enough. If this is true all SQCD theories should exist in a strongly interacting phase. Unfortunately, we are unable to support this conjecture by a quantitative calculation because of the usual infrared problems of strongly interacting non-abelian gauge theories.

7. Discussion and conclusions

In this paper we calculated in detail the instanton induced tunneling amplitude in massless (or almost massless) supersymmetric QCD. We found that, despite of the presence of massless fermions, tunneling is not suppressed. The physical origin of this phenomenon is the instability of the fermionic zero modes in the presence of scalar fields. The scalar fields mix gaugino and matter zero modes. Because of the absence of an energy gap, the result is that all zero modes become resonances. As our explicit calculation shows, the resonances reduce the tunneling amplitude by some power of g but cannot suppress it completely.

We showed that the effective potential of massless SQCD with $N_c \leq N_f$ is negative and monotonically increasing in the weak coupling regime. As a result, massless

SQCD with $N_c \leq N_f$ has a negative vacuum energy and exists in a strongly interacting phase similar to ordinary QCD. The same conclusions apply if small masses are added to the lagrangian and the massless limit is smooth. These theories exhibit a non-perturbative SUSY anomaly as the negative vacuum energy is incompatible with the SUSY algebra (1.3).

Previous treatments of SQCD with $N_c \leq N_f$ predicted a runaway behaviour of the vacua for $m \neq 0$ and an infinite set of inequivalent vacua for $m = 0$. Both of these predictions regarding the peculiar behaviour of SQCD with $N_c \leq N_f$ are seen to be erroneous. In fact, SQCD with $N_c \leq N_f$ turns out to be very similar to ordinary QCD in many respects. It exists in a strongly interacting phase and, most likely, its chiral symmetries are dynamically broken [10]. As in ordinary QCD, non-perturbative effects in SQCD with $N_c \leq N_f$ *lower* the ground state energy compared to its perturbative value.

The instability of the fermionic zero modes leads to an explicit violation of the SUSY algebra. While this is probably the most significant consequence of the zero mode's instability, it is by no means the only one. Another consequence is related to the anomalous R-symmetry of SQCD (which is really nothing but the generalization of the chiral anomaly to the case that scalar fields are present). As is well known, in the absence of scalar fields the anomaly equation is saturated by the fermionic zero modes. In fact, one often finds in the literature the erroneous statement that the index of the Dirac operator can be determined by the anomaly equation. This statement remains true in the presence of scalar fields provided the full background field is rotationally invariant. But for a generic background field there are no zero modes and the anomaly is saturated by the integrated difference between the spectral densities of positive and negative generalized chirality continuum eigenstates [5]. Other contexts in which the instability of the fermionic zero modes may be relevant include superconducting cosmic strings and monopole catalysis of proton decay.

While this paper provides a clear understanding of the reasons why tunneling is not suppressed in massless SQCD, two major questions remain unanswered: (a) what are the most general conditions under which the non-perturbative SUSY anomaly arise, and (b) what is the operator form of the non-perturbative SUSY anomaly. Regarding the first question it is known for instance that monopoles also induce an explicit violation of the SUSY algebra via the zero mode's instability [6]. While a semi-classical treatment of monopoles in N=1 SUSY theories predicts the existence of a supermultiplet containing two bosonic and two fermionic monopoles, a careful calculation reveals that the spectrum contains only a single (bosonic)

monopole. The would-be fermionic monopole is in fact unstable. It is a resonance whose energy is $O(g)$ and whose width is $O(g^2)$, which decays mainly into a bosonic monopole and a photino. (In ref. [6] the resonance behaviour exhibited by the low energy continuum eigenstates was not fully appreciated, and so the quantitative prediction given there for the decay rate of the fermionic monopole turns out to be incorrect. The result quoted here can be easily obtained by redoing the calculation of ref. [6] taking into account the properties of the low energy spectrum as discussed in this paper).

