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q–DEFORMED DIFFERENTIAL OPERATOR ALGEBRA AND NEW BRAID GROUP REPRESENTATION

Lu–yu Wang *
International Centre for Theoretical Physics, Trieste, Italy
and
CCAST (World Laboratory),
P.O. Box 8730, Beijing 100080, People's Republic of China,

Jiang–hui Dai
CCAST (World Laboratory),
P.O. Box 8730, Beijing 100080, People's Republic of China
and
Institute of Theoretical Physics,
P.O. Box 8730, Beijing 100080, People's Republic of China

and

Jun Zhang
Physics Department, Xinjiang University,
Urumqi 830046, People's Republic of China.

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* On leave of absence from: Physics Department, Xinjiang University, Urumqi 830046, Xinjiang, People's Republic of China.

ABSTRACT

It is proved that the q–deformed differential operator algebra introduced in [1] is consistent with quantum hyperplane described by Wess and Zumino. At the same time, a new braid group representation associated with $sl_q(2)$ is obtained by adding the terms of weight conservation to the standard universal R–matrix.

The operator realization of the quantum group is a very interesting problem in the study of quantum group theory. The q-oscillator advocated by Biedenharn and Macfarlane [2] shows that one can give the representations of the quantum group by defining some fundamental relations among the q-oscillators. The quantum, or q-analogue differential operator realization, which is expected to be useful in the study of quantum groups however, fails to be well understood upon to date. In Ref.[3], a new operator D, apart from the usual differential operator, is introduced by its action on function f(Z) as

$$Df(Z) = \frac{f(qZ) - f(q^{-1}Z)}{Z(q - q^{-1})}$$

Obviously, D is a differential operator only when $q \rightarrow 1$ and lacks the geometric meaning.

Recently, a non-commutative q-deformed differential algebra (q-DOA) is introduced in Ref. [1] based upon the basic commutation relations of q-deformed derivatives $\tilde{\partial}_i$ and variables. The novel relation of q-DOA defined in Ref.[1] and the quantum hyperplane described by Wess and Zumino [4] indicate that there exist some deep geometric interpretations in q-DOA.

In this letter, we will show that there indeed exists a braid group structure in q-DOA, and which is consistent with W-Z's quantum hyperplane. The key role played by R, a matrix appearing in q-DOA, is then shown that it could be constructed in standard approach [5,6] with a subtle modification [7] by $sl_q(2)$.

Corresponding to the new $\tilde{R}$, a realization of the B-W algebra [8] is possible. The Yang-Baxterization and related vertex model can also be constructed from the $\tilde{R}$. We shall describe these results in a forthcoming publication.

Before proceeding let's first recall some basic aspects of q-deformed differential algebra introduced in Ref.[1]. In addition to the ordinary differential operator algebra spanned by z_i, $\partial_i = \partial/\partial Z_i$ we introduce the q-deformed or quantum differential operators $\tilde{\partial}_i$ in order to describe the quantum enveloping algebra $sl_q(n)$. Analogus to the commutation of ∂_i, Z_i

$$\partial_i Z_j = \delta_{ij} + Z_j \partial_i \qquad i=1,2, \quad ,n \qquad (1)$$

the q-deformed derivatives are defined as

$$\tilde{\partial}_i Z_j = \delta_{ij} + q^{2\delta_{ij}} Z_j \tilde{\partial}_i \qquad i=1,2, \quad ,n \qquad (2)$$

Relations (1) and (2) are sufficient to realize $A_q(n)$ quantum algebra. For instance, let n=2, define

$$\begin{aligned} J_+ &= q^{-\bar{Z}\partial_{\bar{Z}}} \tilde{\partial}_{\bar{Z}} \\ J_- &= q^{-Z\partial_Z} \tilde{\partial}_Z \\ J_0 &= \frac{1}{2}(Z\partial_Z - \bar{Z}\partial_{\bar{Z}}) \end{aligned} \qquad (3)$$

one has

$$\begin{aligned} [J_+ , J_-] &= [2J_0]_q \\ [J_0 , J_\pm] &= \pm J_\pm \end{aligned} \qquad (4)$$

where

$$[2J_0]_q = \frac{q^{2J_0} - q^{-2J_0}}{q - q^{-1}} \qquad (5)$$

One can also get the the finite representation of $sl_q(2)$ by constructing the corresponding Bergman space [1].

Interestingly enough, the above q-deformed derivatives may be well defined in quantum hyperplane according to the Wess-Zumino approach. Generally, the basic variables { Z_i }, their differential { $\xi_i = dZ_i$ } and their derivatives { $\tilde{\partial}_i$ } have the following commutation relations [4]

$$Z_i Z_j = B^{kl}_{ij} Z_k Z_l \qquad (6.1)$$

$$\xi_i \xi_j = - C^{kl}_{ij} \xi_k \xi_l \qquad (6.2)$$

$$\tilde{\partial}_i \tilde{\partial}_j = F^{lk}_{ji} \tilde{\partial}_k \tilde{\partial}_l \qquad (6.3)$$

$$Z_i \xi_j = C^{kl}_{ij} \xi_k Z_l \qquad (6.4)$$

$$\tilde{\partial}_i \xi_j = (C^{-1})^{kl}_{ij} \xi_k \tilde{\partial}_l \qquad (6.5)$$

$$\tilde{\partial}_i Z_j = \delta_{ij} + C^{kl}_{ij} Z_k \tilde{\partial}_l \qquad (6.6)$$

