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CONNECTIONS AND CURVATURES ON COMPLEX RIEMANNIAN MANIFOLDS *

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ABSTRACT

Characteristic connection and characteristic holomorphic sectional curvatures are introduced on a complex Riemannian manifold (not necessarily with holomorphic metric). For the class of complex Riemannian manifolds with holomorphic characteristic connection a classification of the manifolds with (pointwise) constant holomorphic characteristic curvature is given. It is shown that the conformal geometry of complex analytic Riemannian manifolds can be naturally developed on the class of locally conformal holomorphic Riemannian manifolds. Complex Riemannian manifolds locally conformal to the complex Euclidean space are characterized with zero conformal fundamental tensor and zero conformal characteristic tensor.

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INTRODUCTION

Complex Riemannian manifolds with analytic metrics have been investigated by R.Penrose in connection with the description of theory of non - linear gravitons [8]. A natural step in the construction of Penrose's twistor correspondence is the complexification of the real analytic Riemannian geometry which leads to the notion of complex analytic Riemannian geometry [6,10].

A complex Euclidian 4-space (flat complex analytic Riemannian manifold) has been used by Issenberg et al. [3] and Witten [12] for studying of Yang-Mills fields.

Self-dual complex analytic Riemannian manifolds of complex dimension 4 have been described in [6,8].

In [5] , C. LeBrun proved fundamental relationship between the local complex analytic Riemannian geometry and the global complex analysis.

In this paper we consider complex Riemannian geometry from differential geometric view point. We introduce the notion of characteristic connection and study local differential geometry of complex Riemannian manifolds in terms of this connection and its curvature. It is worth two classes of complex Riemannian manifold to be mentioned: the class of locally conformal holomorphic Riemannian manifolds and the class of complex Riemannian manifolds with holomorphic characteristic connection. In the former we study manifolds of (pointwise) constant characteristic curvature and give a classification theorem. In the latter we study conformally flat manifolds with respect to the characteristic connection.

1. ALGEBRAIC PRELIMINARIES

Let V be a real 2n-dimensional vector space with a complex structure J and $V^C = V \otimes_R \mathbb{C}$ be the complexification of (V,J). The complex linear extension of J over V^C is denoted by the same letter. We set

$$V^{1,0} = \{ Z \in V^C ; JZ = iZ \}, \quad V^{0,1} = \{ Z \in V^C ; JZ = -iZ \}, \quad i = \sqrt{-1};$$

Then $V^C = V^{1,0} \oplus V^{0,1}$ and the complex conjugation $Z = X + iY \longrightarrow \overline{Z} = X - iY$ in V^C defines a linear isomorphism between $V^{1,0}$ and $V^{0,1}$.

Let g be a real inner product in (V,J) and

$$(1.1) \qquad g(JX,JY) = - g(X,Y) \; ; \; X,Y \in V.$$

The proof of the following proposition is straightforward.

PROPOSITION 1.1. *If g is a real inner product in (V,J) satisfying the condition (1.1), then g can be extended uniquely to a symmetric complex nondegenerate bilinear form G of V^C and it satisfies the conditions*

$$(1.2) \quad \begin{cases} (1) & G(\overline{Z},\overline{W}) = \overline{G(Z,W)} \; ; \; Z,W \in V^C, \\ (2) & G(Z,W) = - G(JZ,JW) \; ; \; Z,W \in V^C . \end{cases}$$

Conversely, every symmetric complex bilinear form G on V^C, satisfying (1.2), is the natural extension of a real inner product g in (V,J), satisfying the condition (1.1).

The condition (2) in the above proposition is equivalent to the following

$$(2') \; G(Z,\overline{W}) = 0 \; ; \; Z,W \in V^{1,0}.$$

Thus, every symmetric complex bilinear form with the properties (1.2) is completely determined by its values on $V^{1,0}$.

