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THE CRACK ENERGY ABSORPTIVE CAPACITY OF COMPOSITES WITH FRACTAL STRUCTURE *

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ABSTRACT

This paper discusses the energy absorptive capacity of composites with fibers of fractal stuctures. It is found that this kind of structure may increase the absorption energy during the crack propagation and hence the fracture toughness of composites.

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I. INTRODUCTION

The fractal structure of fillers in composite materials, may play an important role in the way in which a filler absorbs crack energy when cracks attempt to move through a composite material. The energy of the crack was dissipated as it ruptured the interface between the limestone fineparticle and the plastic. Adding fibers to weak materials to increase their crack resistence and hence their effective strength. The addition of the wood flour improves the mechanical properties of the substance at the same time as it lowers costs [1].

If the material has been properly made there is not too much and not too little adhesion between the fiber and the matrix then the fiber will not be broken at that point but the material will crack at the interface between the glass and the resin, the crack will spread along the fiber so that the fiber becomes detached from the matrix often for a considerable distance.

A fractally structured pigment may retain some air internally when incorporated into a plastic. This would give it some compressive resilience with enhanced ability to absorb crack energy both from its springiness and its very large surface area. The performance of even simple crushed limestone–type fillers may be enhanced by coating the fillers with silica flow agents which have fractal structures so that they can weaken the bond between the plastic and the filler and also adsorbs considerable crack energy by their own fractally high surface area.

In this paper, we discuss the energy absorption of crack energy including the pull–out energy of fibers by their own fractally high surface area in composite materials.

II. FIBERS WITH ROUGH AND IRREGULAR SURFACES

The performance of even simple crushed limestone–type fillers may be enhanced by coating the fillers with silica flow agents which have fractal structures so that they can weaken the bond between the plastic and the fillers and also absorbs considerable crack energy by their own fractally high surface area [1]. Rough and irregular surfaces have high surface areas. They can be described by fractals approximately and statistically.

Let us consider the cross–section of a fiber in a composite as a Koch island and its perimeter as a Koch curve (Fig.1). We assume both the matrix and fibers are brittle. For simplicity, we assume that in the direction parallel to the fiber, the surface is composed of regular straight lines. The length of the perimeter of the cross–section may be calculated by

$$\begin{aligned} P(\varepsilon) &= \pi\, L(\varepsilon) \\ L(\varepsilon) &= \varepsilon^{1-D} \\ L(\varepsilon) &= L(\eta)/L_0 \\ \varepsilon &= \eta/L_0 \end{aligned} \qquad (1)$$

where, P is the length of the perimeter, $L(\varepsilon)$ is the length of an elementary Koch curve and ε is the length of the yardstick. $L(\varepsilon)$ and ε are normalized with respect to cetain well defined length,

L_0 (the maximum Feret's diameter, the maximum projected length of the island etc.). They are dimensionless quantities. D is the fractal dimension as defined by B. Mandelbrot [2]. P can be very large when ε is very small. D can be measured by the slope of the P and ε line in log–log plot. Then, the value of the perimeter can be estimated by the fractal dimension D and the length of the smallest step in the Koch curve. P multiplied by the length of the fiber is the fractally high interface area between the fiber and the matrix

$$P = \pi\, L(\eta) = \pi L_0 \left(\frac{\eta}{L_0}\right)^{1-D} = \pi\, L_0^D\, \eta^{1-D} . \tag{2}$$

Their dimensions are [Length] with an appropriate unit

$$A_i = Pl = \pi\, L_0^D\, \eta^{1-D} l . \tag{3}$$

In composites, the transverse tensile strength is much smaller than the corresponding strength parallel to the fibers. The former properties are dominated by the matrix properties whereas the latter are dominated by the fiber properties [3]. The strength of the fiber–matrix interface plays a very important role in the crack propagation though with different mechanism in transverse and longitudinal directions. In the former case, energy is consumed mainly in opening the interfaces whereas the latter is consumed mainly in sliding on the interfaces.

1. The energy absorptive capacity of interfaces

There are three possibilities of different modes of failure of a fiber embedded in a matrix: adhesive failure at the interface, cohesive failure of the matrix close to the interface and cohesive failure of the fiber close to the interface. Clearly, the occurrence of either adhesive or cohesive failure will depend on the relative strengths of the interface and the fiber or matrix. If the strength of the interface is weaker than the fiber or the matrix, the crack prefers to propagate through the interface between the fiber and the matrix. The energy absorptive capacity of the interface from a crack is determined by the specific interface energy and the area of it. The fiber with rough and irregular surface has high surface area. Its own fractally high surface area is the cause of high energy absorptive capacity.

