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ON THE PHYSICAL SOLUTIONS TO THE HEAT EQUATION SUBJECTED TO NONLINEAR BOUNDARY CONDITIONS

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RESUMO

Este trabalho estuda as soluções físicas da equação da transferência de calor em regime permanente, quando esta está sujeita a condições de contorno não lineares. Será apresentado um funcional, cujo mínimo ocorre para a (única) solução física para o problema de transferência de calor considerado, que pode ser empregado a uma grande classe de condições de contorno (representando as perdas por convecção e/ou radiação para o meio ambiente). Será demonstrado que estes problemas admitem sempre uma, e somente uma, solução física (que representa a temperatura absoluta).

ABSTRACT

This work consists of a discussion on the physical solutions to the steady-state heat transfer equation, when it is subjected to nonlinear boundary conditions. It will be presented a functional, whose minimum occurs for the (unique) physical solution to the considered heat transfer problem, suitable for a large class of typical (nonlinear) boundary conditions (representing the radiative/convective loss from the body to the environment). It will be demonstrated that these problems admit always one, and only one, physical solution (which represents the absolute temperature).

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ABSTRACT

This work consists of a discussion on the physical solutions to the steady-state heat transfer equation , when it is subjected to nonlinear boundary conditions. It will be presented a functional, whose minimum occurs for the (unique) physical solution to the considered heat transfer problem, suitable for a large class of typical (nonlinear) boundary conditions (representing the radiative/ /convective loss from the body to the environment). It will be demonstrated that these problems admit always one, and only one, physical solution (which represents the absolute temperature).

INTRODUCTION

Generally, in the current literature of Mechanics and Mathematics, a steady-state heat transfer problem is regarded as a synonym of the Dirichlet Problem [1,2]

$$\Delta u = -r \quad \text{in} \quad \Omega \tag{1}$$

$$u = u_0(X) \quad \text{on} \quad \partial\Omega \tag{2}$$

However, problem (1)+(2) represents a fictitious situation, that possesses didactic importance, but is very far from the physical reality . In addition, these problems are considered without a well defined physical sense (thermodynamical restrictions).

In this paper we shall consider a large and very important class of nonlinear heat transfer problems which describe real heat transfer phenomena. These phenomena are represented by a partial differential equation, subjected to nonlinear boundary conditions, which admits, as its solutions, fields with physical sense and fields without physical sense.

The main objective of this work is to present an adequated mathematical formulation for such class of heat transfer problems (that are characterized by radiative/convective nonlinear boundary conditions). This new formulation takes into account the natural physical restrictions, preserves the physical reality and is mathematically convenient, providing one, and only one, solution (with physical sense).

In this work a solution will be said a "physical solution" when it is nonnegative for all $X \in \bar{\Omega}$ (because the considered solutions will represent an absolute temperature [3]), that means

$$u \geq 0 \quad \text{in} \quad \bar{\Omega} \tag{3}$$

In all situations to be considered the unknown field will be the absolute temperature "u" , there will exist a heat generation r (nonnegative) in Ω and the bounded open set Ω will have a smooth boundary $\partial\Omega$.

As particular cases of the mathematical problem to be considered here we have:

1) Isotropic heat conduction with linear convective boundary conditions [4,5]

$$\begin{gathered} \operatorname{div}(k \operatorname{grad} u) + r = 0 \quad \text{in} \quad \Omega \\ -k \operatorname{grad} u \cdot n = h\,(u - u_\infty) \quad \text{on} \quad \partial\Omega \\ r = \hat{r}(X) \geq 0 \quad \text{in} \quad \Omega \quad \text{(known field)} \\ n = \hat{n}(X)\ ,\ h = \hat{h}(X) > 0\ ,\ u_\infty = \hat{u}_\infty(X) \geq 0 \quad \text{on} \quad \partial\Omega \quad \text{(known fields)} \\ k = \hat{k}(X) > 0 \quad \text{in} \quad \bar{\Omega} \quad \text{(known field)} \end{gathered} \tag{4}$$

