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COMPARISON BETWEEN COASTING AND BUNCHED BEAMS ON OPTIMUM STOCHASTIC COOLING AND SIGNAL SUPPRESSION*

J. Wei

Brookhaven National Laboratory
Upton, NY 11973, USA

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Abstract

A comparison has been performed between coasting and bunched particle beams pertaining to the mechanism of stochastic cooling. In the case that particles occupy the entire sinusoidal rf bucket, the optimum cooling rate for the bunched beam is shown to be the same as that predicted from the coasting-beam theory using local particle density. However, in the case that particles occupy only the center of the bucket, the optimum rate decreases in proportion to the ratio of the bunch area to the bucket area. Furthermore, it has been shown for both coasting and bunched beams that particle motion is stable upon signal suppression if the amplitude of the gain is less than twice the optimum value over the entire frequency bandwidth of the cooling system.

I. INTRODUCTION

Stochastic cooling for both coasting and bunched beams[1-7] has been successfully applied to many accelerators. Theories, mostly using Fokker-Planck approach, have been developed to investigate the particle motion.

This paper provides an analytic comparison between coasting and bunched beams on the cooling mechanism. In section II, the optimum cooling rates for coasting and bunched beams are derived from the Fokker-Planck equations. In the case that particles occupy the entire sinusoidal rf bucket, the optimum rate for the bunched beam is shown to be the same as that predicted from the coasting-beam theory using local particle density. However, in the case that particles occupy only the center of the bucket, the optimum rate decreases in proportion to the ratio of the bunch area to the bucket area, which contradicts the coasting-beam prediction. In section III, the effect of signal suppression is evaluated based on the equations of motion. It is shown that the particle motion under cooling is stable if the amplitude of the gain of the system is less than twice the optimum value over the entire frequency bandwidth.

Although the discussion is restricted to transverse stochastic cooling, it has been found that the conclusion is also true for longitudinal cooling, where the analysis is complicated by the mixing factor that depends on longitudinal particle distribution.

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II. COMPARISON ON OPTIMUM COOLING

A. Coasting Beam

Consider a beam of N charged particles azimuthally distributed along the accelerator. The increment $U_{x'}$ in $x' = dx/ds$ that is experienced by the ith particle per unit time at the kicker, is proportional to the displacement x^P of all the particles at the pick-up,

$$U_{x',i} = \sum_{j=1}^{N} U_{ij} x_j^P, \tag{1}$$

where

$$U_{ij} = \frac{\omega_0}{2\pi\sqrt{\beta_x^P \beta_x^K}} \sum_{m=-\infty}^{\infty} G\left(m^{\pm}\omega_j\right) e^{im(\omega_j-\omega_i)t}. \tag{2}$$

Here, $m^{\pm} = m \pm \nu_x$, ν_x is the transverse tune, $G(\omega)$ is the gain of the cooling system, ω_i is the revolution frequency of the ith particle, ω_0 is the average revolution frequency, and β_x is the Courant-Snyder parameter. The superscript P and K denote values at the pick-up and the kicker, respectively.

It is convenient to describe the transverse motion of the particles in terms of the angle- action variables φ and I that are generated from the original variables x and x' by a generating function

$$F_1(x, \varphi; s) = -\frac{x^2}{2\beta_x}\left(\tan\varphi - \frac{\beta_x'}{2}\right). \tag{3}$$

The equations of motion thus become

$$\varphi' = \frac{1}{\beta_x} + U_\varphi, \quad I' = U_I, \tag{4}$$

where

$$U_I = -\sqrt{2\beta_x I}\sin\varphi U_{x'}, \quad U_\varphi = -\sqrt{\beta_x/2I}\sin\varphi U_{x'}. \tag{5}$$

Evolution of the transverse distribution function Ψ can be described by a transport equation, which is obtained by averaging the two-dimensional Fokker-Planck equation[6] over φ,

$$\frac{\partial\Psi}{\partial t} = -\frac{\partial}{\partial I}(F\Psi) + \frac{1}{2}\frac{\partial}{\partial I}\left(D\frac{\partial\Psi}{\partial I}\right). \tag{6}$$