The critical crack extension force in homogeneous materials is given by

$$G_{1c} = 2\ \nu_m . \tag{4}$$

The effective critical crack extension force of a composite with regular fibers of circular cross–sections is given by

$$G^0_{efc} = 2\pi\ L_0 l\ \nu_{fm}/L_0 l = 2\pi\ \nu_{fm} . \tag{5}$$

In composite materials with fibers of fractal structure, the effective critical crack extension force on the transverse direction of loading is given by

$$\begin{aligned} G_{efc} &= 2\,P\ l\ \nu_{fm}/L_0\, l \\ &= 2\pi\ L_0^D\ \eta^{1-D} l\ \nu_{fm}/L_0\, l \\ &= 2\pi\ L_0^{D-1}\ \eta^{1-D}\ \nu_{fm} . \end{aligned} \tag{6}$$

We obtain from (4), (5) and (6)

$$G_{efc}/G^0_{efc} = L_0^{D-1}\ \eta^{1-D} = \varepsilon^{1-D} . \tag{7}$$

The effective critical crack extension force, or fracture toughness increases with a factor ε^{1-D}. It should be very large when ε is very small.

Furthermore, from Eqs.(5) and (7), we may see that

$$ln\, G_{efc} = ln(2\pi\ \nu_{fm}) + (1 - D)\ ln\, \varepsilon \tag{8}$$

and

$$ln(G_{efc}/G^0_{efc}) = ln\, \varepsilon - D\ ln\, \varepsilon . \tag{9}$$

Because $\varepsilon < 1$, $ln\, G_{efc}$ or $ln(G_{efc}/G^0_{efc})$ increases linearly with the fractal dimension, D.

2. The pull–out energy of a fiber

For fibers embedded in brittle matrices, the stress required to extract the fiber does not drop to zero after debonding has occurred because there are large frictional forces which resist the sliding of the fiber out of the matrix sheath. The frictional forces are usually attributed to residual stresses associated with the matrix shrinkage during curing and to differential thermal contraction. The reduction in the stress on the fibers at debonding results in an increase in fiber diameter owning to the Poisson expansion and this leads to an increase in the pressure and hence in the frictional force on the fiber surface. The frictional forces make a major contribution to the work required to pull the fiber out of the matrix if the distance over which pull–out occurs is large.

In longitudinal direction, the pull–out energy of fibers is important. It is given by [4]

$$\begin{aligned} W_{p0} &= \alpha\ P \\ \alpha &= \tau_i\ l^2/24 \qquad (l < l_c) \end{aligned} \tag{10}$$

where τ_i is a friction parameter of the matrix–fiber interface. The l_c is called the ineffective length or critical fiber length over which the fiber stress is less than $\phi\sigma_{f0}$ where σ_{f0} is the uniform fiber stress undisturbed by the end of a broken fiber and ϕ is a parameter with $0 = \phi < 1$.

$$W_{p0} = \pi(L_0\ \varepsilon^{1-D})\tau_i l^2/24 \tag{11}$$

$$W^0_{p0} = \pi\ d\ \tau_i l^2/24 \tag{12}$$

$$W_{p0}/W^0_{p0} = \pi\ L_0\ \varepsilon^{1-D}/d = \varepsilon^{1-D} \qquad (L_0 \sim d) \tag{13}$$

The pull–out energy per a fiber increases with a factor ε^{1-D}. The same as above, we obtain,

$$ln(W_{p0}/W^0_{p0}) = ln\, \varepsilon - Dln\, \varepsilon . \tag{14}$$

Again, $ln(W_{p0}/W_{p0}^0)$ or W_{p0} increases linearly with the fractal dimension D.

3. The typical numerical values of D and ε

We can find some data on fractal dimension measurements or theoretical analyses in recent published papers:

(1) $Si/600$, Silicon ground on 600 grit paper: $D = 1.04$ [5], (2) $Pt/0.05$, Platinum plate polished with $0.05\mu m$ alumina powder: $D = 1.11$, $\eta = 0.05\mu m$ [5], (3) A537CL1 offshore structural steel under cyclic loading and saltwater spray condition: $D = 1.15$ [6], (4) In corrosion process: $D = 1.465$ [7], (5) Metal grains: $D = 1.26$ [8], (6) Grain shapes of Zn in the recovery and recrystallization process: $D = 1.422; 1.522; 1.358$ [9].

