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The real and the complex Adams e-invariants [1] may be regarded as homomorphisms from the framed cobordism group Ω_n^{fr} into the rational numbers modulo the integers,

$$e\ ,\ e_c : \Omega_n^{fr} \to \mathbf{Q}/\mathbf{Z}\ .$$

When n is of the form $4r-1$ these invariants are specially interesting, and if $r = 2k+1$ is odd one has $e_c = 2e \mod \mathbf{Z}$, by [1]. Hence, if $(M,\mathcal{F})$ is a framed manifold of dimension $8k+3$ then

$$e(M,\mathcal{F}) = \frac{1}{2}\,(e_c(M,\mathcal{F}) + h(M,\mathcal{F})) \mod \mathbf{Z}\ ,$$

where $h(M,\mathcal{F})$ is either 0 or 1, depending on something; Understanding the invariant $h(M,\mathcal{F})$ was the motivation for this work. For example [7], if $(M,\mathcal{F})$ is the framed boundary of an SU-manifold, then $h(M,\mathcal{F})$ is 0. This suggests that $h(M,\mathcal{F})$ may depend on the first relative Chern class $C_1(X,\mathcal{F})$ of a compact U-manifold X with boundary M, which is the obstruction for reducing the structure group of X to SU. This was shown in [24] to be so when M has dimension 3. In that case $h(M,\mathcal{F})$ is essentially the Arf invariant in Rochlin's theorem for $spin^c$ manifolds [23,14], and it can be regarded as the reduction modulo 2 of the dimension of the space of harmonic spinors on a submanifold of X representing $C_1(X,\mathcal{F})$, by [27,16].

Furthermore, let M be the link of a normal Gorenstein surface singularity $(\mathcal{V},P)$. Let $\pi : \tilde{\mathcal{V}} \to \mathcal{V}$ be a resolution of P and let W be a characteristic divisor of $\tilde{\mathcal{V}}$. This means [10] that W is an effective divisor of $\tilde{\mathcal{V}}$ of the form $W = 2D-K$, where K is a canonical divisor and both D and K have support in $E = \pi^{-1}(P)$. By [10], if we let $\mathfrak{h}(W) = h^0(W,\mathcal{O}_W(D))$, then the invariant $[\mathfrak{h}(W) + \frac{1}{8}W^2]$ is constant modulo 2 for all characteristic divisors. The invariant $\mathfrak{h}(W)$ is [10] the mod 2-index of W, and one has (by [24] and [27]),

$$h(M,\mathcal{F}) = [\mathfrak{h}(W) + \frac{1}{8}(W^2-K^2)] \mod 2$$

for every framing $\mathcal{F}$ on M compatible with the complex structure on $\mathcal{V}$. In particular, we may take the resolution to be minimal, so that the canonical class K is negative, and one has

$$h(M,\mathcal{F}) \equiv \mathfrak{h}(-K) \mod 2.$$

Also, if $\mu(M,\mathcal{F})$ is the Rochlin invariant of the link M of $(\mathcal{V},P)$, for the spin structure determined by $\mathcal{F}$, then [17,27]

$$[\mathfrak{h}(W) + \tfrac{1}{8}W^2] \equiv -[\mu(M,\mathcal{F}) + b_2] \mod 2 ,$$

where b_2 is the number of irreducible components of E .

The purpose of this note is two-fold. On the one hand we use [10] and the results of Looijenga in [18] to determine the invariant $h(M,\mathcal{F})$ in higher dimensions for a class of manifolds (see the remark in §1 below). We prove :

Theorem 1. Let $(M,\mathcal{F})$ be a framed manifold of dimension $8k+3$ which is the link of an isolated, normal Gorenstein smoothable singularity $(\mathcal{V},P)$, with $\mathcal{F}$ being compatible with the complex structure on $\mathcal{V}$. Let $\pi : \tilde{\mathcal{V}} \to \mathcal{V}$ be a resolution of P with a canonical divisor $K \leq 0$. Then,

$$e(M,\mathcal{F}) = \frac{1}{2}\,[e_c(M,\mathcal{F}) + \mathfrak{h}(-K)] \mod \mathbf{Z} ,$$

where $\mathfrak{h}(-K) = \sum_{i=0}^{2k} h^{2i}(-K,\mathcal{O})$ is the mod 2 index of $-K$.