Another class of SUSY gauge theories for which there are strong indications that a non-perturbative SUSY anomaly always exist includes SUSY Higgs theories in two and three spacetime dimensions [4]. Beyond that, there are heuristic arguments which suggest that this anomaly may be a very general phenomenon, possibly existing in all asymptotically free SUSY gauge theories. The most important observation of ref. [4] (see in particular their sect. 6) is that in the presence of a non-perturbative object such as an instanton or a monopole the spectra of bosonic and fermionic fluctuations are not supersymmetric. Specifically, the Schrödinger-like operators D_F^2 and $\mathcal{H}_B$ define different quantum mechanical scattering problems except when the background field has an exact self-duality property.

This situation seems remarkably analogous to the case of the anomalous R-symmetry mentioned above. There, the anomaly generically arises because the *continuum* spectra of positive and negative generalized chirality fermions are different. In the case of the SUSY current S_μ, the calculation of certain matrix elements of $\partial_\mu S_\mu$ in a topologically non-trivial background also yields a non-vanishing result which can be directly expressed in terms of the discrepancy between the quantum mechanical scattering matrices of the operators D_F^2 and $\mathcal{H}_B$ [4]. We believe that this calculation provides an important starting point in the search for the operator form of the SUSY anomaly.

One requirement that the operator form of $\partial_\mu S_\mu$ must satisfy is that all its matrix elements vanish when calculated perturbatively, as it has been demonstrated that all perturbative SUSY anomalies can be cancelled by an appropriate choice of counter-terms [20]. Another important difference between the chiral anomaly and the non-perturbative SUSY anomaly is the following. In the case of the chiral anomaly (or the anomalous R-symmetry) there is a local differential operator which maps the spectra of positive generalized chirality fermions into negative generalized chirality ones and vice versa (see eq. (4.5)). As a result, the chiral anomaly can be expressed in terms of the asymptotic properties of the background field. But there is no local operation which maps the spectrum of D_F^2 onto the spectrum of $\mathcal{H}_B$. We interpret this as an indication that the operator form of the SUSY anomaly is not the SUSY transform of a topological density.

In the case of the chiral anomaly, the failure of the conservation equation is intimately related to the fact that the rigorous non-perturbative definition of an asymptotically free gauge theory relies on a lattice regularization which breaks the chiral symmetry explicitly. Supersymmetry too is not respected by the lattice regularization and in that sense it will not be a total surprise to find that the explicit SUSY breaking which exists for finite lattice spacing does not disappear in the continuum limit. (A different opinion on this issue can be found in ref. [21]). A further investigation of this issue and the other open questions mentioned above is left for the future.

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Appendix

SQCD consists of a gauge supermultiplet (A^a_μ, λ^a_L) and an equal number (N_f) of chiral supermultiplets in the fundamental representation and in the complex conjugate one. Suppressing flavour indices which can always be easily inserted we denote them (ϕ^N, ψ^N_L) and $(\phi^{\tilde N}, \psi^{\tilde N}_L)$ respectively. Before we can calculate anything we have to define the SUSY lagrangian in euclidean space, for which we adopt the conventions of ref. [4]. In addition to the above left-handed fermions, the euclidean lagrangian must also contain as independent variables the right-handed fermions that, in Minkowski space, would be obtained from the above left-handed fields by charge conjugation. It is convenient to group left and right handed fermions with the same transformation properties under the action of the gauge group into four component Majorana fermions. We thus have $\lambda = (\overline{\lambda}^a_R, \lambda^a_L)$, $\psi^N = (\overline{\psi}^{\tilde N}_R, \psi^N_L)$ and $\psi^{\tilde N} = (\overline{\psi}^N_R, \psi^{\tilde N}_L)$.