Neglecting the thermal noise of the cooling system, the coefficients of coherent correction F and diffusion D can be evaluated

$$F(I) = F^0 I, \quad D = D^0 I \langle I \rangle, \tag{7}$$

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where $\langle\ \rangle$ denotes the average, and

$$F^0 = -\frac{\omega_0 \sin\nu_z \Delta\theta^{PK}}{2\pi} \sum_{m=-\infty}^{\infty} G\left(m^{\pm}\omega_i\right) e^{-im\Delta\omega_i \Delta\theta^{PK}/\omega_0}$$

$$D^0 = \frac{\omega_0^2 \rho(\omega_i)}{4\pi} \sum_{m=-\infty}^{\infty} \frac{|G(m^{\pm}\omega_i)|^2}{|m|}. \tag{8}$$

Here, $\rho(\omega_i)$ is the density in frequency seen by the test particle i, and $\Delta\theta^{PK}$ is the azimuthal distance in radian between the kicker and the pick-up. The factor containing the difference $\Delta\omega_i$ in revolution frequency represents "bad mixing". Because F and D are both independent of Ψ, the cooling rate can be obtained by integrating Eq. (6) using[6] relevant boundary conditions,

$$\tau^{-1} = \frac{1}{\langle I\rangle}\frac{d\langle I\rangle}{dt} = F^0 + \frac{D^0}{2}. \tag{9}$$

The average gain G_{opt} to achieve the optimum cooling rate τ_{opt}^{-1} is then

$$G_{opt}^{-1} = \frac{\omega_0\langle\rho(\omega_i)\rangle}{2\langle n\rangle}, \quad \tau_{opt}^{-1} = \frac{\Delta n\langle n\rangle}{\pi\langle\rho(\omega_i)\rangle}, \tag{10}$$

where $\Delta n f_0$ is the frequency bandwidth, and $\langle n\rangle$ is the average harmonic of the cooling system.

B. Bunched Beam

Consider a bunch of N_0 particles performing synchrotron oscillation with frequency Ω_i and amplitude τ_i. U_{ij} can now be expressed

$$U_{ij} = \frac{\omega_0}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \frac{(-i)^l e^{il\phi_j^0}}{\sqrt{\beta_z^P \beta_z^K}} J_l\left(m\omega_0\tau_j\right) G\left(m^{\pm}\omega_0 - l\Omega_j\right)$$
$$\times \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-i)^k J_k\left(n\omega_0\tau_i\right) e^{ik\phi_i^0} e^{it(m\omega_0 - l\Omega_j + n\omega_0 - k\Omega_i)}, \tag{11}$$

where J_l is the Bessel function of lth order, and ϕ^0 is the initial phase. The coefficients of the transport equation (Eq. 6) in terms of the action variable can be evaluated

$$F^0 = -\frac{\omega_0 \sin\nu_z \Delta\theta^{PK}}{2\pi} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} G\left(m^{\pm}\omega_0 - l\Omega_i\right)$$
$$\times e^{il\Omega_i \Delta\theta^{PK}/\omega_0} J_l^2\left(m\omega_0\tau_i\right) \tag{12}$$

$$D^0 = \frac{\omega_0^2}{4\pi} \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left.\frac{\rho(J')}{|l|\left|\frac{d\Omega(J')}{dJ'}\right|}\right|_{\Omega(J')=k\Omega(J)/l}$$
$$\times \left\{|G_D(-)|^2 + |G_D(+)|^2 + 2Re\left[G_D(-)\, G_D(-)\right]\right\}, \tag{13}$$

where $e^{il\Omega_i\Delta\theta^{PK}/\omega_0}$ represents "bad mixing", and

$$G_D(\pm) = \sum_{m=1}^{\infty} G\left(m^{\pm}\omega_0 \pm l\Omega_j\right) J_{\mp l}\left(m\omega_0\tau_j\right) J_{\mp k}\left(m\omega_0\tau_i\right). \tag{14}$$

Here, $\rho(J)$ is the particle density, and J represents the longitudinal phase-space area enclosed by the particle trajectory. The rate of change in synchrotron-oscillation frequency under a sinusoidal rf voltage is[6]

$$\frac{d\Omega(J)}{dJ} \approx -\frac{C_W}{8\pi}, \quad C_W = \frac{h^2\omega_0^2\eta}{2E\beta^2}, \tag{15}$$

where h is the rf harmonic number.

The average gain to achieve the optimum cooling rate τ_{opt}^{-1} can be similarly obtained. Because the number of significant synchrotron sideband $\langle k\rangle$ ($\langle k\rangle = \langle n\rangle\omega_0\langle\tau\rangle$) is typically much larger than 1, $J_{\mp k}$ in Eq. (14) may be replaced by their asymptotic forms. Employing the identity

$$J_0^2(x) + 2\sum_{k=1}^{\infty} J_k^2(x) = 1, \tag{16}$$

the optimum gain can be derived

$$G_{opt}^{-1} = \frac{\Delta n\omega_0\langle\rho(\Omega_i)\rangle}{\pi\langle k\rangle^2}, \quad \tau_{opt}^{-1} = \frac{\langle k\rangle^2}{2\langle\rho(J)\rangle}\left|\frac{d\Omega}{dJ}\right|. \tag{17}$$

C. Comparison

In the case that particles in the bunch occupy the entire rf bucket,

$$\rho(J) \approx \frac{N_0}{8}\sqrt{\frac{C_W}{C_\phi}}, \quad C_\phi = \frac{qe\hat{V}}{\pi h}. \tag{18}$$

The spread $\Delta\Omega$ in synchrotron-oscillation frequency is comparable to the zero- amplitude frequency Ω_s. Expressing the oscillation amplitude

$$\langle\tau\rangle = \frac{2}{h\omega_0}\sqrt{\frac{C_W}{C_\phi}}\langle W\rangle \approx \frac{1}{h\omega_0}, \tag{19}$$

Eq. (17) becomes

$$\tau_{opt}^{-1} \approx \frac{\langle n\rangle^2\langle\Delta\omega_i\rangle}{2\pi h N_0}, \tag{20}$$

where $\langle\Delta\omega_i\rangle$ is the mean spread in revolution frequency. Compared with Eq. (10), the optimum cooling rate for the bunched beam is the same as that predicted from the coasting-beam theory if $4hN_0$ is considered the effective number of particles in the ring.

In the case that particles in the bunch occupy only the center of the bucket, the spread $\Delta\Omega$ is small compared with Ω_s. Consequently, coasting-beam prediction is no longer applicable. In order to demonstrate the difference, consider the situation when the peak rf voltage is increased while the bunch area is kept constant. According to the coasting-beam theory (Eq. 10), the optimum cooling rate is unchanged because the effective density $\langle\rho(\omega_i)\rangle$ is unchanged. However, according to the bunched-beam theory (Eq. 17), the rate decreases in proportion to $\Gamma^{-1/2}$. Cooling becomes difficult because of the effectively higher particle density in synchrotron sideband. Define the bucket filling ratio as the ratio of the bunch area to the bucket area, Fig. 1 shows the change of optimum cooling rate calculated by assuming constant bunch area, and by including synchrotron sideband overlapping.

III. CONDITION FOR BEAM INSTABILITY

A. Coasting Beam

In the previous section, the undisturbed value x^{P0} has been used (Eq. 1) as an approximation for the transverse displacement x^P. In reality, the solution x^P to the equations of motion valid to the first order in U_x, is

$$x_i^P = x_i^{P0} - \sqrt{\beta_x^P \beta_x^K} \int_0^t dt' \cos \nu_x \omega_i (t - t') \sum_{j=1}^{N} x_j^P U_{ij}(t'). \quad (21)$$

The center-of-mass displacement can be solved[3] by using the Laplace-Fourier transformation,

$$X_m(t) = \sum_{j=1}^{N} e^{-im\omega_j t} x_j^P = \sum_{j=1}^{N} \frac{e^{im\omega_j t} x_j^{P0}}{1 + \tilde{F}_m(m^{\pm}\omega_j)}, \quad (22)$$

where $\tilde{F}_m$ is transformed from

$$F_m(t) = \frac{\omega_0 G(m^{\pm}\omega_0)}{2\pi} \sum_{j=1}^{N} e^{im\omega_j t} \cos \nu_x \omega_j t. \quad (23)$$

Instability occurs if the denominator vanishes, i.e.

$$1 = \frac{\omega_0 G(m^{\pm}\omega_0)\rho(\omega_j)}{4m^{\pm}} + \frac{i\omega_0 G(m^{\pm}\omega_0)}{4\pi m^{\pm}} \mathcal{P} \int \frac{\rho(\omega)\, d\omega}{\omega + \omega_j}, \quad (24)$$

where $\mathcal{P}$ denotes the Cauchy principle value. Compared with Eq. (10), Eq. (24) implies that particle motion is stable upon signal suppression if the magnitude of the gain is less than twice the optimum value G_{opt} over the entire frequency bandwidth of the cooling system.

B. Bunched Beam

Signal suppression for the bunched beam is complicated due to the coupling between different harmonics. Define the center-of-mass displacement

$$X_n(t) = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{N} (-i)^k J_k(n\omega_0\tau_j)\, e^{ik\phi_j^0} e^{-ik\Omega_j t} x_j^P. \quad (25)$$

The Laplace-Fourier transform satisfies

$$\tilde{X}_n(\nu) = \tilde{X}_n^0(\nu) - \sum_{m=-\infty}^{\infty} \tilde{X}_m(\nu)\tilde{F}_{mn}(\nu), \quad (26)$$

where $\tilde{F}_{mn}(\nu)$ is transformed from

$$F_{mn}(t) = \frac{\omega_0 G(m^{\pm}\omega_0)}{2\pi} \sum_{l=-\infty}^{\infty} \sum_{j=1}^{N} J_l(n\omega_0\tau_j) J_l(m\omega_0\tau_j) \times e^{-il\Omega_j t} \cos \nu_x \omega_0 t. \quad (27)$$

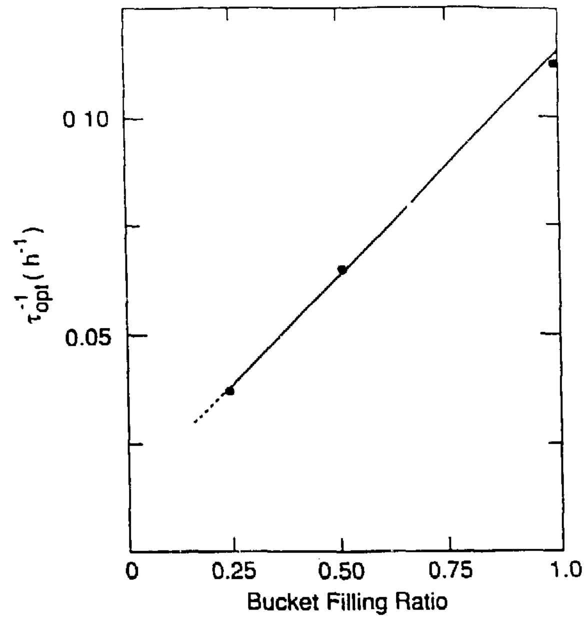

Figure 1: Optimum cooling rate for the bunched beam as a function of the bucket filling ratio.

Solving Eq. (26) is equivalent to obtaining the eigenvalues of a matrix **F**, which is defined with $\tilde{F}_{mn}$ as its elements. The instability that most likely occurs corresponds to the largest eigen-value being equal to 1. Using Eq. (16), this criterion is

$$1 \approx \frac{\Delta n G(m^{\pm}\omega_0)\rho(\Omega_j)}{2\pi k m^{\pm}\tau_j} + i\frac{\Delta n G(m^{\pm}\omega_0)}{2\pi^2 k m^{\pm}\tau_j} \mathcal{P} \int \frac{d\Omega \rho(\Omega)}{\Omega - \Omega_j}. \quad (28)$$

Compared with Eq. (17), this again implies that particle motion is stable if the magnitude of the gain is less than twice the optimum value over the entire frequency bandwidth.

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