If we can choose the divisor $-K$ to be non-singular, then $-K$ is canonically a spin manifold [23,27,10], and $\mathfrak{h}(-K)$ is congruent modulo 2 to the dimension of the space of harmonic spinors on $-K$, by [5].

On the other hand we observe that the invariant $\mathfrak{h}(W)$ is the reduction modulo 2 of an integer, also denoted by $\mathfrak{h}(W)$. It follows from [10,27] that if M is 3-dimensional and $W = 2D-K$ is choosen so that D is *effective* and $-D$ is *relatively ample* (see §2 for the definition), then $(\mathfrak{h}(W) + \frac{1}{8}W^2)$ is actually constant for all such divisors. So this yields to "an integral lifting" of the Rochlin μ invariant, and it is natural to ask whether this is related to the Casson invariant [11] when the link is a homology sphere. In §2 below we use the theorem of [11,22], relating the Casson invariant of the link to the signature of the Milnor fibre, to prove:

Theorem 2. *If the germ of $\mathcal{V}$ at P is a quasi-homogeneous germ and the link M is an integral homology sphere, then*

$$[\mathfrak{h}(W) + \frac{1}{8}W^2] = -[\lambda(M) + b_2] ,$$

for every characteristic divisor $W = 2D-K$ with $D > 0$ and $-D$ relatively ample, where $\lambda(M)$ is the Casson invariant of M .

We also give in §2 a partial positive answer to a question in [22] concerning the geometric genus of a Gorenstein surface singularity. Namely, we prove that if the link is a homology sphere, then the geometric genus modulo 2 is determined by the topology of the link. This is a slight improvement of a result in [10] related to the behaviour of the geometric genus under deformations.

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§1. On the e-invariant.

In this work a ***framed manifold*** means a pair $(M,\mathcal{F})$, consisting of a closed, C^∞ , stably parallelizable manifold M , together with a trivialization $\mathcal{F}$ of its stable tangent bundle.

Let $(M,\mathcal{F})$ be a stably framed n-manifold. The framing $\mathcal{F}$ induces (stably) a spin structure and a U-structure on M . This gives rise to well defined group homomorphisms from Ω_n^{fr} into Ω_n^{spin} and Ω_n^{U} , the spin and the complex cobordism groups, respectively. (See [28] for basic material on cobordism.) The complex cobordism groups are 0 in odd dimensions, and the image of Ω_{4k-1}^{fr} in Ω_{4k-1}^{spin} is zero for all k. Hence, when $n = 4k-1$ the manifold $(M,\mathcal{F})$ can be regarded as bounding both, a compact spin manifold Y and a compact U-manifold X . In the first case, we let $p_i(Y,\mathcal{F}) \in H^{4i}(Y,M;\mathbf{Z})$, $i = 1,\ldots,k$, be [13] the Pontryagin classes of Y relative to the framing on M , and we define [6] the real Adams e-invariant of $(M,\mathcal{F})$ by ,

$$e(M,\mathcal{F}) = \begin{cases} \mathbf{A}(p_1(Y,\mathcal{F}),\ldots,p_k(Y,\mathcal{F}))[Y] \mod \mathbf{Z} , & \text{if } k \text{ is even} \\ \frac{1}{2}\,\mathbf{A}(p_1(Y,\mathcal{F}),\ldots,p_k(Y,\mathcal{F}))[Y] \mod \mathbf{Z} , & \text{if } k \text{ is odd} \end{cases}$$

where $\mathbf{A}(p_1(Y,\mathcal{F}),\ldots,p_k(Y,\mathcal{F}))[Y]$ denotes the k^{th} $\mathbf{A}$-polynomial of Hirzebruch [12] in the relative Pontryagin classes, evaluated on the orientation cycle of Y . The complex e-invariant is defined similarly, we consider the U-manifold X and we let

$$e_c(M,\mathcal{F}) = Td(C_1(X,\mathcal{F}),\ldots,C_{2k}(X,\mathcal{F}))[X] \mod \mathbf{Z} ,$$

where $Td(C_1(X,\mathcal{F}),\ldots,C_{2k}(X,\mathcal{F}))$ denotes the value of the $2k^{th}$ Todd polynomial [12] on the Chern classes of X relative to the framing. We denote this number by $Td[X,\mathcal{F}]$. If the first Chern class of X relative to $\mathcal{F}$ vanishes, then X is also a spin manifold, and in fact an SU-manifold ; In this case, if the dimension of M is $8k+3$, then [7;p.104] :

$$e(M,\mathcal{F}) = \frac{1}{2}\,Td(C_1(X,\mathcal{F}),\ldots,C_{2k}(X,\mathcal{F}))[X] \mod \mathbf{Z} .$$

In dimensions of the form $8k+7$, the real and the complex e-invariants coincide [1].