For $m=0$, the fermionic part of the euclidean SUSY lagrangian is [4]

$$\begin{aligned} \mathcal{L}_F \;=\; & \frac{i}{2}\overline{\lambda}\gamma_5 \not{D}\lambda + \frac{i}{2}\overline{\psi}^N \gamma_5 \not{D}\psi^N + \frac{i}{2}\overline{\psi}^{\tilde N}\gamma_5 \not{D}\psi^{\tilde N} \\ & + ig/\sqrt{2}\left(\overline{\lambda}^a \gamma_5 (\tilde\phi^N)^\dagger T^a \psi^N - \overline{\lambda}^a\gamma_5(\tilde\phi^{\tilde N})^\dagger (T^a)^* \psi^{\tilde N}\right) \\ & - ig/\sqrt{2}\left(\overline{\psi}^N \gamma_5 T^a (\tilde\phi^{\tilde N})^* \lambda^a - \overline{\psi}^{\tilde N}\gamma_5 (T^a)^* (\tilde\phi^N)^* \lambda^a\right)\,, \qquad \text{(A.1)} \end{aligned}$$

where

$$\tilde\phi = \frac{1}{2}(1-\gamma_5)\phi + \frac{1}{2}(1+\gamma_5)\mathcal{C}\phi^*\,, \qquad (A.2)$$

and ϕ stands for either ϕ^N or $\phi^{\tilde N}$. The mapping $\mathcal{C}$ is defined as follows. In the bosonic sector $\mathcal{C}_B A^a_\mu = A^a_\mu$, $\mathcal{C}_B\phi^N = \phi^{\tilde N}$ and $\mathcal{C}_B\phi^{\tilde N} = \phi^N$. In the fermionic sector $\mathcal{C}_F = \gamma_1\gamma_3\mathcal{C}_B$. We take the anti-hermitian gamma matrices to be in the chiral representation.

We also define $D_F(\beta)$ by rewriting $\mathcal{L}_F$ as

$$\mathcal{L}_F = \frac{1}{2}\overline{\chi} D_F(\beta)\chi\,. \qquad (A.3)$$

Here β is a generic name for all Bose fields and χ is a generic name for all Fermi fields. $\overline{\chi}$ is not an independent variable. Rather, it is related to χ through the Majorana condition

$$\overline{\chi} = \chi^T \mathcal{C}\,. \qquad (A.4)$$

The following identities always hold

$$D^T_F(\beta) = \mathcal{C}^{-1} D_F(\beta)\mathcal{C}\,, \qquad (A.5)$$

$$D^\dagger_F(\beta) = D_F(\mathcal{C}\beta^*)\,. \qquad (A.6)$$

In addition, for scalar configurations which satisfy the reality condition

$$\phi^{\tilde N} = (\phi^N)^*\,, \qquad (A.7)$$

the Dirac operator $D_F(\beta)$ is hermitian and satisfies

$$D_F(\beta) = \mathcal{C}^{-1} D^*_F(\beta)\mathcal{C}\,. \qquad (A.8)$$

Notice that $D_F(\beta)$ can be expressed as $\gamma_5 D'_F(\beta)$. It is $D'_F(\beta)$ rather than $D_F(\beta)$ which is obtained by analytically continuing the minkowskian lagrangian to euclidean space. The formulation in terms of $D_F(\beta)$ has exactly the same physical content because $Det\{D_F(\beta)\} = Det\{D'_F(\beta)\}$. It is more convenient because of the hermiticity property of $D_F(\beta)$ mentioned above.

We give below some of the technical detail needed for the derivation of the results of sect. 3 and 4. We consider a background field which consists of the gauge field of eqs. (2.3) and (2.4) plus a singe destabilizing mode. The four relevant channels consist of the two channels supporting the gaugino and matter zero modes which are coupled through the destabilizing mode, as well as their antimode channels. The radial equations in these channels take the form

