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Magnetic Monopoles without String in the Kähler–Clifford Algebra Bundle: A Geometrical Interpretation(*)

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Abstract: In substitution for Dirac monopoles with string (and for topological monopoles) we have recently introduced "monopoles without string" on the basis of a generalized potential. the sum of a vector A and a pseudo-vector $\gamma_5 B$ potential. By making recourse to the Clifford bundle $\mathcal{C}(\tau M, g)[(T_x M, g) = \mathbb{R}^{1,3}; \mathcal{C}(T_x M, g) = \mathbb{R}_{1,3}]$. which just allows adding together for each $x \in M$ tensors of different ranks. in a previous paper we succeeded in constructing a lagrangian and hamiltonian formalism for interacting monopoles and charges that can be regarded as satisfactory from various points of view. In the present note. after having "completed" our formalism. we put forth a purely *geometrical interpretation* of it within the Kähler–Clifford bundle $\mathcal{K}(\tau^* M, \hat{g})$ of differential forms. essential ingredients being a generalized curvature and the Hodge decomposition theorem. We thus pave the way for the extension of our "monopoles without string" to non abelian gauge groups. The analogy with supersymmetric theories is apparent.

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It is wellknown that, when describing the electromagnetic field $F_{\mu\nu}$ produced by a Dirac monopole[1] in terms of one single potential A_μ only, such a potential has to be singular along an arbitrary line starting from the monopole and going to infinity. This "string" has been considered - since long[2] - as unphysical, because the singularity in A_μ does not correspond to any singularity in $F_{\mu\nu}$.

It is also wellknown that, in the $U(1)$ gauge theory of electromagnetism which has as mathematical model a Principal Fiber Bundle (PFB) $\pi : P \to M$ with group $U(1)$, monopoles appear only if we consider a non trivial bundle. M is in general a four dimensional Lorentzian manifold modeling the space time. The standard model is obtained by taking $M = \mathbb{R}^{1,3}$ and deleting from $\mathbb{R}^{1,3}$ the world line of the monopole. We then have as model the PFB $\pi : P \to \mathbb{R}^2 \times S^2$ with group $U(1)$ and the monopole charges appears as the Chern numbers characterizing the PFB. These observations show that the topological theory does not put on equal footing the electric charge and the monopole, since the former is introduced through the electric current and the latter is a hole moving in space time[3,4]. Notice that the topology of space time becomes even more exotic when generalized monopoles are present[5].

A way out has been looked for by many authors[2,6] via the introduction of a second potential B_μ. But they did not completely succeed in dispensing with an exotic space time, whenever they wanted to stick to ordinary vector-tensor algebra. However (just on the basis of both a vector potential $A \in \sec \Lambda^1 \tau M \subset \sec \mathcal{C}(\tau M, g)$, [where $\mathcal{C}(\tau M, g)$ is the Clifford bundle constructed in the tangent bundle τM of the Lorentz manifold M equipped with the Lorentz metric g, and sec means a section of the bundle] and a pseudovector potential $\gamma_5 B \in \sec \mathcal{C}(\tau M, g)$), we recently constructed[7] a rather satisfactory formalism for magnetic monopoles without strings (i.e., living in ordinary Minkowski space time, $\mathbb{R}^{1,3}$), by making recourse to the Clifford algebra $\mathbb{R}_{1,3}$ or more precisely to the Clifford-bundle $\mathcal{C}(\tau M, g)$ [where $(T_x M, g) = \mathbb{R}^{1,3}$]. $\mathbb{R}^{1,3}$ is an algebra sufficiently powerful to allow adding together tensors of different ranks (grades). In ref.[8], for example, both the electric and the magnetic current are vectorial, whilst in our approach they are represented by a vectorial and a pseudovectorial current, respectively (and nevertheless we can add them together[7]). Our formalism can be considered satisfactory for the reasons we shall see bellow. See also ref.[9]. Some analogous, but non equivalent, results have been gotten in refs.[10,11].

From Clifford to Kähler. In this paper we want, first of all, to pass from the $\mathcal{C}(\tau M, g)$–language, used in ref.[7], to the $\mathcal{K}(\tau^* M, \hat{g})$–language, i.e., to the language of the differential forms in $\tau^* M$, the cotangent bundle with metric $\hat{g}$ (equipped with the Kähler algebra [12,13]) #1. This paves the way, incidentally, for a generalization of our "monopoles without string" to non abelian gauge groups.

The new language will allow approaching the question of a suitable formalism for interacting charges and monopoles without string from a *geometrical* point of view in the space time manifold. #2.