If g is a real inner product in (V,J) with the property (1,1), then $\tilde{g}$ defined by $\tilde{g}(X,Y) = g(JX,Y)$; $X,Y \in V$, is also an inner product in (V,J), satisfying the condition (1.1). The inner product $\tilde{g}$ is said to be associated with g. If G and $\tilde{G}$ are the complex extensions of g and $\tilde{g}$,respectively, then $\tilde{G}(Z,W) = G(JZ,W)$ for all $Z,W \in V^{C}$. The complex bilinear form $\tilde{G}$ is said to be associated with G.

Let $Z = (z^1,\dots,z^n) \in \mathbb{C}^n$. If $z^k = x^k + iy^k$; $x^k, y^k \in \mathbb{R}$; $k = 1,\dots,n$, then the standard identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ is given by $(z^1,\dots,z^n) \to (x^1,\dots,x^n;y^1,\dots,y^n)$. The canonical complex structure J_o of $\mathbb{R}^{2n}$ maps $(x^1,\dots,x^n;y^1,\dots,y^n)$ onto $(y^1,\dots,y^n;-x^1,\dots,-x^n)$. The canonical inner product g_o in $\mathbb{R}^{2n}$ with the property (1.1) is defined by

$$(1.3) \qquad g_o(Z,Z) = (x^1)^2 + \dots + (x^n)^2 - (y^1)^2 - \dots - (y^n)^2,$$

where $Z = (x^1,\dots,x^n;y^1,\dots,y^n) \in \mathbb{R}^{2n}$. The associated inner product $\tilde{g}_o$ is determined by the equality

$$\tilde{g}_o(Z,Z) = 2(x^1y^1 + \dots + x^ny^n).$$

We denote the real representation of $GL(n,\mathbb{C})$ and $O(n,\mathbb{C})$ by the same symbols. By a direct verification we have

PROPOSITION 1.2. *There is a natural one-to-one correspondence between the set of inner products in* $\mathbb{R}^{2n}$*, satisfying the property (1.1) with respect to the canonical complex structure* J_o*, and the homogeneous space* $GL(n,\mathbb{C})/O(n,\mathbb{C})$*; the coset represented by an element* $A \in GL(n,\mathbb{C})$ *corresponds to the inner product* g *defined by*

$$g(X,Y) = g_o(AX,AY) \; ; \; X,Y \in \mathbb{R}^{2n},$$

where g_o *is the canonical inner product* (1.3) *with the property* (1.1).

Let M be an n-dimensional complex manifold. We denote by $(\mathbf{M},J)$ the manifold considered as a real 2n-dimensional manifold with the induced complex structure J. The tangential space to $(\mathbf{M},J)$ at $p \in M$ and its complexification are denoted by T_pM and T_p^CM, respectively. The algebras of real differentiable vector fields, complex differentiable vector fields and vector fields of type (1,0) on $\mathbf{M}$ are denoted by $\mathfrak{X}M$, $\mathfrak{X}^CM$ and $\mathfrak{X}^{1,0}M$, respectively.

If $z^1, \ldots, z^n$ are holomorphic coordinate functions in a coordinate neighbourhood U in M and $z^\alpha = z^\alpha + iy^\alpha$, $(\alpha = 1,\ldots,n)$ then the complex vector fields $Z_\alpha = \dfrac{\partial}{\partial z^\alpha} = \dfrac{1}{2}\left(\dfrac{\partial}{x^\alpha} - i\dfrac{\partial}{\partial y^\alpha}\right)$,

(resp. $Z_{\overline{\alpha}} = \dfrac{\partial}{\partial z^{\overline{\alpha}}} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^{\overline{\alpha}}} + i\dfrac{\partial}{\partial y^{\overline{\alpha}}}\right)$), form a basis for $V^{1,0}$ (resp. $V^{0,1}$), where $T_pM = V$, $T_p^CM = V^C = V^{1,0} \oplus V^{0,1}$.