Assuming the diameter of the fiber being $10\mu m$ [3] and $\eta = 0.05\mu m$ [4]; then, $\varepsilon = 0.005$. Therefore, the values of the factor ε^{1-D} are 14.14; 3.76 and 1.939 for D values of 1.5; 1.25 and 1.125, respectively. It seems that the energy absorption from crack including the pull-out energy of fibers would be increased a large amount by their own fractally high surface area in composite materials. Fig.2 shows the relation between them and the yardstick with different fractal dimensions.

III. THE FIBER DIVIDED AS A CANTOR SET

Suppose we have a fiber in the matrix, the specific interface energy is smaller than the specific surface energy of the matrix. The crack may propagate through the interface between the fiber and the matrix. The condition for absorbing more energy in cracking by using fibers is that

$$2\ \nu_{fm} \cdot \pi d \cdot l > 2\ \nu_m \cdot d \cdot l$$
$$\nu_{fm} > \nu_m/\pi\ . \tag{15}$$

That means,

$$\nu_m > \nu_{fm} > \nu_m/\pi\ . \tag{16}$$

Eq.(16) is the necessary condition for the crack propagating through the interfaces and absorbing more energy. It is independent on the radius of the fiber.

The energy absorbed by a fiber from the crack is given by

$$G_{efc}^{(1)} = \pi(\nu_{fm}/\nu_m)G_{1c}\ . \tag{17}$$

The energy absorbed by n_f fibers is given by

$$G_{efc}^{(n)} = n_f G_{efc}^{(1)} = n_f\pi(\nu_{fm}/\nu_m)G_{1c}\ . \tag{18}$$

The energy absorptive capacity depends on the number of fibers. However, if we consider the volume fraction of fibers in the composite, we may prefer to divide the fiber into finer ones under constant volume fraction. But, we cannot increase the number of fibers infinitely. We shall explain it later (Fig.3).

A fiber of length l is divided into two finer ones of the same length (Fig.3). The diameter of fibers in the first generation is d_1, the second generation, $2^{-1/2}\ d_1$ and the n-th generation $d_n = 2^{-(n-1)/2}\, d_1$.

The fractal dimension of the Cantor set can be calculated by [2],

$$D = \log N/\log(1/r) \tag{19}$$

where,

$$N = 2$$
$$1/r = d_1/d_2 = 2^{1/2}\ .$$

Therefore

$$D = 2\ .$$

The diameter of the fiber in the n-th generation is

$$2^{-\frac{n-1}{2}}\ d_1\ .$$

The number of fibers in the n-th generation is

$$2^{(n-1)}\ .$$

The total perimeters (or interfaces) between fibers and the matrix is

$$\pi\sum_n d_n = \pi\ d_1(d_n/d_1)^{1-D} = 2^{\frac{(n-1)}{2}}\ \pi\ d_1\ .$$

As above, the energy absorptive capacity for the transverse loading case and the pull-out energy of fibers for the longitudinal case are increased by a factor $2^{(n-1)/2}$, respectively.

The necessary condition is still Eq.(16). If the finer fibers are distributed only on the crack plane, there is an upper limit of n at which,

$$2^{(n-1)}\, d_n = L_1 \quad \text{(the distance between fibers of diameter } d_1\text{)}$$

or

$$2^{\frac{n-1}{2}}\, d_1 = L_1$$
$$n_m = 1 + 2\,ln(L_1/d_1)/ln\,2\ . \tag{20}$$

The contribution to the critical crack extension force is given by

$$G_{efc}^{(n)} = \pi\, 2^{\frac{n_m-1}{2}}\ (d_1/L_1)\ (\nu_{fm}/\nu_m)G_{1c}\ . \tag{21}$$

If some of the finer fibers are distributed on other planes near the crack, the crack would branch away and the situation is complicated. Anyhow, the contribution to the critical crack extension force would be more.

In general, if the fiber is divided into N fibers as a Cantor set with the total cross–section area being constant, the diameter of fibers in the first generation is d_1, the second generation, $N^{-1/2} d_1$ and the n–th generation, $d_n = N^{-(n-1)/2} d_1$.

The fractal dimension of this Cantor set can be calculated by

$$D = \log N / \log N^{1/2} = 2 \ .$$

The diameter of the fiber in the n–th generation is

$$N^{-\frac{(n-1)}{2}} d_1 \ .$$

The number of fibers in the n–th generation is

$$N^{(n-1)} \ .$$

The total perimeters (or interfaces) between fibers and the matrix is

$$\pi \sum_n d_n = \pi d_1 (d_n/d_1)^{(1-D)} = N^{\frac{n-1}{2}} \pi d_1 \ .$$

The energy absorption capacity for the transverse loading case and the pull–out energy of fibers for the longitudinal loading case are increased by a factor $N^{n-1/2}$, respectively.