We recall that $\mathcal{K}(T_x^* M, \hat{g}) = \mathcal{C}(T_x M, g) = \mathbb{R}_{1,3}$, the so called space time algebra [#3]. Now $\mathcal{K}(T_x^*M, \hat{g})$, as a linear space over the real field, can be written

$$\Lambda^0(T_x^*M) + \Lambda^1(T_x^*M) + \Lambda^2(T_x^*M) + \Lambda^3(T_x^*M) + \Lambda^4(T_x^*M), \tag{1}$$

where $\Lambda^k(T_x^*M)$ is the $\binom{4}{k}$-dimensional space of the k-forms. $\Lambda(T_x^*M) = \sum \Lambda^k(T_x^*M)$ is called the Cartan algebra, and the pair $[\Lambda(T_x^*M), \hat{g}_x]$ is called the Hodge algebra. An analogous terminology exists for the vector bundles associated with these algebras.[9]

In $\mathcal{K}(\tau^*M, \hat{g})$ there is a particular differential operator ∂ odd in the $\mathbb{Z}_2$-gradation of the algebra [#4]. To introduce ∂, consider first, for any $t^* \in \sec \tau^*M \subset \sec \mathcal{K}(\tau^*M, \hat{g})$ and any $t \in \sec \tau M$, the bilinear tensorial map of type $(1,1)$ given by

$$\Psi \rightarrow t^* \nabla_t \Psi, \tag{2}$$

where Ψ is any element of $\sec \mathcal{K}(\tau^*M, \hat{g})$ and ∇_t is the covaraint derivative of Ψ (considered as an element of the tensor bundle). Then ∂ is defined as the tensorial trace of the map:

$$\partial = Tr(t^* \nabla_t). \tag{3}$$

In terms of a local basis $\{\gamma^\mu\}$ of 1-form fields and its dual basis $\{e_\mu\}$ of vector fields, we can write

$$\partial = \gamma^\mu \nabla_{e_\mu}. \tag{3'}$$

In particular, taking any local neighborhood $U \subset M$ with a local basis $\{dx^\mu\}$ $\partial = \gamma^\mu \nabla_\mu$, we can show[9,13] that for any $\Psi \in \sec(\Lambda\tau^*M, \hat{g}) \subset \sec \mathcal{K}(\tau^*M, \hat{g})$:

$$\partial\Psi = dx^\mu \wedge (\nabla_\mu \Psi) + \partial_\mu \rfloor (\nabla_\mu \Psi), \tag{4}$$

where $\rfloor$ is the usual contraction operator of the theory of differential forms. We have:

$$dx^\mu \wedge (\nabla_\mu \Psi) = d\Psi \tag{5}$$

$$\partial_\mu \rfloor (\nabla_\mu \Psi) = -\delta\Psi \tag{6}$$

where d is the usual differential, and δ is the Hodge coderivative operator, here defined as:

$$\delta\Psi_k = (-1)^k *^{-1} d * \Psi_k \tag{7}$$

where $*$ is the Hodge star operator and $\Psi_k \in \sec \mathcal{K}(\tau^*M, \hat{g})$. The power of the Kähler bundle formalism appears clearly once we add to the fundamental formula

$$\partial\Psi = (d - \delta)\Psi \tag{8}$$

the result[9,13,15,16]

$$\gamma^5 \Psi_k = (-1)^\ell * \Psi_k, \tag{9}$$

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is the volume element. #5 and where $t = 1$ for $k = 1, 2, 3$ and $t = 2$ for $k = 0, 4$ in the particular case of the space time algebra $\mathbb{R}_{1,3}$ and with our conventions. We also have that $\partial^2 = (d - \delta)^2$ is the D'Alambertian operator.

Generalized potential and field: A satisfactory formalism. Before going on, observe that the "completed" Maxwell equations. $\delta F = -J_e, dF = - \star J_m$. where $F \in \sec(\Lambda^2\tau^* M, \hat{g} \subset \sec \mathcal{K}(\tau^* M, \hat{g})$ is the electromagnetic field and $J_e, J_m \in \sec(\Lambda^1\tau^* M, \hat{g}) \subset \sec \mathcal{K}(\tau^* M, \hat{g})$ are respectively the electric and magnetic currents, can be written as[17]

$$\partial F = J_e - \star J_m = J_e + \gamma^5 J_m \equiv \overline{J}. \tag{10}$$