$$-\partial_r v_\lambda(r) - r^{-1}[3+2(l_\lambda - a(r)(t_\lambda+1))]\,v_\lambda(r) + g\xi C_1\overline{\delta\phi}^*(r)u_\psi(r) = Eu_\lambda(r), \quad \text{(A.9a)}$$
$$\partial_r u_\lambda(r) - 2r^{-1}(l_\lambda - a(r)(t_\lambda+1))\,u_\lambda(r) - g\xi C_1\overline{\delta\phi}^*(r)v_\psi(r) = Ev_\lambda(r), \quad \text{(A.9b)}$$
$$-\partial_r v_\psi(r) - r^{-1}[3+2(l_\psi - a(r)(t_\psi+1))]\,v_\psi(r) + g\xi C_1\overline{\delta\phi}(r)u_\lambda(r) = Eu_\psi(r), \quad \text{(A.9c)}$$
$$\partial_r u_\psi(r) - 2r^{-1}(l_\psi - a(r)(t_\psi+1))\,u_\psi(r) - g\xi C_1\overline{\delta\phi}(r)v_\lambda(r) = Ev_\psi(r). \quad \text{(A.9d)}$$

The normalized destabilizing mode is given by

$$\overline{\delta\phi}^N(x) = \left(\overline{\delta\phi}^N(x)\right)^* = \frac{1}{2\sqrt{2C}}\mathcal{U}^\dagger_{\lambda a}(x)T^a\mathcal{U}_\psi(x)\,, \tag{A.10}$$

where

$$4C^2 = \int d^4x\,\left|\mathcal{U}^\dagger_{\lambda a}(x)T^a\mathcal{U}_\psi(x)\right|^2\,, \tag{A.11}$$

and $\mathcal{U}_\lambda(x)$ and $\mathcal{U}_\psi(x)$ are the unperturbed gaugino and matter zero modes respectively. (Eq. (A.10) gives the destabilizing mode when the original background field contains scalar fields with non-zero expectation values as well. In this case $\mathcal{U}_\lambda(x)$ stands for the gaugino component of the λ_{SS} zero mode). The radial function $\overline{\delta\phi}(r)$ that enters eq. (A.9) is obtained by writing

$$\overline{\delta\phi}^N(x) = \overline{\delta\phi}(r)Y(x_\mu/r)\,, \tag{A.12}$$

where $Y(x_\mu/r)$ is a normalized angular eigenstate which carries the colour and flavour indices of $\overline{\delta\phi}^N(x)$. Making an analogous separation of the radial dependence of the zero modes

$$\mathcal{U}_\lambda(x) = \mathcal{U}_\lambda(r)Y_\lambda(x_\mu/r)\,, \tag{A.13a}$$
$$\mathcal{U}_\psi(x) = \mathcal{U}_\psi(r)Y_\psi(x_\mu/r)\,, \tag{A.13b}$$

one has

$$Y(x_\mu/r) = \frac{1}{\sqrt{2C_1}}Y^\dagger_{\lambda a}(x_\mu/r)T^aY_\psi(x_\mu/r)\,. \tag{A.14}$$

Notice that the scalar perturbation considered in eq. (3.1) has the same quantum numbers as the destabilizing mode, but apriori the only requirement from $\delta\phi(r)$ of eq. (3.1) is that it be normalizable.

We now discuss the low energy continuum eigenstates which solve eq. (A.9), but first in the absence of the destabilizing mode. In this case there is no coupling between the gaugino and matter channels and so we drop the subscripts λ and ψ. For $r \ll E^{-1}$ one can solve for $u(E,r)$ and $v(E,r)$ by expanding in powers of E. Thus, $u(E,r) = u^{(0)}(r) + E^2u^{(2)}(r) + \cdots$ and $v(E,r) = Ev^{(1)}(r) + E^3v^{(3)}(r) + \cdots$. The individual terms in this expansion can be obtained by successively integrating,

say, eqs. (A.9a) and (A.9b) starting with $u^{(0)}(r)$ which coincides with the zero mode up to a normalization constant (compare eq. (4.7))

$$u^{(0)}(r) = \mathcal{F}(E)b^{-1}\mathcal{U}(r)\,, \tag{A.15}$$

The solution $v(E,r)$ must be regular at the origin. Recalling the properties of the homogeneous solution in the antimode channel this gives in the first step

$$v^{(1)}(r) = -E\mathcal{F}(E)b^{-1}h(r)\int_0^r dr'\,(r')^3\mathcal{U}^2(r')\,. \tag{A.16}$$