The numerical values are in Table 1.

Table 1

The values of $\left[\left(G^{(n)}_{efc}/G^{0}_{efc}\right)\right]$ or $\left[\left(W^{(n)}_{p0}/W^{0}_{p0}\right)\right]$ (*)

n \ N (*)	2	3	4
1	1.000	1.000	1.0
2	1.414	1.732	2.0
3	2.000	3.000	4.0
4	2.828	5.196	8.0
5	4.000	9.000	16.0
6	5.656	15.588	32.0

IV. NON–UNIFORM FRACTAL STRUCTURE OF FIBERS

The fiber can be divided non–uniformly. Suppose we divide the fiber into two fibers with one of diameter $(3/5) d_1$ and another one of diameter $(4/5) d_1$ (Fig.4). The total area of the two fibers is $\frac{\pi}{4}\left[\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2\right] d_1^2 = \frac{\pi}{4} d_1^2$. The volume fraction of fibers does not change after this division. This process may continue and this multifractal is obtained after infinitely many recursions.

1. The multifractal structure

According to Feder [10], the weighted number of objects in boxes, $N(q, \delta)$ has the form

$$N(q,\delta) = \left[\left(\frac{\frac{3}{5}}{\frac{3}{5}+\frac{4}{5}}\right)^q + \left(\frac{\frac{4}{5}}{\frac{3}{5}+\frac{4}{5}}\right)^q\right]^n = \left[\left(\frac{3}{7}\right)^q + \left(\frac{4}{7}\right)^q\right]^n , \quad \delta = \frac{d_n}{d_1} = 2^{-\frac{n}{2}} \tag{22}$$

$$\tau(q) = -\lim_{\delta \to 0} \frac{\ell n\, N(q,\delta)}{\ell n\, \delta} = \left(\frac{2}{\ell n\, 2}\right) \ell n \left[\left(\frac{3}{7}\right)^q + \left(\frac{4}{7}\right)^q\right] \tag{23}$$

$$\alpha(q) = -\frac{d\tau(q)}{dq} = -\left(\frac{2}{\ell n\, 2}\right) \frac{1}{\left(\frac{3}{4}\right)^q + 1} \left[\left(\frac{3}{4}\right)^2 \ell n \left(\frac{3}{7}\right) + \ell n \left(\frac{4}{7}\right)\right] \tag{24}$$

$$\begin{aligned} f(q) &= \alpha q + \tau(q) \\ &= -\left(\frac{2}{\ell n\, 2}\right) \frac{q}{\left(\frac{3}{4}\right)^q + 1} \left[\left(\frac{3}{4}\right)^q \ell n \left(\frac{3}{7}\right) + \ell n \left(\frac{4}{7}\right)\right] \\ &\quad + \left(\frac{2}{\ell n\, 2}\right) q\, \ell n \left(\frac{4}{7}\right) \\ &\quad + \left(\frac{2}{\ell n\, 2}\right) \ell n \left[\left(\frac{3}{4}\right)^q + 1\right] \ . \end{aligned} \tag{25}$$

The above two equations, (24) and (25) give a parametric representation of the $f(\alpha)$ curve, i.e., the fractal dimension, $f(\alpha)$ of the support of "singularities" in measure with the Lipschitz–Holder exponent α. The $f(\alpha)$ curve characterizes the measure and is equivalent to the sequence of "mass" exponent $\tau(q)$. Using the pair of Eqs.(24) and (25) for the following simple example of the binomial multiplicative process with $\tau(q)$ given by (23), we can obtain the $f(\alpha)$ curve.

The values of the sequence of $\alpha(q)$ and $f(\alpha(q))$ are in Table 2.

Table 2

The values of $\alpha(q)$ and $f(\alpha)$

q	$\alpha(q)$	$f(\alpha(q))$
$-\infty$	2.45	0
-4	2.23	1.65
-3	2.18	1.79
-2	2.13	1.90
-1	2.07	1.98
0	2.03	2.0
1	1.96	1.96
2	1.90	1.87
3	1.85	1.79
4	1.8	1.54
∞	1.62	0

The $f(\alpha) - \alpha(q)$ curve is shown in Fig.5.