With the introduction of the generalized potential[17] $\overline{A} \equiv A + \gamma^5 B$, where $A, B \in \sec(\Lambda^1\tau^* M, \hat{g}) \subset \sec \mathcal{K}(\tau^* M, \hat{g})$, we get $F = \partial\overline{A} = \partial \wedge A + \partial.(\gamma^5 B)$, once we impose the Lorentz gauge $\partial \circ \overline{A} = 0$ #6. Then we can write eq.(10) as:

$$\partial^2 A = J_e \; ; \; \partial^2 B = J_m. \tag{11}$$

In our previous work[7] we wrote eqs.(10) and (11) in $C(\tau M, g)$, instead of $\mathcal{K}(\tau^* M, \hat{g})$. There we succeeded in introducing a non coventional lagrangian which yields the correct field equations when varied with respect to the generalized potential. Our approach, however, cannot overcome the "no - go theorems" by Rosenbaum et al.[8]; for instance Rohrlich[8] showed that a single Lagrangian can yield both the field equations and the charge and pole motion–equations *only* in the trivial case when $J_m = kJ_e$, where k is a constant. Nevertheless in our approach we need apply the variational principle just once, since our lagrangian[7] *implies* even the correct coupling of the currents to the field. In the sense that, as showed in detail in refs.[9,18], the "completed" Maxwell equations [eq.(10)] *imply*, if $S^\mu \equiv -\frac{1}{2}F\gamma^\mu F$, that:

$$\partial_\mu S^\mu = F.J_e + (\gamma^5 F).J_m. \tag{12}$$

where $S^\mu\gamma^\nu = E^{\mu\nu}$ is the symmetric energy- momentum of the electromagnetic field. Calling $K_e = F.J_0$ and $K_m = -(\gamma^5 F).J_m$, and by projecting on the Pauli algebra$\mathbb{R}_{3,0}$, one does *consequently* find the expected expressions for the forces (in particular the *Lorentz forces*) acting on a charge and a monopole:

$$\vec{K}_e = \rho_e \vec{E} + \vec{J}_e \times \vec{H} \tag{13a}$$

$$\vec{K}_m = -\rho_m \vec{E} + \vec{J}_m \times \vec{E} \tag{13b}$$

Generalized connection and curvature. As is wellknown, in a gauge theory[19] the potentials are pull–backs of *connections* in the PFB $\pi : P \to M$ with group G, and the associated field is the pull–back of the connection *curvature*. In the case of standard electromagnetism, the field $F \in \sec(\Lambda^2\tau^* M, \hat{g})$ is derived from a potential $A \in \sec(\Lambda^1\tau^* M, \hat{g})$, i.e.:

$$F = dA. \tag{14}$$

However the Hodge decomposition theorem[20] (valid for compact spaces) assures us that more generally, if $F \in sec(\Lambda^2\tau^*M,\hat{g})$, then there always exist $A \in sec(\Lambda^1\tau^*M,\hat{g})$, $*B \in sec(\Lambda^3\tau^*M,\hat{g})$ and $C \in sec(\Lambda^2\tau^*M,\hat{g})$, with $dC = \delta C = 0$, such that F can be ***uniquely decomposed*** into:

$$F = dA + \delta * B + C. \tag{15}$$

The Hodge decomposition naturally suggests naming ***generalized connection*** the quantity

$$\overline{A} = A - *B \in sec(\Lambda^1\tau^*M,\hat{g}) + sec(\Lambda^3\tau^*M,\hat{g}) \tag{16}$$

and ***generalized curvature*** #7 the quantity

$$F = \partial\overline{A} = (d - \delta)\overline{A} = dA + \delta * B - d * B - \delta A. \tag{17}$$

Then $F \in sec(\Lambda^0\tau^*M,\hat{g}) + sec(\Lambda^2\tau^*M,\hat{g}) + sec(\Lambda^4\tau^*M,\hat{g})$. If we want F to be still a 2-form, then the last two addenda in eq.(17) have to vanish, and we automatically end up with the Lorentz gauge condition

$$d * B = \delta A = 0. \tag{18}$$

and are left with

$$F = dA + \delta * B. \tag{17'}$$

The field equations are obtained by evaluating ∂F, with $\partial \equiv d - \delta$:

$$(d - \delta)(dA + \delta * B - d * B - \delta A) = \partial^2 A - \partial^2 * B. \tag{19}$$

which writes

$$\partial F = J_e - *J_m \tag{20}$$

once we identity $\partial^2 A \equiv J_e \; ; \; \partial^2 B \equiv J_m$. Eqs.(19) are of course the "completed" Maxwell equations, now deduced within a geometrical context via a natural generalization of the definitions of connection and curvature: a generalization inspired by the "correspondences" $\partial = d - \partial$ and $* = (-1)^\epsilon\gamma^5$ and by the Hodge decomposition theorem.

Further remarks: (i) A rather interesting consequence of the geometrical interpretation just presented is that eq.(17) can be assumed as a new definition of F, without imposing any longer the Lorentz gauge, since even in this case we get the right "completed" Maxwell equations [as it is clear from eq.(18) and eq.(19)].

(ii) The introduction of our "monopoles without string" for the more general case of non abelian groups is discussed in refs.[14,21]. Here we want to emphasize once more that, for our aims, the ordinary tensorial language is too poor, since - among the others - it does not satisfactorily distinguish between scalar and pseudo scalar quantities, as on the contrary it is strictly required by physics. For instance, it is an essential character of the lagrangian density of ref.[7] to be the sum of a scalar and a pseudo scalar part[7,22].

(iii) At last, let us take advantage of the present opportunity for pointing out some misprints appeared in the previous paper.[7] that might make difficult for the interested reader to rederive those results of ours: (1) at page 234, column 2, line 18: the two expressions $\partial \, \overline{J}$ ought rather to read $\partial \circ \overline{J}$; (2) at page 235, eqs.(24) and (15): all the three expressions should we written $\overline{J} \circ \overline{A}$; (3) at page 235: the last term in the r.h.s. of eq.(17) ought to be eliminated; (4) at page 236, column 1, line 22: "pseudoscalars" should be corrected into "pseudovectors". Let us stress that the "ball product" $\circ$ is <u>*not*</u> a new fundamental product since in terms of the Clifford product we have, for $A, B \in \mathrm{sec}\, C\ell(\tau M, g)$, that $A \circ B \equiv \frac{1}{2}(A \tilde{B} + B \tilde{A})$.

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FOOTNOTES

#1 Let us consider that the metric tensor $g \in \sec(\tau^* M \times \tau^* M)$ induces the "dual metric" $\hat{g}$ in the spaces $\Lambda^k(\tau^* M)$:[3]

$$\hat{g}(\varphi_1, \varphi_2)\gamma^5 = \varphi_1 \wedge *\varphi_2.$$

where $\varphi_1, \varphi_2 \in \sec(\Lambda^k \tau^* M, \hat{g})$ is the so-called Hodge bundle. For future reference note that, in the particular case in which $\varphi_1 = \varphi_2 = \varphi \in \sec(\Lambda^2 \tau^* M, \hat{g})$,

$$\hat{g}(\varphi, \varphi) = -\hat{g}(*\varphi, *\varphi).$$

#2 For a *completely geometric* formulation (and generalization to arbitrary gauge groups) of the theory, we ought however to make recourse to a spliced bundle $\pi : P \circ P \to M$ with group $G \times G$ where M is an arbitrary space time with non zero Lorentzian curvature[14]: cf. ref.[14].

#3 By adopting Hestenes' notations (cf. the second one of refs.[13]), we call *space-time algebra* the Clifford algebra $\mathbb{R}_{1,3}$ that we called "Dirac algebra" in ref.[7]. More correctly we shall reserve the name Dirac algebra for $\mathbb{R}_{4,1} \simeq \mathbb{C}(4)$. Notice, incidentally, that the *Majorana algebra* $\mathbb{R}_{3,1}$ is quite different from $\mathbb{R}_{1,3}$, so that *two* algebras $[\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, and $\mathbb{R}_{3,1} \simeq \mathbb{R}(4)]$ can be naturally associated with Minkoswski space time; and this can have a bearing on physics (even for the mathematical problems with tachyons, for instance). At last, the *Pauli algebra* is $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$.

#4 Recall that we denote the Clifford product in $\mathcal{C}(\tau M, g)$, as well as in $\mathcal{K}(\tau^* M, \hat{g})$, by mere *justaposition* of symbols.

#5 Recall that, whereas γ^5 is the volume element in $\mathcal{K}(\tau^* M, \hat{g})$, in ref.[7] $\gamma_5 = e_0 e_1 e_2 e_3 \in \mathcal{C}(\tau M, g)$ and $\{e_\mu\}$ is an orthonormal basis of $\mathbb{R}^{1,3}$.

#6 Note that the scalar product between $\Psi_r \in \Lambda^r(T_x^* M)$ and $\Psi_k \in \Lambda^k(T_x^* M)$ is defined by $\Psi_r \cdot \Psi_k = \langle \Psi_r \Psi_k \rangle_{|r-s|}$; i.e., it is the component in $\Lambda^{|r-s|}(T_x^* M)$ of the Clifford product of Ψ_r and Ψ_k.
Sometimes we make recourse also the *ball product* ($\circ$) which, in terms of the Clifford product, is defined as follows: $A \circ B = \frac{1}{2}(A\tilde{B} + B\tilde{A})$. The tildle operation, in its turn, is defined as follows: $D = d_1 d_2 \ldots d_r$; $\tilde{D} = d_r \ldots d_2 d_1$ where the $d_i \in \mathbb{R}^{1,3}$, $i = 1, 2 \ldots r$.

#7 Such a terminology is, of course, acceptable only when working in the base manifold. Despite this fact, the theory of electromagnetism with monopoles without string, containing the potentials A and B, can be formulated as a PFB $\pi : P \circ P \to M$ with group $U(1) \times U(1)$, where A and B are "parts" of a genuine connection in the sense of a PFB theory[9,14].

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