Definition. A complex Riemannian metric on a complex manifold M is a covariant symmetric 2-tensor G, which is nondegenerate at each point and

(1) $G(\overline{Z},\overline{W}) = \overline{G(Z,W)}$; $Z,W \in \mathfrak{X}^CM$

(2) $G(JZ,JW) = -G(Z,W)$; $Z,W \in \mathfrak{X}^CM$

Thus every complex Riemannian metric on M induces in each T_p^CM, $p \in M$, a symmetric complex bilinear form satisfying the condition (1.2). Hence, the condition *(2)* in the definition is equivalent to the condition:

$\langle 2' \rangle$ $G(Z,\bar{W}) = 0$; $Z, W \in \mathfrak{X}^{1,0}M$.

Thus every complex Riemannian metric is comletely determined by its values on $\mathfrak{X}^{1,0}M$.

Let $z^1, \dots, z^n$ be a local holomorphic coordinate system in M. Unless otherwise stated, Greek indices α, β, γ, ... run from 1 to n, while Latin capitals A,B,C.... run through $1, \dots, n, \bar{1}, \dots \bar{n}$. In terms of local coordinates we set

$$G_{AB} = G(Z_A, Z_B).$$

Then, the defining conditions for a complex Riemannian metric become

$$(1.4) \qquad \langle 1 \rangle \; G_{\bar{\alpha}\bar{\beta}} = \overline{G_{\alpha\beta}} \qquad\qquad \langle 2 \rangle \; G_{\bar{\alpha}\beta} = G_{\alpha\bar{\beta}} = 0.$$

Given a complex Riemannian metric G on M , we define the tensor field $\tilde{G}$ on M by the equality

$$\tilde{G}(Z,W) = G(JZ,W) \; , \; Z,W \in \mathfrak{X}^{C}M \; .$$

It is clear that $\tilde{G}$ is a complex Riemannian metric on M. This metric is said to be associated with G. In local coordinates

$$(1.5) \qquad \tilde{G}_{\alpha\beta} = \iota G_{\alpha\beta} \; ; \qquad \tilde{G}_{\bar{\alpha}\bar{\beta}} = - \iota G_{\bar{\alpha}\bar{\beta}}.$$

We will refer to the pair (M,G), where M is a complex manifold and G is a complex Riemannian metric on M, as a complex Riemannian manifold.

Complex Riemannian manifolds with holomorphic metrics (i.e. $\bar{\partial} G_{\alpha\beta} = 0$ in local coordinates) have been studied in [5,6,8,9,10] Here we call such manifolds *holomorphic (complex analytic)* Riemannian manifolds.

Given a complex Riemannian manifold (M,G), the complex Riemannian metric G induces a real pseudo-Riemannian metric g on the manifold $(\mathbf{M},J)$. Thus, every n-dimensional complex Riemannian manifold (M,G) can be considered as a real 2n-dimensional manifold $(\mathbf{M},J,g)$ with a complex structure J and a metric g of signature (n,n) such that

$$g(JX,JY) = -g(X,Y), \quad X,Y \in \mathfrak{X}M .$$

We call the triple $(\mathbf{M},J,g)$ *the realization* of (M,G).

Denoting by $C(M)$ the bundle of complex linear frames and by $C(M)/O(n,\mathbb{C})$ the associated bundle with standard fibre $GL(n,\mathbb{C})/O(n,\mathbb{C})$, we have

PROPOSITION 1.3. *Let M be an n-dimensional complex manifold. Then there is a natural one-to-one correspondence between any two of the sets:*

(1) the complex Riemannian metrics on M;

(2) the reductions of the structure group $GL(n,\mathbb{C})$ *of* $C(M)$ *to* $O(n,\mathbb{C})$;

(3) the cross sections of the associated bundle $C(M)/O(n,\mathbb{C})$ *over M.*

2. CONNECTIONS ON COMPLEX RIEMANNIAN MANIFOLDS

Given any linear connection ∇ on a complex manifold M, we set

$$\nabla_{Z_A} Z_B = \Gamma^C_{AB} Z_C$$

in a local holomorphic coordinate system. Notice that covariant differentiation, which is defined for real vector fields in $\mathfrak{X}M$, is extended by complex linearity to act on complex vector fields in $\mathfrak{X}^{C}M$. Then $\Gamma^{\bar{C}}_{\bar{A}\bar{B}} = \overline{\Gamma^{C}_{AB}}$ with the convention $\bar{\bar{\alpha}} = \alpha$. A linear connection ∇ with local components Γ^C_{AB} is said to be almost complex if $\nabla J = 0$. A connection ∇ is almost complex iff $\Gamma^{\bar{\lambda}}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\bar{\beta}} = 0$.