2. The sum total of the perimeters of fine fibers

For 1st generation, πd_1

For 2nd generation, $\pi(3/5 + 4/5)d_1 = \pi(7/5)d_1$

For 3rd generation, $\pi(3/5 + 4/5)^2 d_1 = \pi(7/5)^2 d_1$

For nth generation, $\pi(3/5 + 4/5)^{n-1} d_1 = \pi(7/5)^{n-1} d_1$

In general, if the fiber is divided into two fibers, the diameters of which are pd_1 and $(1-p^2)^{1/2}d_1$, respectively, the sum of the perimeters of fine fibers for the n–th generation is given by

$$\pi\left(p + (1-p^2)^{1/2}\right)^{n-1} d_1 . \tag{26}$$

3. The enhancement of the energy absorptive capacity

For n–th generation fibers, it is given by

$$G^{(n)}_{efc}/G^{(1)}_{efc} \quad \text{or} \quad W^{(n)}_{p0}/W^{(1)}_{p0} = \left[p + (1-p^2)^{1/2}\right]^{n-1} . \tag{27}$$

Fig.6 shows the relationship of Eq.(27). We may see that the homogeneous division is the most effective one from an energy absorption point of view.

V. SUMMARY

1. If the surface of the fiber is rough and irregular,the energy absorption from crack or the pull–out energy of fibers would increase a factor ε^{1-D} by their own fractally high surface area in composite materials.

2. If the fiber is divided into N fibers as a Cantor set with the total area being constant, both the energy absorptive capacity for the transverse loading case and the pull–out energy of fibers for the longitudinal loading case would increase by a factor $N^{n-1/2}$.

3. If the fiber is divided into two non–uniform finer fibers, the energy absorptive capacity or the pull–out energy of the composite would increase by a factor $\left[p + (1-p^2)^{1/2}\right]$.

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FIGURE CAPTIONS

Fig.1 The schematic figure of the cross–section of fibers with fractal perimeter.

Fig.2 (G_{efc}/G_{efc}^0) or (W_{p0}/W_{p0}^0) vs ε relationship.

Fig.3 The schematic figure of the fiber divided into two finer fibers as a Cantor Set.

Fig.4 The schematic figure of the fiber divided into two non–uniform finer fibers as a multifractal.

Fig.5 The $f(\alpha)$ vs α curve of the multifractal model.

Fig.6 (G_{efc}/G_{efc}^0) or (W_{p0}/W_{p0}^0) vs n relationship.

(1) $p = \frac{1}{\sqrt{2}}$, $q = \frac{1}{\sqrt{2}}$; (2) $p = \frac{3}{5}$, $q = \frac{4}{5}$;

(3) $p = \frac{2}{5}$, $q = \frac{\sqrt{21}}{5}$; (4) $p = \frac{1}{5}$, $q = \frac{2\sqrt{6}}{5}$