The consistent differential calculus may be defined if the required properties of the exterior differentials are satisfied

$$\begin{cases} \tilde{d} = \xi_i \tilde{\partial}_i \\ \tilde{d}^2 = 0 , \quad \tilde{d}(fg) = (\tilde{d}f)g + f\tilde{d}g \end{cases} \tag{7}$$

These lead to the following consistency constraints on the matrices B, E and F

$$\begin{aligned} &(E_{12} - B_{12})(E_{12} + C_{12}) = 0 \\ &(E_{12} + C_{12})(E_{12} - F_{12}) = 0 \\ &B_{12} C_{23} C_{12} = C_{23} C_{12} B_{23} \\ &C_{12} C_{23} F_{12} = F_{23} C_{12} C_{23} \\ &C_{12} C_{23} C_{12} = C_{23} C_{12} C_{23} \end{aligned} \tag{8}$$

where E is the unit matrix. A nontrivial solution of (8) is given by Wess And Zumino [4] as follows

$$B = F = \frac{1}{q} \hat{R} , \quad C = q \hat{R} \tag{9}$$

where $\hat{R}$ is the standard R-matrix of $sl_q(2)$. With the solution (8), (6) defined a consistent differential calculus on the hyperplane.

However, there exist some other nontrivial solutions to the equations (8). Among them, a interesting one, which is obtained from relations (2) and (6.6), is

$$B = F = E , \quad C = \tilde{R} \tag{10}$$

where

$$\tilde{R}^{ii}_{ii} = q^2 , \quad \tilde{R}^{ij}_{ji} = 1 \quad (i \neq j) \qquad \tilde{R}^{ij}_{kl} = 0 \quad \text{otherwise} \tag{11}$$

It is easily verified that $\tilde{R}$ satisfies the Yang-Baxter equation (see appendix)

$$\tilde{R}_{12} \tilde{R}_{23} \tilde{R}_{12} = \tilde{R}_{23} \tilde{R}_{12} \tilde{R}_{23} \tag{12}$$

and (10) is indeed a solution of (8). Therefore, the original q-deformed derivatives (2) are just the quantum derivatives well

defined on quantum hyperplane (6) where variables $\{z_i\}$, derivatives $\{\tilde{\partial}_i\}$ are commutative with themselves, differentials $\{\xi_i\}$ are anti-commutative, however, the commutation relations of $\{z_i, \xi_j\}$, $\{\tilde{\partial}_i, z_j\}$ and $\{\tilde{\partial}_i, \xi_j\}$ are deformed by the parameter q.

The R satisfied (12) leads to a new representation of braid group (BGR). For n=2, we can easily prove that this new BGR can be constructed from $sl_q(2)$. Following Drinfeld theory [5,6], the standard BGRs or R-matrices have been constructed in terms of the quantum groups as q-analogues of the universal enveloping algebras of Lie algebras. In the case of $sl_q(2)$, the generators h and $E_\pm$ satisfy

$$\begin{aligned} &[h, E_\pm] = \pm E_\pm \\ &[E_+, E_-] = [2h]_q \end{aligned} \tag{13}$$

and the standard R-matrix is

$$\begin{aligned} R &= \sum_\alpha e_\alpha \otimes e^\alpha \\ &= q^{2(h \otimes h)} \sum_i q^{\frac{1}{2} i(i-1)} \{ (q - q^{-1})^i / [i]_q \} E_+^i q^{ih} \otimes E_-^i q^{-ih} \end{aligned} \tag{14}$$

For a given irreducible representation of $sl_q(2)$, e.g. $j=\frac{1}{2}$, [2,9]

$$E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \quad h = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{15}$$

and R-matrix is

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{16}$$

Recently, the investegation shows that we can add some terms ΔR satisfying weight conservation to the standard R-matrix such that $R' = R + \Delta R$, where

$$\Delta R = \sum_{m+m'=n+n'} C^{mnr}_{m'n'r'} E_+^m E_-^n h^r \otimes E_+^{m'} E_-^{n'} h^{r'} \tag{17}$$

may also obtain a solution of Yang-Baxter equation [7]. In our case, let

$$\Delta R = (q^{-\frac{3}{2}} - q^{\frac{1}{2}}) E_+ \otimes E_- \tag{18}$$

we get a new R-matrix

$$R' = R + \Delta R$$

$$R' = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \qquad (19)$$

which leads to a BGR is equivalent to (11).

The discussion about the generation of B_n, C_n, D_n cases based on this letter will be described in our forthcoming publications.

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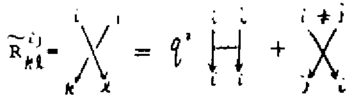
APPENDIX

According to the Kauffman diagram approach [10], the BGR (11) can be written as

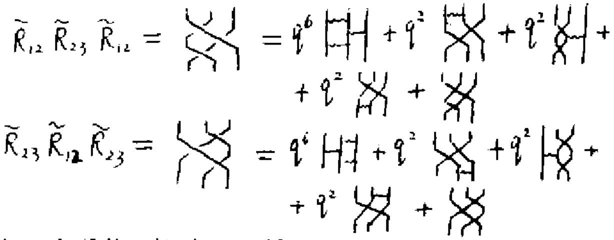


(A1)

The braid relation is expressed as

(A2)

With (A1) we have



(A3)

(A4)

From (A3) and (A4), it is easily seen that

$$\tilde{R}_{12}\tilde{R}_{23}\tilde{R}_{12} = \tilde{R}_{23}\tilde{R}_{12}\tilde{R}_{23} \qquad (A5)$$

ie. the $\tilde{R}$ given by (11) is a solution of Yang-Baxter equation (12).

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