Let (M,G) be a complex Riemannian manifold and ∇ be the Levi-Civita connection of G. Then the local components of ∇ are given by

$$\Gamma^C_{AB} = \frac{1}{2}G^{CS}\left(\partial_A G_{BS} + \partial_B G_{AS} - \partial_S G_{AB}\right)$$

where G^{AB} are components of the inverse matrix $(G_{AB})^{-1}$ and summation convention is supposed.

Similarly, if $\tilde{\Gamma}^C_{AB}$ are components of the Levi-Civita connection $\tilde{\nabla}$ of the associated metric $\tilde{G}$, we have

$$\tilde{\Gamma}^C_{AB} = \frac{1}{2}\tilde{G}^{CS}\left(\partial_A \tilde{G}_{BS} + \partial_B \tilde{G}_{AS} - \partial_S \tilde{G}_{AB}\right)$$

Taking into account (1.4) and (1.5), by direct computation we obtain

$$\tilde{\Gamma}^{\lambda}_{\alpha\beta} = \Gamma^{\lambda}_{\alpha\beta} = \frac{1}{2}G^{\lambda\sigma}\left(\partial_{\alpha} G_{\beta\sigma} + \partial_{\beta} G_{\alpha\sigma} - \partial_{\sigma} G_{\alpha\beta}\right)$$

(2.1) $$\tilde{\Gamma}^{\bar{\lambda}}_{\alpha\beta} = -\Gamma^{\bar{\lambda}}_{\alpha\beta} = \frac{1}{2}\overline{G^{\lambda\sigma}}\partial_{\bar{\sigma}} G_{\alpha\beta}$$

$$\tilde{\Gamma}^{\lambda}_{\bar{\alpha}\beta} = \Gamma^{\lambda}_{\bar{\alpha}\beta} = \frac{1}{2}G^{\lambda\sigma}\partial_{\bar{\alpha}} G_{\beta\sigma}$$

The fundamental tensor Φ on a complex Riemannian manifold is defined by

(2.2) $$\Phi(Z,W) = \tilde{\nabla}_Z W - \nabla_Z W \;;\; Z,W \in \mathfrak{X}^C M$$

From the definition of Φ it follows that

(2.3) $$\Phi(\bar{Z},\bar{W}) = \overline{\Phi(Z,W)} \;;\; Z,W \in \mathfrak{X}^C M .$$

Using (2.2), (2.1) and (2.3) we find the essential components (which may not be zero) of the fundamental tensor

$$(2.4)\qquad \Phi^{\bar{\lambda}}_{\alpha\beta} = G^{\overline{\lambda\sigma}}\partial_{\bar{\sigma}}G_{\alpha\beta}\ , \qquad \left(\Phi^{\lambda}_{\overline{\alpha\beta}} = \overline{\Phi^{\bar{\lambda}}_{\alpha\beta}}\right)$$

From (2.2) and (2.4) it follows that the fundamental tensor Φ of a complex Riemannian manifold (M,G) has the properties:

(1) $\Phi(Z,W) = \Phi(W,Z)$;

(2) $\Phi(JZ,W) = -J\Phi(Z,W)$

for all complex vector fields Z,W.

Let $(\mathbf{M},J,g)$ be the realization of a complex Riemannian manifold (M,G). The real fundamental tensor on (M,G) is defined by the equality (2.2) for real vector fields [1]. The property (2.3) of Φ implies that Φ is the complex linear extension of the real fundamental tensor on $(\mathbf{M},J,g)$.

Now we shall construct the linear connection on a complex Riemannian manifold which is our main tool for studying such manifolds.

We denote the fundamental tensor of type (0,3) by the same letter:

$$\Phi(X,Y,Z) = G(\Phi(X,Y),Z)\ ;\ X,Y,Z \in \mathfrak{X}^{C}M$$

In local coordinates we have

$$\Phi_{AB,C} = \Phi^{S}_{AB}G_{SC}$$

Note that the essential components of $\Phi_{AB,C}$ are

$$(2.5)\qquad \Phi_{\alpha\beta,\bar{\gamma}} = \partial_{\bar{\gamma}}G_{\alpha\beta}\left(\Phi_{\overline{\alpha\beta},\gamma} = \overline{\Phi_{\alpha\beta,\bar{\gamma}}}\right)$$

THEOREM 2.1. *On a complex Riemannian manifold* (M,G) *there exists a unique linear connection* D *with components* D^{A}_{BC} *such that*

(1) $D^{C}_{AB} = D^{C}_{BA}$, i.e. D is symmetric;

(2) $D^{\gamma}_{\alpha\bar{\beta}} = D^{\bar{\gamma}}_{\alpha\beta} = 0$, i.e. D is almost complex;

(3) $D_{\alpha}G_{\beta\gamma} = 0$.

Proof. Existence. We define

$$D^{C}_{AB} = \Gamma^{C}_{AB} + \frac{1}{2}\Phi^{C}_{AB} - \frac{1}{2}G^{CS}\left(\Phi_{SA,B} + \Phi_{SB,A}\right) \tag{2.6}$$

where Γ^{C}_{AB} are the components of the Levi-Civita connection ∇ of G. By direct computations we check that D satisfies the conditions of the theorem.

Uniqueness. Let D′ be another connection satisfying the conditions *(1)*, *(2)* and *(3)* of the theorem. Denote $S^{C}_{AB} = D^{C}_{AB} - D'^{C}_{AB}$. Then

$$S^{C}_{AB} = S^{C}_{BA}; \qquad S^{\bar{\gamma}}_{\alpha\beta} = S^{\gamma}_{\alpha\bar{\beta}} = 0; \qquad S^{\sigma}_{\alpha\beta}G_{\sigma\gamma} + S^{\sigma}_{\alpha\gamma}G_{\sigma\beta} = 0.$$

From these equalities it follows immediately that $S^{C}_{AB} = 0$, i.e. D′ = D. □

Further we call the linear connection D from Theorem 2.1 *the characteristic connection* of the complex Riemannian manifold (M,G).

Taking into account the defining equality (2.6) of the characteristic connection and the properties of the fundamental tensor, we obtain

COROLLARY 2.2. *On a complex Riemannian manifold* (M,G) *there* *exists a unique linear connection D such that*

(1) D is symmetric

(2) D is almost complex;

(3) $D_A G_{BC} = \Phi_{BC,A}$ *, i.e. the covariant derivative of the metric G is equal to the fundamental tensor* Φ.

The third condition of the statement and (2.5) imply that the essential components of the tensor $D_A G_{BC}$ are

$$D_{\bar\alpha} G_{\beta\gamma} = \Phi_{\beta\gamma,\bar\alpha} = \partial_{\bar\alpha} G_{\beta\gamma}\ , \qquad \left(D_\alpha G_{\overline{\beta\gamma}} = \overline{D_{\bar\alpha} G_{\beta\gamma}}\right)$$

On the realization of a complex Riemannian manifold we have

COROLLARY 2.3. *Let* (**M**,J,g) *be the realization of a complex Riemannian manifold* (*M*,G)*. Then the characteristic connection* D *on* (**M**,J,g) *is the unique connection satisfying the condition*

(1) D *is symmetric;*

(2) D *is almost complex;*

(3) $(D_X g)(Y,Z) = (D_{JX} g)(JY,Z)$ *for* $X,Y,Z \in \mathfrak{X}M$*, i.e. the covariant derivative* $(D_X g)(Y,Z)$ *of* g *is hybrid in* X,Y.

The defining equality (2.6) and (2.5) imply that the essential components of the characteristic connection are

$$(2.7) \qquad D^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} = \frac{1}{2} G^{\lambda\sigma}\left(\partial_\alpha G_{\beta\sigma} + \partial_\beta G_{\alpha\sigma} - \partial_\sigma G_{\alpha\beta}\right), \quad \left(D^{\bar\lambda}_{\overline{\alpha\beta}} = \overline{D^\lambda_{\alpha\beta}}\right)$$

This means that D is completely determined on $\mathfrak{X}^{1,0}M$.

Definition. The characteristic connection D is said to be *holomorphic* if $D_X Y$ is a holomorphic vector field for arbitrary holomorphic vector fields X, Y.

It follows immediately that D is holomorphic iff the components $D^{\lambda}_{\alpha\beta}$ of D are holomorphic functions.

3. THE FOUR CLASSES OF COMPLEX RIEMANNIAN MANIFOLDS

In this section we consider the classes of complex Riemannian manifolds with respect to the fundamental tensor and study conformal properties of these classes.

Let (M,G) be a complex Riemannian manifold with fundamental tensor Φ. In local holomorphic coordinates we set

$$(3.1) \qquad \theta_{\alpha} = \frac{1}{n}\Phi_{\overline{\beta\gamma},\alpha} G^{\overline{\beta\gamma}}; \qquad \theta_{\bar{\alpha}} = \overline{\theta_{\alpha}} = \frac{1}{n}\Phi_{\beta\gamma,\bar{\alpha}} G^{\beta\gamma}.$$

We define the *Lee form* θ and the *conjugate Lee form* $\bar{\theta}$ by the equalities

$$\theta = \theta_{\alpha} dz^{\alpha}; \qquad \bar{\theta} = \theta_{\bar{\alpha}} dz^{\bar{\alpha}} .$$

From (3.1) and (2.5) it follows immediately that in local holomorphic coordinates

$$\theta_{\alpha} = \frac{1}{n}\partial_{\alpha} G_{\overline{\beta\gamma}} G^{\overline{\beta\gamma}} = \frac{1}{n}\partial_{\alpha} \log \bar{G} ,$$

where $G = \det(G_{\alpha\beta})$. Hence,

$$(3.2) \qquad \theta = \frac{1}{n}\partial \log \bar{G}$$

As a consequence of this formula we have

PROPOSITION 3.1. *The Lee form of a complex Riemannian manifold is $\bar{\partial}$-closed.*

Almost complex manifolds (M,J,g) with metric g and almost complex structure J satisfying the condition $g(JX,JY) = - g(X,Y)$ for all $X,Y \in \mathfrak{X}M$ have been classified with respect to the fundamental tensor Φ in [1]. We shall apply this classification to describe the four classes W_o, W_1, W_2, W of complex Riemannian manifolds in terms of conditions for the fundamental tensor Φ :

$$W_o : \Phi = 0 ;$$

$$W_1 : \Phi_{\alpha\beta,\bar{\gamma}} = \theta_{\bar{\gamma}} G_{\alpha\beta} ;$$

$$W_2 : \theta = 0 ;$$

W : no conditions (W is the class of all complex Riemannian manifolds)

The following inclusions are valid :

$$W_o \subset W_2 \subset W ; \quad W_o \subset W_1 \subset W , \; (W_1 \cap W_2 = W_o)$$

As a direct consequence of (2.5), (2.1), Corollary 2.2 and (2.6) we obtain

PROPOSITION 3.2. *Let (M,G) be a complex Riemannian manifold. The following statements are equivalent:*

(1) The fundamental tensor Φ is zero;

(2) The components $G_{\alpha\beta}$ of the metric G are holomorphic functions;

(3) The Levi-Civita connection ∇ of G is almost complex, i.e. $\nabla J = 0$;

(4) The characteristic connection D is metric, i.e. $DG = 0$;

(5) The Levi-Civita connection ∇ coincides with the characteristic connection D.