In the intermediate r-range one has

$$v^{(1)}(r) = -E\mathcal{F}(E)b^{-2}r^{2l_T-1} + O(r^{2l_T-3}) \quad , \quad 1 \ll r \ll E^{-1}\,. \tag{A.17}$$

For a zero mode $l_T = t - l \geq 1/2$. In going from eq. (A.16) to eq. (A.17) we have used the fact that the zero mode is normalized as well as eqs. (2.12) and (4.7). Notice that the boundary conditions implicit in eq. (A.16) are equivalent to choosing $r_0 = 0$ and $c = 0$ in eq. (3.3). No homogeneous term arises in the second step which gives

$$u^{(2)}(r) = b^{-2}E^2r^{2l_T}/(4l_T+2) + O(r^{2l_T-2}) \quad , \quad 1 \ll r \ll E^{-1}\,. \tag{A.18}$$

Summing to two leading terms of $u(E,r)$ we arrive at eq. (4.9).

Let us now turn on the destabilizing mode. The four radial channels of eq. (A.9) are now all coupled. Using similar (up to normalization) boundary conditions at $r = 0$ as before we find two solutions. In the intermediate range $1 \ll r \ll E^{-1}$ one solution is

$$u_\lambda(E,r) \approx r^{-2-2l_T} - b^{-2}E^2r^{2l_T}/(4l_T+2)\,, \quad \text{(A.19a)}$$
$$v_\lambda(E,r) \approx -Eb^{-2}r^{2l_T-1}\,, \quad \text{(A.19b)}$$
$$u_\psi(E,r) \approx b^{-2}gC\xi Er^{2l'_T}/(4l'_T+2)\,, \quad \text{(A.19c)}$$
$$v_\psi(E,r) \approx b^{-2}gC\xi r^{2l'_T-1}\,. \quad \text{(A.19d)}$$

Corrections to eq. (A.19) are either of higher order in E and ξ, or involve lower powers of r compared to the ones shown for zeroth and second order in E and ξ. In eq. (A.19) l_T refers to the gaugino zero mode and l'_T to the matter zero mode. The second solution is obtained by interchanging the role of λ and ψ as well as l_T and l'_T in eq. (A.19).

The two solutions found above are not the physical scattering states. The physical states are their (properly normalized) linear combinations that minimize the mixing between channels which decouple for $\xi = 0$. The modified physical eigenstate of e.g. $\mathcal{D}^\dagger\mathcal{D}$ which exists mainly in the gaugino zero mode channel is equal to $\mathcal{F}(\xi, E)$ times the first solution (see eqs. (A.19a) and (A.19d)) plus $\mathcal{F}(\xi, E) gC\xi/E$ times the second solution. In the intermediate range $1 \ll r \ll E^{-1}$ the solution in the positive generalized chirality channels is

$$\begin{pmatrix} u_\lambda(\xi, E, r) \\ v_\psi(\xi, E, r) \end{pmatrix} \approx \mathcal{F}(\xi, E) \begin{pmatrix} r^{-2-2l_T} + b^{-2} \left(|gC\xi|^2 - E^2 \right) r^{2l_T}/(4l_T + 2) \\ b^{-2} gC\xi r^{-1-2l_T}/(4l_T) \end{pmatrix} . \quad (A.20)$$

Eq. (4.12) is simply the upper component of eq. (A.20).

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Figure Captions

(1) Qualitative structure of the spectrum of the Dirac operator (a) on a compact space, (b) in an open space when all fermions are massive and (c) in an open space when some fermions are massless. In all cases the existence of generalized chirality is assumed which implies that the spectrum is symmetric around $E = 0$. In an open space the continuous spectrum (shaded area) exists for $|E| > m_0$ where m_0 is the smallest mass. In case (b) there may in general be bound states with non-zero energy as well.

(2) Homogeneous solutions in antimode channels. The asymptotic behaviour is r^{-3-2l} at the origin and r^{2l_T-1} at infinity, where l and l_T are the quantum numbers of the corresponding zero mode. The solid line corresponds to $l_T > 1/2$ and the dashed line to $l_T = 1/2$.