It is worth noting that a parallelism on a (closed) manifold induces a riemannian metric. In that case the e-invariant can be expressed [4] in terms of the eta-invariant, the harmonic spinors and the Chern-Simons invariant of the flat connexion defined by the parallelism.

Let us now suppose that M is the link of an isolated, normal, smoothable Gorenstein singularity P in a complex analytic space $\mathcal{V}$ of $\mathbb{C}$-dimension $2n$. Let F be a Milnor fibre of P (see [18]). If the framing $\mathcal{F}$ is compatible with the complex structure on $\mathcal{V}$, then $(M,\mathcal{F})$ can be regarded as the framed boundary of F. The first Chern class of F relative to $\mathcal{F}$ lives in $H^2(F,M;\mathbb{Z})$, which is isomorphic to $H_{4n-2}(F;\mathbb{Z})$ by Poincaré-Lefschetz duality. If $n > 1$ this group vanishes, by the theorem of Andreotti and Frankel [2]. Hence we have:

1.1 Lemma. (c.f.[15]) If $n > 1$, the Milnor fibre F has an SU-structure, and also a spin structure, compatible with the framing $\mathcal{F}$. Hence, if $n = 2k+1$ is odd then the real e-invariant of $(M,\mathcal{F})$ is

$$e(M,\mathcal{F}) = \frac{1}{2}\, Td[F,\mathcal{F}] \mod \mathbb{Z} .$$

We now observe [13] that the Chern classes of F relative to $\mathcal{F}$ map to the usual Chern classes. Hence the Chern numbers defined by these classes agree with the Chern numbers defined by Looijenga [18], except those involving the top dimensional Chern class. That Chern number is defined in [18] to be the Euler characteristic of F; Here, that number depends on $\mathcal{F}$: it equals the degree of $\mathcal{F}$ in F. Similar considerations apply if we replace the smoothing F by a compact tubular neighbourhood of the exceptional set $E = \pi^{-1}(P)$ of a resolution $\mathcal{V}$ of P; for simplicity, we also denote this neighbourhood of E by $\mathcal{V}$. The following lemma is an easy extension of the index formula in [18].

1.2 Lemma. Let $Td[F,\mathcal{F}]$ and $Td[\mathcal{V},\mathcal{F}]$ denote respectively the value on the orientation class of F, or $\mathcal{V}$, of the $4k+2$ Todd polynomial evaluated on the Chern classes of F, or $\mathcal{V}$, relative to the framing $\mathcal{F}$. Then,

$$Td[F,\mathcal{F}] = Td[\mathcal{V},\mathcal{F}] + p_g ,$$

where $p_g = \dim H^{4k+1}(\mathcal{V},\mathcal{O})$ is the (geometric) genus of P.

Proof. Let $Td[F]$ and $Td[\mathcal{V}]$ be the Todd genera of F and $\mathcal{V}$, respectively, obtained by using the Chern numbers of Looijenga. Then [18;p.298] says

$$Td[F] = Td[\mathcal{V}] + \sum_{i\geq 1} (-1)^{i-1} h^i(\mathcal{O}_{\mathcal{V}}) - \dim_{\mathbb{C}} \pi_* {}^{\mathcal{O}}\mathcal{V}/\mathcal{O}_{\mathcal{V}} .$$

Since the germ of $\mathcal{V}$ at P is normal and Gorenstein, the last term in the right hand side vanishes and the groups $H^i(\mathcal{O}_{\mathcal{V}})$ are 0 for $i \leq 4k$. (See for instance [10;3.2.b].) Thus Looijenga's formula reads

$$Td[F] = Td[\mathcal{V}] + p_g ,$$

where p_g is the geometric genus. Lemma 1.2 will be proved if we show

$$Td[F,\mathcal{F}] - Td[F] = Td[\mathcal{V},\mathcal{F}] - Td[\mathcal{V}] .$$

Since the Chern numbers involving lower (relative) Chern classes are unique [15,18], one has

$$Td[F,\mathcal{F}] - Td[F] = q_{4k+2}\,\{C_{4k+2}(F,\mathcal{F})[F] - \chi(F)\} ,$$

$$Td[\mathcal{V},\mathcal{F}] - Td[\mathcal{V}] = q_{4k+2}\,\{C_{4k+2}(\mathcal{V},\mathcal{F})[\mathcal{V}] - \chi(\mathcal{V})\} ,$$

where q_{4k+2} is the coefficient of C_{4k+2} in the $4k+2^{th}$ Todd polynomial and $\chi(*)$ denotes the topological Euler-Poincaré characteristic. The proof of lemma 1.2 is completed by standard arguments: First we note that up to homotopy, $\mathcal{F}$ is defined by $4k+2$, $\mathbb{C}$-linearly independent smooth sections $f_1,\dots,f_{4k+2}$ of $T\mathcal{V}|_M$, because the Stiefel manifold $W_{r,n+r}$ of complex r-frames in $\mathbb{C}^{n+r}$, $r > 0$, is $2n$-connected. Hence, if we think of $\mathcal{F}$ as being a trivialization of $TF|_M$, then $C_{4k+2}(F,\mathcal{F})[F]$ is the total index in F of one of the f_i's; Similarly for $C_{4k+2}(\mathcal{V},\mathcal{F})[\mathcal{V}]$. Hence 1.2 follows from the theorem of Poincaré-Hopf for manifolds with boundary [19] and the elementary lemma below, that we state without proof. (See [26] for an easy proof.)

1.3 Lemma. Let X and Y be compact, smooth and oriented manifolds with diffeomorphic boundary M, and let α, β be non-singular vector fields defined on a neighbourhood of M in X. Denote also by α, β the induced vector fields on a neighbourhood of M in Y. If $Ind(\alpha,X)$ denotes the total index of α in X, and similarly for β and Y, then

$$Ind(\alpha,X) - Ind(\beta,X) = Ind(\alpha,Y) - Ind(\beta,Y) .$$

We are now in a position to prove Theorem 1 stated in the introduction: For $k = 0$, this is the theorem in [24] together with the identification in [16,27,10] of Rochlin's Arf invariant with the invariant $\mathfrak{b}(W)$. In higher dimensions, Lemmas 1.1 and 1.2 above imply

$$e(M,\mathcal{F}) = \frac{1}{2}\,(Td[\mathcal{V},\mathcal{F}] + p_g) \mod \mathbb{Z} .$$

By the definition of the complex e-invariant one has

$$e_c(M,\mathcal{F}) = Td[\mathcal{V},\mathcal{F}] \mod \mathbb{Z} ,$$

hence,

$$e(M,\mathcal{F}) = \frac{1}{2}\,(e_c(M,\mathcal{F}) + p_g) \mod \mathbb{Z} ,$$

and Theorem 1 follows from this equation and the fact [10;3.5] that the geometric genus is congruent to $\mathfrak{b}(-K)$ modulo 2.

1.4 Remarks. **a)** We note that we can replace the divisor $-K$ in Theorem 1 by any characteristic divisor $W = 2D-K$ of $\mathcal{V}$, though the expression we get is more complicated [10]. We also note that if $h(M,\mathcal{F})$ is the invariant in the introduction, i. e. the difference between $e(M,\mathcal{F})$ and $\frac{1}{2}e_c(M,\mathcal{F})$, then Theorem 1 determines $h(M,\mathcal{F})$. However, this answer is not entirely satisfactory because it is not intrinsic, as it makes use of a U-manifold with boundary $(M,\mathcal{F})$. There must be a way for expressing $h(M,\mathcal{F})$ purely in terms of $(M,\mathcal{F})$.

b) It would be interesting to know whether Theorem 1, formulated in terms of harmonic spinors, still holds when we replace the divisor $-K$ by a C^∞ smoothing of it. As noted in [27], such smoothings always exist for divisors, by Thom's transversality. When M is a 3-manifold we can indeed replace $-K$ by a smoothing [24], and we can also define $\mathfrak{h}(-K)$ as the dimension of the space of harmonic spinors on the smoothing modulo 2. These two facts are essentially consequences of Rochlin's theorem for spin^c manifolds. In higher dimensions they would be consequences of the following conjecture (c.f. [10;2.4.b]).

Conjecture. Let X be a C^∞, closed, spin^c manifold of dimension $8k+4$, and let D_X be the Dirac operator on X. If W is an oriented, characteristic submanifold of X, and if D_W is the Dirac operator on W for the induced spin structure [23], then:

$$\dim \operatorname{Ker} D_X - \dim \operatorname{Coker} D_X \equiv \dim \operatorname{Ker} D_W \mod 2 .$$

This conjecture is true if $\dim X = 4$, by Rochlin's theorem, or if X is spin and W is empty, where it says that the $\hat{A}$-genus of a closed, spin $8k+4$-manifold is an even integer [3], or if X and W are both complex analytic [10].

§2. On the Casson invariant.

In this section we prove Theorem 2. This follows easily from proposition 2.1 below together with the theorem in [11,22], saying that if $(\mathcal{V},P)$ is a quasi-homogeneous surface germ with link M a homology sphere, then

$$\lambda(M) = -\frac{1}{8}\sigma(F) ,$$

where λ is the Casson invariant and $\sigma(F)$ is the signature of the Milnor fibre.

2.1 Proposition. Let $\pi : \tilde{\mathcal{V}} \to \mathcal{V}$ be a resolution of a normal Gorenstein smoothable surface singularity $(\mathcal{V},P)$, and let F be the Milnor fibre of a smoothing of P. Then for every characteristic divisor $W = 2D - K$ of $\tilde{\mathcal{V}}$ one has :

$$\sigma(F) \equiv \sigma(\tilde{\mathcal{V}}) - (8\mathfrak{h}(W) + W^2) \mod 2 ,$$

where σ denotes the signature and $\mathfrak{h}(W) = h^0(W,\mathcal{O}_W(D))$.

Moreover, if we choose W so that $D \geq 0$ and $-D$ is relatively ample for π, then one has :

$$\sigma(F) = \sigma(\tilde{\mathcal{V}}) - (8\mathfrak{h}(W) + W^2) .$$

We recall that $-D$ is relatively ample for π if $-D\cdot E_i > 0$ for all non-singular curves contained in the exceptional divisor.

Proof of 2.1. We know from [25] that the tangent bundle of F is trivial. Hence Durfee's formula for the signature of the Milnor fibre says,

$$\sigma(F) = -\frac{1}{3}(2\chi(F) - 2 + K^2 + 2b_1 + b_2) ,$$

where b_1 and b_2 are the corresponding Betti numbers of $\tilde{\mathcal{V}}$ and χ is the topological Euler-Poincaré characteristic. The formula of Laufer-Steenbrink for $\chi(F)$ says (see [18]),

$$\chi(F) = 1 - b_1 + b_2 + K^2 + 12p_g ,$$

where p_g is the geometric genus. Hence,

$$\sigma(F) = -(b_2 + K^2 + 8p_g) ,$$

and $-b_2$ equals the signature of $\tilde{\mathcal{V}}$, because the intersection matrix of $\tilde{\mathcal{V}}$ is negative definite [20]. Therefore

$$\sigma(F) = \sigma(\tilde{\mathcal{V}}) - (K^2 + 8p_g) ,$$

and the proposition follows from this equation together with the formulae in [10;4.1] relating p_g and $\mathfrak{h}(W)$.

Now we have the following improvement of [10;4.4] :

2.2 Proposition. Let $(\mathcal{V},P)$ be a normal Gorenstein surface singularity. If the graph Γ of the minimal good resolution is a tree of rational curves, then the geometric genus of P modulo 2 is determined by the orientation preserving homeomorphism type of M. Moreover, if the singularity is smoothable, then the same statement applies to the signature of a smoothing of P.

Proof. The statement concerning the signature of a smoothing follows from the statement for the geometric genus together with the formula

$$\sigma(F) = \sigma(\mathcal{V}) - (K^2 + 8p_g) ,$$

because the graph Γ is determined by the oriented homeomorphism type of the link [21], and Γ determines $\sigma(\mathcal{V})$ and K^2. To prove the first statement in 2.2 we use again the formula of [10],

$$p_g \equiv \mathfrak{h}(W) + \frac{1}{8}(W^2 - K^2) \quad \text{mod } 2 ,$$

for every characteristic divisor W. In particular W can be taken to be K_{red}, the union of all components of the exceptional set $E = \pi^{-1}(P)$ that appear in K with odd multiplicity. As noted in [17], the fact that the graph Γ is a tree of rational curves implies that $W = K_{red}$ is a disjoint union of such curves. Hence $\mathfrak{h}(W) = 0$, and the result follows from the fact [21] that the link M determines Γ, K^2 and W^2.

We remark that if the germ $(\mathcal{V},P)$ is quasi-homogeneous, then [27] the quotient $\chi(\mathcal{V},P)/E(\mathcal{V},P)$ is an integer, where $\chi(\mathcal{V},P)$ is the Euler-Poincaré characteristic of the base orbifold and $E(\mathcal{V},P)$ is the Euler number of the link as a Seifert manifold. If this integer is odd, then [27] the reduced canonical divisor K_{red} is a disjoint union of rational curves. Hence the concusions of 2.2 apply to these singularities, with the same proof. This includes, for instance, the canonical singularities of Dolgachev [8].

If the link M of a surface singularity is a homology sphere, then [22,17] the graph of a resolution of P is necessarily a tree of rational curves, hence 2.2 implies :

2.3 Corollary. Let $(\mathcal{V},P)$ be a normal Gorenstein surface singularity with link a homology sphere. Then the geometric genus modulo 2 is determined by the orientation preserving homeomorphism type of the link.

2.4. Remarks. It is clear that our proof of Theorem 2 is rather ad hoc and relies heavily on [11,22]. Still, the result is interesting because it points out aspects of Casson's invariant not yet regarded, and the signature of a resolution is easier to handle than the signature of the Milnor fibre.

If one chooses the characteristic divisor W to be non-singular, then W is a characteristic submanifold in the sense of Rochlin (see [14,23,24]). In general, a characteristic divisor W is singular and reducible. However [27, p. 355], we can always approximate it by a family of C^∞, oriented, characteristic submanifolds W_t. Each W_t is canonically spin [23] and the dimension of the space of harmonic spinors on W_t equals $\mathfrak{h}(W)$ mod 2, as noted before. If the link M is a homology sphere, if $(\mathcal{V},P)$ is quasi-homogeneous, and if the W_t's are complex analytic submanifolds (of the form $2D_t-K$ with $D_t \geq 0$ and $-D_t$ relatively ample), then Theorem 1 shows that the dimension of the space of harmonic spinors on the W_t's is actually constant, not only modulo 2, and it equals the algebraic invariant $h^0(W,\mathcal{O}_W(D))$. This is surprizing and it arises some questions:

i) Is this statement still true when the W_t's are not complex analytic ? .

ii) What happens if the germ $(\mathcal{V},P)$ is not quasi-homogeneous ?.

iii) Is the Casson invariant of the link given in general by the formula

$$\lambda(M) = \sigma(\mathcal{V}) - \frac{1}{8}(K^2 + \dim \operatorname{Ker} \mathcal{D}_K)$$

where $\mathcal{D}_K$ is the Dirac operator on a smooth, C^∞ representative of the anti-canonical class $-K$ of a resolution $\mathcal{V}$?.

A positive answer to these questions would imply the conjecture in [22]. One may also consider the following question, arising from similar questions in quantum field theories : Let $\{W_t\}$ be a 1-parameter family of compact Riemann surfaces that degenerates to a singular (reducible) curve W with a dualizing sheaf ω_W. For each t we have a holomorphic square root $\mathcal{D}_t$ of the canonical bundle of W_t, depending holomorphically on t, which degenerate to a square root $\mathcal{D}$ of ω_W. Is $h^0(W_t,\mathcal{D}_t)$ congruent to $h^0(W,\mathcal{D})$ modulo 2 ?. If the W_t's and W appear as characteristic divisors in a family of complex surfaces, with the appropiate line bundles, then [10] gives a positive answer to this question.

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