2) Isotropic heat conduction with (black body) radiative boundary conditions [6,7]

$$\begin{gathered} \operatorname{div}(k \operatorname{grad} u) + r = 0 \quad \text{in} \quad \Omega \\ -k \operatorname{grad} u \cdot n = u^4 \quad \text{on} \quad \partial\Omega \\ r = \hat{r}(X) \geq 0 \quad \text{in} \quad \Omega \quad \text{(known field)} \\ n = \hat{n}(X) \quad \text{on} \quad \partial\Omega \quad \text{(known field)} \\ k = \hat{k}(X) > 0 \quad \text{in} \quad \bar{\Omega} \quad \text{(known field)} \end{gathered} \tag{5}$$

THE HEAT TRANSFER PROBLEM TO BE CONSIDERED

We shall turn our attention to the anisotropic (or isotropic, when K=k1) heat conduction with combined convective/radiative boundary conditions.

This phenomenon is mathematically represented by

$$\mathrm{div}(K \text{ grad } u) + r = 0 \quad \text{in} \quad \Omega \tag{6}$$

$$-K \text{ grad } u \cdot n = \gamma u^4 + h (u - u_\infty) \quad \text{on} \quad \partial\Omega \tag{7}$$

in which the (known) fields K, n, γ, h, u_∞ and r represent, respectively, the thermal conductivity ($K=\hat{K}(X), X\in\bar{\Omega}$ - symmetric positive-definite) , the unit outward normal ($n=\hat{n}(X), X\in\partial\Omega$) , a positive field associated to body's emittance ($\gamma=\hat{\gamma}(X)>0, X\in\partial\Omega$) [7], the convective coefficient ($h=\hat{h}(X)\geq0, X\in\partial\Omega$) [5], an (absolute) temperature of reference ($u_\infty=\hat{u}_\infty(X)\geq0, X\in\partial\Omega$) and an internal heat supply ($r=\hat{r}(X)\geq0, X\in\Omega$).

The operators "grad" and "div" are defined as follows

$$\text{grad } u \equiv \frac{\partial u}{\partial x} i + \frac{\partial u}{\partial y} j + \frac{\partial u}{\partial z} k \tag{8}$$

$$\text{div } v \equiv \frac{\partial}{\partial x}(v\cdot i) + \frac{\partial}{\partial y}(v\cdot j) + \frac{\partial}{\partial z}(v\cdot k) \tag{9}$$

PHYSICAL SOLUTIONS' UNIQUENESS FOR PROBLEM (6)+(7)

Let us consider the following particular form of problem (6)+(7), which represents the conduction/radiation heat transfer in an infinite slab of thickness 2,

$$
\begin{aligned}
\frac{d^2u}{dx^2} + 2 &= 0 \qquad -1 < x < 1 \\
\frac{du}{dx} &= u^4 \qquad x=-1 \\
-\frac{du}{dx} &= u^4 \qquad x= 1
\end{aligned}
\tag{10}
$$

Problem (10) admits the following solutions

(1^{st})	$u = (1 - x^2) + 2^{0.25}$	$-1 \le x \le 1$	(11)
(2^{nd})	$u = (1 - x^2) - 2^{0.25}$	$-1 \le x \le 1$	(12)
(3^{rd})	$u = (1 - x^2) + i\ 2^{0.25}$	$-1 \le x \le 1$	(13)
(4^{th})	$u = (1 - x^2) - i\ 2^{0.25}$	$-1 \le x \le 1$	(14)

in which "i" is the imaginary unit [2].

This simple case shows us that problem (6)+(7) is not a good mathematical description, once that its solution is not unique and it provides solutions without a physical meaning ((13) and (14) are not real and (12) is not nonnegative for all x).

Before presenting an adequated way for describing mathematically problems like (6)+(7) (preserving their physical sense and eliminating the occurrence of solutions without physical sense), we shall demonstrate that such problems (when admit a solution) admit only one physical solution.

Aiming to this, let us assume that u_1 and u_2 are physical solutions to (6)+(7). Hence,

$$
\begin{aligned}
\operatorname{div}(K\ \operatorname{grad}(u_1-u_2)) &= 0 \qquad \text{in} \quad \Omega \\
-K\ \operatorname{grad}(u_1-u_2)\cdot n &= \gamma\ (u_1^4 - u_2^4) + h\ (u_1 - u_2) \qquad \text{on} \quad \partial\Omega
\end{aligned}
\tag{15}
$$

Since the field (u_1-u_2) assumes its maximum and its minimum on $\partial\Omega$ (since K is positive-definite) we conclude that, at $X_1 \in \partial\Omega$, where $(u_1 - u_2) = \min [(u_1 - u_2)]$,

$$\operatorname{grad}(u_1 - u_2) \cdot n \le 0 \Leftrightarrow \gamma\,(u_1^4 - u_2^4) + h(u_1 - u_2) \ge 0 \quad \text{at } X_1 \in \partial\Omega \qquad (16)$$

and that, at $X_2 \in \partial\Omega$, where $(u_1 - u_2) = \max [(u_1 - u_2)]$,

$$\operatorname{grad}(u_1 - u_2) \cdot n \ge 0 \Leftrightarrow \gamma\,(u_1^4 - u_2^4) + h(u_1 - u_2) \le 0 \quad \text{at } X_2 \in \partial\Omega \qquad (17)$$

Therefore (since $u_1 \ge 0$, $u_2 \ge 0$, $\gamma > 0$ and $h \ge 0$)

$$\min [(u_1 - u_2)] \ge 0 \quad \text{and} \quad \max [(u_1 - u_2)] \le 0 \qquad (18)$$

and, consequently,

$$u_1 \equiv u_2 \quad \text{in } \bar{\Omega} \qquad (19)$$

Hence the physical solution to (6)+(7), when exists, is unique.

Now we shall present a new mathematical formulation for the considered heat transfer phenomena. In this formulation the phenomena are mathematically represented by the minimization of a functional. This functional will admit a minimum, which is unique and corresponds to the physical solution to problems like (6)+(7). Thus, undesirable solutions like (12),(13) and (14) will be automatically excluded.

A MINIMUM PRINCIPLE FOR THE PHYSICAL SOLUTIONS TO (6)+(7)

Proposition : " The physical solution to (6)+(7) is the field "v" which minimizes the functional

$$I[w] = \frac{1}{2}\int_{\Omega} \text{grad } w \cdot K \text{ grad } w \, dV - \int_{\Omega} r \, w \, dV + \int_{\partial\Omega} \left(\frac{1}{5}\gamma |w|^5 + \frac{1}{2}h(w-u_{\infty})^2\right) dS \tag{20}$$

in which $w \in H^1(\Omega) \cap L^5(\partial\Omega)$ ".

In order to calculate the first variation of I, let us represent the admissible fields [8] w as follows

$$w = v + \alpha\eta \qquad w \in H^1(\Omega) \cap L^5(\partial\Omega) \tag{21}$$

in which α is a real-valued parameter , η is an admissible variation and v is a field that minimizes I.

Considering (21) we have

$$\frac{dI}{d\alpha} = \int_{\Omega} \text{grad}(v+\alpha\eta) \cdot K \text{ grad } \eta \, dV - \int_{\Omega} r \, \eta \, dV + \\ + \int_{\partial\Omega} (\gamma |v+\alpha\eta|^3 (v+\alpha\eta) + h(v+\alpha\eta - u_{\infty})) \, \eta \, dS \tag{22}$$

and, consequently (imposing $\alpha=0$, $dI/d\alpha=0$ and employing Green's Identity), the Euler-Lagrange equation and Natural boundary conditions associated to I are given as follows

$$\text{div}(K \text{ grad } v) + r = 0 \quad \text{in } \Omega \tag{23}$$

$$-K \text{ grad } v \cdot n = \gamma |v|^3 v + h (v - u_{\infty}) \quad \text{on } \partial\Omega \tag{24}$$

Calculating the second derivative of I with respect to α we obtain

$$\frac{d^2I}{d\alpha^2} = \int_{\Omega} \text{grad}\ \eta \cdot K\ \text{grad}\ \eta\ dV + \int_{\partial\Omega} (4\gamma |v+\alpha\eta|^3 + h)\ \eta^2\ dS \qquad (25)$$

Since K is positive-definite, h is nonnegative and γ is positive, we have that $d^2I/d\alpha^2 \geq 0$ (for any α and η) . Therefore the functional I is convex and the field "v" (solution to (23)+(24)) corresponds to a minimum of I.

In order to demonstrate the existence of the minimum (solution's existence), we shall prove that I is a coercive functional. Aiming to this, let us define the following norm (to $H^1(\Omega) \cap L^5(\partial\Omega)$) [9]

$$\|w\| = \left(\int_{\Omega} \text{grad}\ w \cdot \text{grad}\ w\ dV \right)^{0.5} + \left(\int_{\Omega} |w|^2 dV \right)^{0.5} + \left(\int_{\partial\Omega} |w|^5\ dS \right)^{0.2} \qquad (26)$$

Since K is positive-definite and γ is positive we have

$$\lim_{\|w\| \to \infty} \frac{I(w)}{\|w\|} = + \infty \qquad (27)$$

Therefore I is coercive and the solution to (23)+(24) exists [10].

Now we shall demonstrate that the solution to (23)+(24) is always positive and, consequently, a physical solution.

Since $r \geq 0$ we conclude that v assumes its minimum on $\partial\Omega$. Hence, at the point $X_1 \in \partial\Omega$ in which $v = \min\{v\}$, we have

$$\text{grad}\ v \cdot n \leq 0 \iff \gamma\ |v|^3 v + h(v - u_{\infty}) \geq 0\ , \quad \text{at}\ X_1 \in \partial\Omega \qquad (28)$$

Therefore (once that $\gamma>0$, $h \geq 0$ and $u_\infty \geq 0$) we have

$$\min [v] \geq 0 \quad \Leftrightarrow \quad v \geq 0 \ \text{in} \ \bar{\Omega} \tag{29}$$

and, consequently, the solution to (23)+(24) is a physical solution. Hence, the solution to (23)+(24) is always a solution to (6)+(7) , once that,

$$|v|^3 v = v^4 \quad \text{on} \ \partial\Omega \tag{30}$$

Since (6)+(7) admits only one physical solution, the field which minimizes I is unique, being the one which satisfies (6)+(7)

A MORE GENERAL HEAT TRANSFER PROBLEM AND ITS PHYSICAL SOLUTION

Let us consider the following boundary condition to (6)

$$-K \ \text{grad} \ u \cdot n = \sum_{j=1}^{N} a_j \, (u)^{p_j} - \beta \quad \text{on} \ \partial\Omega \tag{31}$$

in which a_j and β are known fields that satisfy the following inequalities

$$a_j \geq 0 \ (\text{for all } j), \ a_N > 0 \ \text{and} \ \beta \geq 0 \quad \text{for all } X \in \partial\Omega \tag{32}$$

and the real constants p_j are such that

$$p_{j+1} > p_j > 0 \tag{33}$$

Problem (6)+(31) generalizes problem (6)+(7). The physical solution to (6)+(31) is the field $w \in H^1(\Omega)\cap L^M(\partial\Omega)$ $(M=p_N+1)$ which minimizes the functional I given as follows

$$I[w] = \frac{1}{2}\int_\Omega \text{grad } w \cdot K \text{ grad } w \, dV - \int_\Omega r \, w \, dV - \int_{\partial\Omega} \beta \, w \, dS +$$

$$+ \int_{\partial\Omega} \sum_{j=1}^{N} a_j \left(\frac{|w|^{(p_j+1)}}{p_j+1} \right) dS \quad , \qquad w \in H^1(\Omega)\cap L^M(\partial\Omega) \quad (34)$$