(3) Qualitative behaviour of the potential of the Schrödinger-like operator $D_F^2(A_\mu)$ in a left-handed channel supporting a zero mode. The dashed line shows the energy that an imaginary particle living in five dimensional spacetime will acquire as a result of a scalar perturbation if its wave function coincides with a zero mode at $t = 0$. The right end of this line is the point where the particle starts free propagation after it has tunneled through the centrifugal barrier.

(4) (a) The linear vertex of the Feynman rules of sect. 5. (b) A forbidden diagram.

(5) The effective potential of massless SQCD with $N_c \leq N_f$. For $m \neq 0$, the effective potential can be made arbitrarily close to the massless case over an arbitrarily large part of the weak coupling regime if we take m small enough. As $v \to \infty$ the effective potential will eventually be dominated by the classical term $2N_f m^2 v^2$ and will change sign. For both $m = 0$ and $m \neq 0$ the minimum of the effective potential is negative and it occurs in the strongly interacting regime $v \sim \Lambda_{SQCD}$.

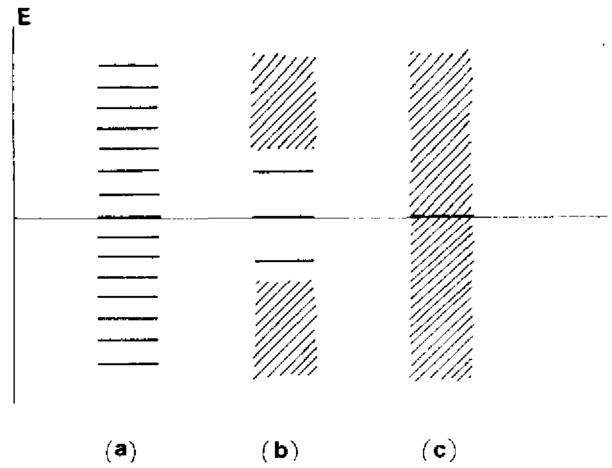

Fig. 1

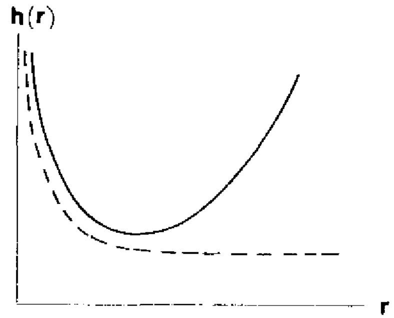

Fig. 2

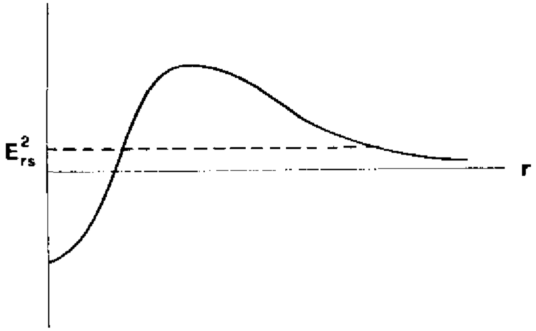

Fig.3

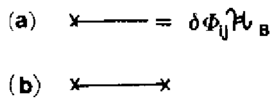

Fig.4

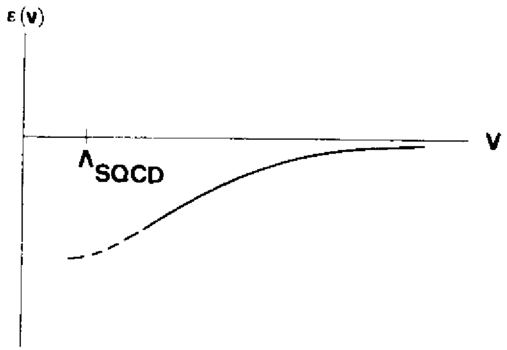

Fig.5

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