$(q = (1 - p^2)^{1/2})$

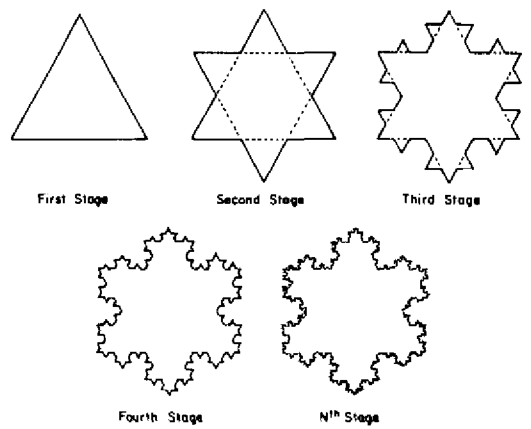


Fig.1

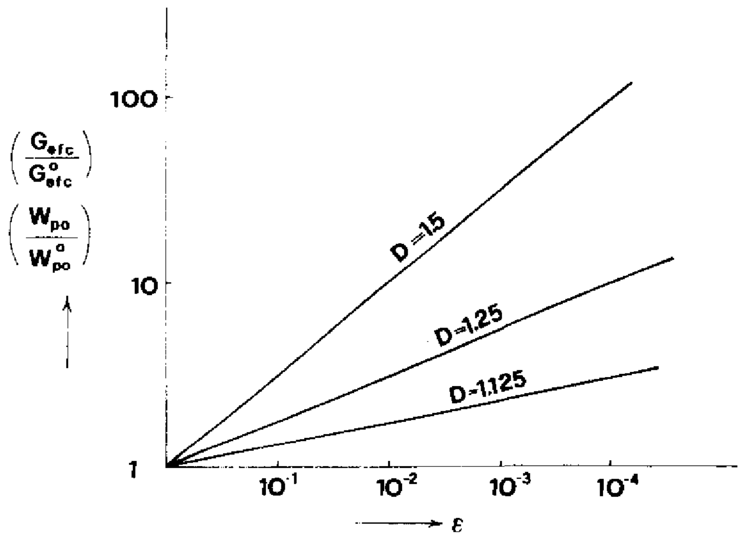


Fig.2

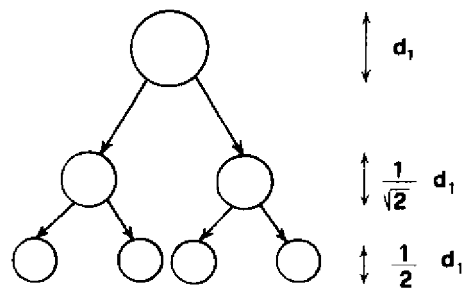


Fig.3

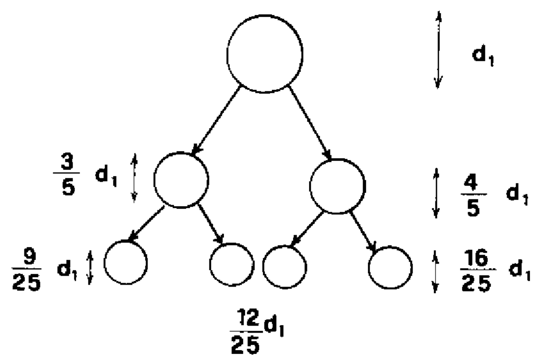


Fig.4

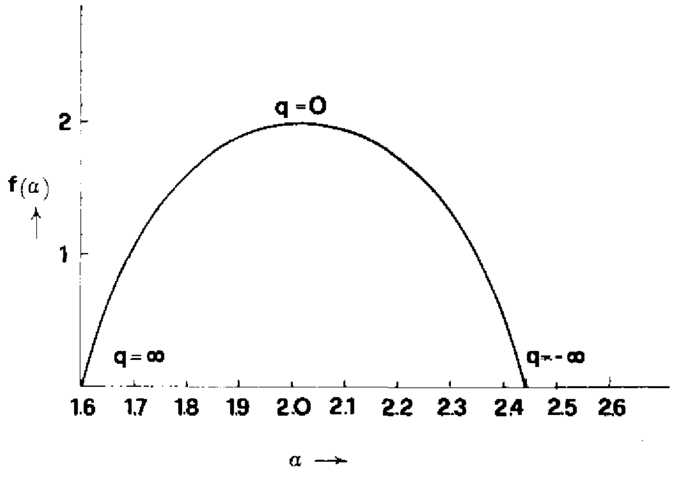
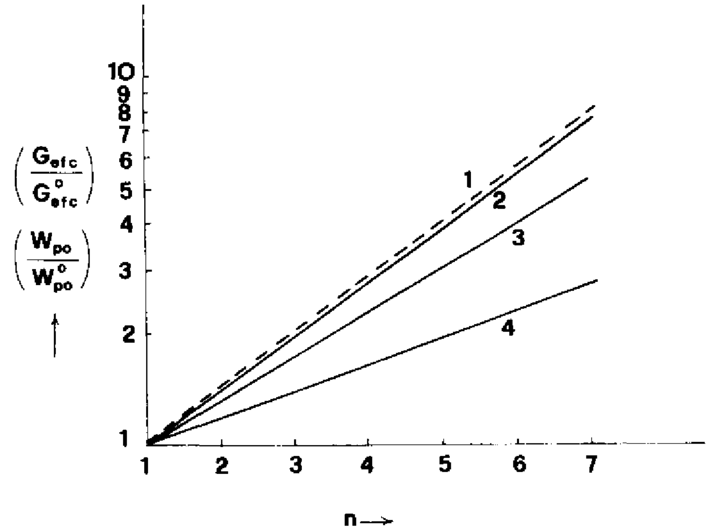

q =0
2
f(α)
1
q=∞
q=-∞
1.6 1.7 1.8 1.9 2.0 2.1 2.2 2.3 2.4 2.5 2.6
α →

Fig.5


10
9
8
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1
(Gefc / G°efc)
(Wpo / W°po)
1
2
3
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1 2 3 4 5 6 7
n →

Fig.6

Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica