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P-COMMUTATIVE TOPOLOGICAL *-ALGEBRAS

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ABSTRACT

If $P(A)$ denotes the set of all continuous positive functionals on a unital complete lmc *–algebra and $S(A)$ the extreme points of $P(A)$, and if the spectrum of an element $x \in A$ coincides with the set $\{f(x) : f \in S(A)\}$, then A is shown to be P–commutative. Moreover, if A is unital symmetric Fréchet Q lmc *–algebra, then this spectral condition is, in fact, necessary. Also, an isomorphism theorem between symmetric Fréchet P–commutative lmc *–algebras is established.

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1. INTRODUCTION

The notion of P–commutativity in Banach *–algebras was introduced by W. Tiller [11]. Precisely, let $P(A)$ denote the set of continuous positive functionals on a Banach *–algebra A and $P = \cap\{L_f : f \in P(A)\}$ with $L_f = \{x \in A : f(x^*x) = 0\}$, then A is, by definition, P–commutative if $xy - yx \in P$ for every $x, y \in P$. He showed [10] that various results of commutative Banach *–algebras remain valid in this class of non–commutative Banach *–algebras, the so–called P–commutative Banach *–algebras.

M. Fragoulopoulou [4] generalized this concept of P–commutativity to the class of (non–normed) topological *–algebras and obtained several results analogous to those of Banach *–algebras. Among other things, it was shown in [4; Theorem 5.3] that a non–zero continuous positive functional f on a P–commutative lmc *–algebra A having a bounded approximate identity is an extreme point if and only if it is multiplicative. Moreover, if A is a unital P–commutative symmetric Fréchet lmc *–algebra satisfying $r_A(x) < \infty$ for all $x \in H(A)$ (self–adjoint elements of A) and $S(A)$ denotes the non–zero extreme point of $P(A)$, then the spectrum sp_A of $x \in A$ coincides with the set $\{f(x) : f$ is a multiplicative linear functional on $A\}$ (cf. [4; Theorem 6.1]). In other words, $sp_A(x) = \{f(x) : f \in S(A)\}$ for every $x \in A$.

In this paper we show that if A is any unital complete lmc *–algebra and if the spectrum $sp_A(x)$, for any $x \in A$, is given by $sp_A(x) = \{f(x) : f \in S(A)\}$, then this condition is sufficient to ensure the P–commutativity of A (cf. Theorem 3.2 below). Consequently, we obtain a spectral characterization of P–commutativity for a particular class of lmc *–algebras. Precisely, if A is a unital symmetric Fréchet lmc *–algebra with $r_A(x) < \infty$ for every $x \in H(A)$, then A is P–commutative if and only if the above spectral condition holds (cf. Corollary 3.1 and Corollary 3.2 below). We also establish an isomorphism theorem between unital P–commutative symmetric Fréchet lmc *–algebras. Thus, we extend Theorems 1 and 2 of [2] for Banach *–algebras to the more general situation of lmc *–algebras. We prove these results in Section 3 of this paper, while in Section 2 we review some technical preliminaries needed in the sequel.

Regarding the general theory of topological algebras we refer to [8].

2. PRELIMINARIES

We give here only the basic definitions and facts needed in our further discussion. Throughout this paper we assume, without mentioning explicitly, that the algebras under consideration are over the field $\mathbb{C}$ of complex numbers, while the topological spaces involved are always Hausdorff.

By a *locally m–convex* (lmc) *algebra* we mean a topological algebra A whose topology is defined by a family $\{p_\alpha\}$, $\alpha \in I$ (directed index set), of submultiplicative seminorms, i.e., $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$ for all $x, y \in A$ and $\alpha \in I$. An lmc–algebra A with an involution * satisfying the property $p_\alpha(x^*) = p_\alpha(x)$, for all $\alpha \in I, x \in A$, will be called an lmc *–*algebra* (i.e., involution of A is continuous).

Let A be an lmc *–algebra and $N_\alpha = ker(p_\alpha)$ $\alpha \in I$. Then A/N_α is a normed algebra with respect to the norm $\|\cdot\|_\alpha$ defined by

$$\|x_\alpha\|_\alpha := p_\alpha(x), \;\; x_\alpha = x + N_\alpha \in A/N_\alpha, \;\; \alpha \in I \; .$$

Denote by A_α the completion of A/N_α with respect to the above norm; A_α is a Banach *–algebra. In particular, the families $(A/N_\alpha), (A_\alpha), \alpha \in I$, form projective systems of normed resp. Banach (*–) algebras, such that

$$A \underset{\to}{\subset} \varprojlim_\alpha A/N_\alpha \underset{\to}{\subseteq} \varprojlim_\alpha A_\alpha$$

up to topological algebraic (*–) isomorphisms. If A is moreover complete the preceding imbeddings become equalities [8; p.88, Theorem 3.1].

Now, given an algebra A and $x \in A$, denote by $sp_A(x), r_A(x)$ the *spectrum* and the *spectral radius* of x respectively. If moreover, A is involutive denote by $H(A)$ the set of all self–adjoint $(x^* = x, x \in A)$ elements of A. An involutive algebra A is said to be *symmetric* if every element of the form $-x^*x, x \in A$ is quasi–regular.

A topological algebra A is called a *Q–algebra* if the set of its quasi–regular elements is open in A [9; p.80, Definition E.1]. Every element x of a Q–algebra A has a compact spectrum (hence $r_A(x) < \infty$) [9; p.80, Lemma E.3]. A sequentially complete Q lmc *–algebra is called an MQ *–algebra [6; Definition 3]. Clearly, every Fréchet Q lmc *–algebra is an MQ *–algebra.

Let A be an lmc *–algebra. We denote by $M(A)$ the set of all non–zero continuous multiplicative linear functionals on A, and by $P(A)$ the set of all continuous positive functionals on A. If A is moreover unital, we also suppose that $f(e) = 1$ for every $f \in P(A)$. Every positive functional is continuous on a unital MQ *–algebra [6; Theorem 2]. A non–zero element $f \in P(A)$ is said to be *pure* or *indecomposable* if for every $g \in P(A)$ with $g \leq f (\Leftrightarrow g(x^*x) \leq f(x^*x), \forall\, x \in A)$ one has $g = \lambda f$, for some $\lambda \in [0,1]$. Denote by $S(A)$ the set of all non–zero extreme points of $P(A)$.

Let now $U_\alpha = \{x \in A : p_\alpha(x) \leq 1\}$ be the unit semiball of A corresponding to p_α, and U_α^0 the polar of $U_\alpha, \alpha \in I$, in A'_s, where A'_s denotes the weak topological dual of A. Then, we have

$$P(A) = \bigcup_\alpha P_\alpha(A) \quad \text{and} \quad S(A) = \bigcup_\alpha S_\alpha(A)$$

with $P_\alpha(A) = \{f \in P(A) : |f(x)| \leq 1, x \in U_\alpha\}$ and $S_\alpha(A)$ the extreme points of $P_\alpha(A)$, $\alpha \in I$. The preceding sets being subsets of A'_s are considered endowed with the relative topology; moreover, since $P_\alpha(A) = P(A) \cap U_\alpha^0$, $P_\alpha(A)$ (and therefore $S_\alpha(A), \alpha \in I$) is an equicontinuous subset of $P(A)$. Furthermore, for each $\alpha \in I$

$$P(A/N_\alpha) = P_\alpha(A) = P(A_\alpha) \; ,$$

within homeomorphisms [3; Theorem 3.1]. The same relationship also holds for the respective extreme points, i.e.,

$$S(A/N_\alpha) = S_\alpha(A) = S(A_\alpha) ,$$

within homeomorphisms [3; Corollary 3.1]. Thus, for each $f \in P(A)$ one can define, for some $\alpha \in I$, a continuous positive functional f_α on the Banach *–algebra A_α, by $f_\alpha(x_\alpha) := f(x), \forall x_\alpha \in A/N_\alpha$, and vice versa. Besides, if A is also unital, we have $S(A) = \{f \in P(A); f$ pure and $||f_\alpha|| = 1\}$.

Finally, we give some basic facts concerning the representation theory of lmc *–algebras, needed in the sequel. By a **–representation* of an involutive algebra A we shall mean a *–morphism of A into the C^*–algebra $B(H_\pi)$ of all bounded linear operators on some Hilbert space H_π. If A is furthermore an lmc *–algebra, the continuity of a *–representation π of A will be always considered with respect to the norm topology of $B(H_\pi)$. It is well–known that each *–representation of an MQ *–algebra is continuous [6; Theorem 3]. Moroever, each continuous positive functional f on a unital lmc *–algebra A is *representable* (Gel'fand–Naimark-Segal (GNS) construction) in the sense that, there exists a continuous *–representation π_f of A, on a Hilbert space H_f, and a cyclic vector $\xi_f \in H_f$ of π_f, such that

$$f(x) = < \pi_f(x)(\xi_f),\ \xi_f >,\ \forall x \in A .$$

For a detailed treatment of the GNS–construction, the reader is referred to [3; Theorem 3.4] (see also, [1; Theorem 6.1]).

We recall that a *–representation π of an lmc *–algebra A is said to be *irreducible* if the only closed subspaces of H invariant under π are H and $\{0\}$. Thus, it follows from the GNS–construction that for a unital lmc *–algebra A an element $f \in P(A)$ is an extreme point or pure if and only if the respective *–representation π_f is non–trivial (continuous) irreducible.

For a given topolgical *–algebra A, the **–radial* of A, denoted by $R^*(A)$, is defined to be $R^*(A) = \bigcap_\pi ker(\pi)$ with π running over the set of all continuous irreducible *–representations of A. In particular, if A is an lmc *–algebra, then $R^*(A) = \bigcap_\pi ker(\pi)$ with π running over the set of all continuous (not necessarily irreducible) *–representations of A (cf. [4; p.192, (3.2)]). If $R^*(A) = \{0\}$, then A is said to be **–semisimple.*

3. P–COMMUTATIVE TOPOLOGICAL *–ALGEBRAS

Let A be an lmc *–algebra. For each $f \in P(A)$, put

$$L_f = \{x \in A : f(x^*x) = 0\} = \{x \in A : f(yx) = 0 \text{ for every } y \in A\} .$$

Then L_f is a closed left ideal of A. Set

$$P = \cap\{L_f : f \in P(A)\} .$$

Then P is a closed two–sided ideal of A. To see this, we note that P is trivially a closed left ideal of A. We only show that P is a right ideal of A. For $f \in P(A)$ and for fixed $y \in A$ define $f_y : A \to \mathbb{C}$ by $f_y(x) = f(y^*xy)$. Clearly, $f_y \in P(A)$. Moreover, if $x \in P$ then

$$f((xy)^*xy) = f_y(x^*x) = 0 \text{ for every } f \in P(A) .$$

Hence P is also a right ideal. The (two–sided) ideal P is called the *reducing ideal* of A and A is called *reduced* if $P = \{0\}$.

Following [4; Definition 4.1] (see also [10]), we now define P–commutatity in lmc *–algebras.

Definition 3.1 An lmc *–algebra A is said to be *P–commutative* if $xy - yx \in P$ for any $x, y \in A$.

Several examples of (non–commutative) P–commutative topological *–algebras can be found in [3; p.196].

The following proposition connects the reducing ideal P of an lmc *–algebra with its *–radical. This result is well–known, however for the sake of completeness we give the proof (cf. [4; p.194, (4.5)]).

Proposition 3.1 Let A be a unital lmc *–algebra. Then

$$P = R^*(A) .$$

In other words,

$$R^*(A) = \cap\{ker(f) : f \in P(A)\} .$$

Proof Given a continuous *–representation π of A on a Hilbert space H_π and a cyclic vector ξ of π in H_π, the relation

$$f_\pi(x) = < \pi(x)\xi, \xi >, \quad x \in A ,$$

defines a positive functional on A which is continuous, i.e., $f_\pi \in P(A)$. Thus, $P \subset R^*(A)$.

On the other hand, each $f \in P(A)$ is representable [3; Theorem 3.4]. Therefore, $L_f = ker(\pi_f)$, where π_f is the continuous *–representation of A corresponding to f. Besides, for an lmc *–algebra A we have

$$R^*(A) = \bigcap_\pi ker(\pi)$$

with π running over all continuous *–representations of A [4; p.192, (3.2)], hence we conclude that $R^*(A) \subset P$, i.e., $P = R^*(A)$. Since for each $f \in P(A)$

$$|f(x)|^2 \leq f(e)\ f(x^*x), \quad \forall\, x \in A ,$$

it follows that $R^*(A) = \cap\{ker(f) : f \in P(A)\}$.

Remark 3.1 As a consequence of Proposition 3.1, we obtain that every *–semisimple lmc *–algebra is P–commutative if and only if it is commutative. Furthermore, a unital lmc *–algebra A is P–commutative if and only if $A/R^*(A)$ is commutative.

The next theorem, due to Kahane and Zelazko, gives the characterization of multiplicative linear functionals on lmc *–algebras.

Theorem 3.1 ([7; pp.339, 343], [12]). Let A be a unital complete lmc *–algebra. Then a continuous linear functional f on A is multiplicative if and only if

$$f(x) \in sp_A(x) \quad \text{for every} \quad x \in A .$$

Similarly, the next result establishes the existence of extreme points in $P(A)$.

Lemma 3.1 Let A be a unital lmc *–algebra and suppose that for every $x \in H(A)$ there is some $f \in P(A)$ with $f(x) \neq 0$. Then there exists an extreme point g in $P(A)$ such that $g(x) \neq 0$.

Proof By Theorem 3.1 [3], for each $\alpha \in I$

$P(A/N_\alpha) = P_\alpha(A) = P(A_\alpha)$, upto homeomorphisms, where

$$P_\alpha(A) = \{f \in P(A) : |f(x)| \leq 1,\ x \in U_\alpha\}(U_\alpha = \{x \in A : p_\alpha(x) \leq 1\}) .$$

The same relationship is also valid for the respective extreme points, i.e.,

$$S(A/N_\alpha) = S_\alpha(A) = S(A_\alpha), (\forall\, \alpha \in I)$$

upto homeomorphisms. The assertion now follows immediately from [10; Lemma 4.6.6, p.225].

We now show that if the spectrum of an element x of a unital complete lmc *–algebra A is given by

$$sp_A(x) = \{f(x) : f \in S(A)\} , \qquad (*)$$

then the spectral condition (*) is, in fact sufficient to ensure the P–commutativity of A. Thus, we extend a result of Doran and Tiller [2; Theorem 1] for Banach *–algebras to the more general framework of lmc *–algebras.

Theorem 3.2 A unital complete lmc *–algebra A satisfying the spectral condition

$$sp_A(x) = \{f(x) : f \in S(A)\} \text{ for every } x \in A ,$$

is P–commutative.

Proof By Proposition 3.1, $R^*(A) = \cap\{ker(f) : f \in P(A)\}$. Moreover, by Lemma 3.1, $R^*(A) = \cap\{ker(f) : f \in S(A)\}$. The hypothesis and Theorem 3.1 now imply that each $f \in S(A)$ is multiplicative. Therefore, $f(xy - yx) = 0$ for every $x, y \in A$ and for every $f \in S(A)$. Consequently, $xy - yx \in R^*(A) = P$ for every $x, y \in A$, and hence A is P–commutative.

It is shown in [4; Theorem 6.1] that if A is a unital P–commutative symmetric Fréchet lmc *–algebra satisfying the property $r_A(x) < \infty$ for all $x \in H(A)$, then the Gel'fand space $M(A)$ is non–empty, hermitian (i.e., $f(x^*) = \overline{f(x)}$, $\forall\, f \in M(A), x \in A$) and $sp_A(x) = \hat{x}(M(A))$, where $\hat{x}$ denotes the Gel'fand transform of x. In other words,

$$sp_A(x) = \{f(x) : f \in M(A)\} .$$

Furthermore, by [4; Theorem 5.3] a non–zero continuous positive functional f on a unital P–commutative lmc *–algebra A is an extreme point of $P(A)$ if and only if it is multiplicative. Thus for a unital P–commutative symmetric Fréchet lmc *–algebra A satisfying $r_A(x) < \infty$ for all $x \in H(A)$, one has

$$sp_A(x) = \{f(x) : f \in S(A)\} .$$

Hence we get

Corollary 3.1 A unital symmetric Fréchet lmc *–algebra with $r_A(x) < \infty$, for all $x \in H(A)$, is P–commutative if and only if it satisfies the spectral condition $(^*)$.

Corollary 3.2 Let A be a unital symmetric Fréchet Q lmc *–algebra. Then the following statements are equivalent:

(1) $M(A) \neq \emptyset, r_A(x) = \sup\{|f(x)| : f \in M(A)\},\ r_A(x^*x) \leq r_A(x)^2,\ \forall\, x \in A$;

(2) A is P–commutative;

(3) $sp_A(x) = \{f(x) : f \in S(A)\}$.

Proof The equivalence of (1) and (2) is established in [4; Corollary 7.4], while the equivalence of (2) and (3) follows from Corollary 3.1, since each Q–algebra satisfies the property $r_A(x) < \infty$ for every $x \in A$.

We now establish an isomorphism theorem.

Theorem 3.3 Let A and B be two unital P–commutative symmetric Fréchet lmc *–algebras with $r_A(x) < \infty$ and $r_B(y) < \infty$ for all $x \in H(A)$ and $y \in H(B)$. Let $R^*(A)$ and $R^*(B)$ be the *–radicals of A and B respectively. If T is a self–adjoint topological vector space isomorphism from A onto B satisfying the property $sp_B(Tx) \subset sp_A(x)$ for every $x \in A$, then $A/R^*(A)$ and $B/R^*(B)$ are topologically *–isomorphic (as algebras). Moreover, the correspondence $f \leftrightarrow f \circ T$ is a bijection of $S(A)$ onto $S(B)$, where $S(A)$ and $S(B)$ denote the sets of extreme points of $P(A)$ and $P(B)$.

Proof By Remark 3.1, $A/R^*(A)$ and $B/R^*(B)$ are commutative *–semisimple lmc *–algebras; further, these are also Fréchet (cf. [5; pp.113, 300]). Clearly, $sp_{A/R^*(A)}(x + R^*(A)) \subseteq sp_A(x)$ for all $x \in A$. We recall that a non–zero continuous positive functional on a unital P–commutative lmc *–algebra is an extreme point if and only if it is multiplicative [4; Theorem 5.3]. We now show that $T(R^*(A)) = R^*(B)$. To see this, consider the mapping g given by:

$$A \xrightarrow{T} B \xrightarrow{\varphi_B} B/R^*(B) \xrightarrow{f} \mathbb{C}, \quad g = f \circ \varphi_B \circ T ,$$

where φ_B is the natural quotient map and $f \in S(B/R^*(B))$ i.e., f is multiplicative. If there is no such f, then it follows that $B = R^*(B)$ and $T(R^*(A)) \subset R^*(B)$. Otherwise, $g = f \circ \varphi_B \circ T$ is a continuous self–adjoint complex linear map on A satisfying the relation

$$\begin{aligned} g(x) &= (f \circ \varphi_B \circ T)(x) = f(Tx + R^*(B)) \\ &\in sp_{B/R^*(B)}(Tx + R^*(B)) \quad (\text{cf. [8; Corollary 6.4, p.104]}) \\ &\subseteq sp_B(Tx) \subset sp_A(x) \quad \text{for all } x \in A\,. \end{aligned}$$

Thus by Theorem 3.1, g is multiplicative and hence $g \in S(A)$ [4; Theorem 5.3]. Besides,

$$\begin{aligned} R^*(A) &= \cap\{ker(h) : h \in S(A)\} \\ &= \cap\{ker(h) : h \in M(A)\}\,, \end{aligned}$$

therefore $g(R^*(A)) = \{0\}$. Moreover, since the elements of $S(B)$ separate the points of $B/R^*(B)$, it follows that $T(R^*(A)) \subset R^*(B)$. By symmetry, we obtain $T^{-1}(R^*(B)) \subset R^*(A)$ or $R^*(B) \subset T(R^*(A))$; consequently $T(R^*(A)) = R^*(B)$.

Now consider the following diagram

$$\begin{array}{ccc} A & \xrightarrow{T} & B \\ \varphi_A \downarrow & & \downarrow \varphi_B \\ A/R^*(A) & \xrightarrow{\tilde{T}} & B/R^*(B) \end{array}$$

where φ_A and φ_B are quotients maps (continuous) and $\tilde{T}$ is defined by $\tilde{T}(x + R^*(A)) = Tx + R^*(B)$. Using the relation $T(R^*(A)) = R^*(B)$, it can be verified that $\tilde{T}$ is a well–defined linear map. Furthermore, $\tilde{T}$ is a topological vector space isomorphism of $A/R^*(A)$ onto $B/R^*(B)$. We show that $\tilde{T}$ is, in fact, multiplicative. To see this, consider $g(xy) = f \circ \varphi_B \circ T(xy) = f(T(xy) + R^*(B))$, and $g(x)g(y) = f(T(x)+R^*(B))f(T(y)+R^*(B))$ for any $f \in S(B/R^*(B))$. Since g is multiplicative, $f(T(x)T(y) + R^*(B)) = f(T(xy) + R^*(B))$. This implies that $f((T(xy) - T(x)T(y)+R^*(B)) = 0$ for every $f \in S(B/R^*(B))$. Hence, $(T(xy)-T(x)T(y))+R^*(B) \in \cap\{ker(f) :\in S(B/R^*(B))\} = \{0\}$ (because $B/R^*(B)$ is *–semisimple). Therefore, $(T(xy) - T(x)T(y) + R^*(B) = 0$. In other words, $\tilde{T}(xy) + R^*(B) = \tilde{T}(x)\tilde{T}(y) + R^*(B)$; and hence $\tilde{T}$ is multiplicative.

The correspondence between $S(A)$ and $S(B)$ is now immediate since the elements of $S(A)$ and $S(B)$ vanish on $R^*(A)$ and $R^*(B)$ respectively. This completes the proof.

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