This functional is convex and has the following associated Euler-Lagrange equation and Natural boundary conditions

$$\text{div}(K \text{ grad } v) + r = 0 \quad \text{in } \Omega \qquad (35)$$

$$-K \text{ grad } v \cdot n = \sum_{j=1}^{N} a_j \left(|v|^{(p_j-1)} \right) v - \beta \quad \text{on } \partial\Omega \qquad (36)$$

Considering the following norm (to $H^1(\Omega)\cap L^M(\partial\Omega)$)

$$\|w\| = \left(\int_\Omega \text{grad } w \cdot \text{grad } w \, dV\right)^{0.5} + \left(\int_\Omega |w|^2 \, dV\right)^{0.5} + \left(\int_{\partial\Omega} |w|^M \, dS\right)^{1/M} \quad (37)$$

we conclude that I is coercive ((27) holds) and, therefore, the field which minimizes I exists.

In addition, once that $r \geq 0$, we conclude that the solution to (35)+(36) is a physical solution. Hence,

$$a_j \, (v)^{p_j} \equiv a_j \left(|v|^{(p_j-1)} \right) v \quad , \text{ for all } j \, , \quad \text{on } \partial\Omega \qquad (38)$$

and, consequently, the solution to (35)+(36) is also a solution to (6)+(31).

Since (18) holds for any two physical solutions (u_1 and u_2) to problem (6)+(31), the field which minimizes I is the (unique) solution to (6)+(31).

It is to be noticed that, when $p_1=1$, $p_2=\infty$, $a_1=h$, $a_2=\gamma$ and $\beta=a_1 u_\infty$ problem (6)+(31) reduces to problem (6)+(7).

AN INTERESTING FEATURE

An interesting feature of the formulation presented in this work arises when the considered heat transfer problem do not admit a physical solution (that means an unrealistic phenomenon). In such case the field that minimizes I is such that

$$v < 0 \quad \text{for some} \quad X \in \bar{\Omega} \qquad (39)$$

and hence, we have the sufficient information for concluding that the considered problem makes no physical sense.

This information is not obtained when such "unrealistic problems" are numerically simulated directly from (6)+(7) (or (6)+(31)).

In order to ilustrate this fact, let us consider the following problem (obtained when "2" is replaced by "-2" in (10))

$$\begin{aligned} \frac{d^2u}{dx^2} - 2 = 0 \qquad & -1 < x < 1 \\ \frac{du}{dx} = u^4 \qquad & x=-1 \\ -\frac{du}{dx} = u^4 \qquad & x=1 \end{aligned} \qquad (40)$$

Problem (40) admits the following solutions (not real)

(1^{st}) $\quad u = (x^2 - 1) + 2^{-0.25} + i\ 2^{-0.25} \qquad -1 \le x \le 1$ (41)

(2^{nd}) $\quad u = (x^2 - 1) - 2^{-0.25} + i\ 2^{-0.25} \qquad -1 \le x \le 1$ (42)

(3^{rd}) $\quad u = (x^2 - 1) - 2^{-0.25} - i\ 2^{-0.25} \qquad -1 \le x \le 1$ (43)

(4^{th}) $\quad u = (x^2 - 1) + 2^{-0.25} - i\ 2^{-0.25} \qquad -1 \le x \le 1$ (44)

In such case the (unique) field that minimizes I is the solution to

$$\frac{d^2u}{dx^2} - 2 = 0 \qquad -1 < x < 1$$
$$\frac{du}{dx} = |u|^3 u \qquad x=-1 \tag{45}$$
$$-\frac{du}{dx} = |u|^3 u \qquad x= 1$$

being given as follows

$$u = (x^2 - 1) - 2^{0.25} \qquad -1 \le x \le 1 \tag{46}$$

FINAL REMARKS

The results presented in this work provide an adequated way for simulating a large class of nonlinear heat transfer phenomena (through the minimization of I) , simplifying the computational implementation of such problems and excluding undesirable solutions (that have no physical meaning).

In addition, the formulation presented here provides sufficient information about the "physical reality" of the considered problem.

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