Thus the class W_o is exactly the class of holomorphic Riemannian manifolds.

In order to describe more precisely the class W_1 we consider complex conformal transformations of complex Riemannian manifolds.

Let (M,G) be a complex Riemannian manifold. For arbitrary real C^{∞}- functions u, v on *M* we set $f = u + iv$ and consider the tensor field

$$(3.3) \qquad G' = \exp(u)\cos(v)\, G + \exp(u)\sin(v)\, \tilde{G},$$

which is a complex Riemannian metric on *M*. The equality (3.3) defines a conformal transformation of the metric G. In local holomorphic coordinates (3.3) is equivalent to

$$(3.4) \qquad G'_{\alpha\beta} = \exp(f)\, G_{\alpha\beta}$$

The metric G′ is said to be *conformal to the metric* G.

LEMMA 3.3. *Let* G, G′, *be complex Riemannian metrics, related as in* (3.4). *If* Φ, Φ' *are the corresponding fundamental tensors and* θ, θ' *– the corresponding Lee forms of* G, G′, *then*

$$\Phi'_{\alpha\beta,\bar{\gamma}} = \exp(f)\, \Phi_{\alpha\beta,\bar{\gamma}} + \partial_{\bar{\gamma}} f \exp(f)\, G_{\alpha\beta};$$

$$\theta'_{\bar{\gamma}} = \theta_{\bar{\gamma}} + \partial_{\bar{\gamma}} f$$

The proof of the lemma follows by straightforward computations.

From Lemma 3.3 it follows immediately

$$\Phi'^{\bar{\lambda}}_{\alpha\beta} - G'_{\alpha\beta}\, \theta'^{\bar{\lambda}} = \Phi^{\bar{\lambda}}_{\alpha\beta} - G_{\alpha\beta}\theta^{\bar{\lambda}}$$

Thus we obtained

PROPOSITION 3.4. *On a complex Riemannian manifold* (*M*,G) *with fundamental tensor* Φ *the tensor defined by*

$$(3.5) \qquad \Phi^{\bar{\lambda}}_{\alpha\beta} - G_{\alpha\beta}\, \theta^{\bar{\lambda}}$$

is a conformal invariant.

We call the tensor given by (3.5) the *conformal fundamental tensor* of (M,G).

Since the class W_1 is characterized by the condition $\Phi^{\bar\lambda}_{\alpha\beta} - G_{\alpha\beta}\,\theta^{\bar\lambda} = 0$. Proposition 3.4 implies

PROPOSITION 3.5. *The class* W_1 *is closed with respect to conformal transformations.*

The next theorem means that every complex Riemannian manifold can be considered locally (up to a conformal transformation) as a W_2-manifold.

THEOREM 3.6. *Every complex Riemannian manifold admits locally a conformal metric with zero Lee form.*

Proof. Let p be an arbitrary point in M and U be a simply connected holomorphic coordinate neighbourhood of p. Define in U the metric $G'_{\alpha\beta} = G^{-1/n}\,G_{\alpha\beta}$ where $G = \det(G_{\alpha\beta})$. Applying Lemma 3.3, (3.1) and (3.2) we find the Lee form θ' of G' is zero and this proves the assertion. □

We shall precise this result for the class W_1.

Definition. A complex Riemannian manifold (M,G) is said to be *a locally conformal holomorphic Riemannian manifold* if every point $p \in M$ has an open neighbourhood U such that the metric $\exp(f)\,G_{\alpha\beta}$ is holomorphic for some function $f = u + iv$ (u,v-real differentiable function on U).

THEOREM 3.7. *Every locally conformal holomorphic Riemannian* *manifold is in the class* W_1 *and vice versa.*

Proof. Let (M,G) be a locally conformal holomorphic Riemannian manifold and the metric $G'_{\alpha\beta} = \exp(f)\, G_{\alpha\beta}$ be holomorphic in a coordinate neighbourhood U of a point p in M. Applying Lemma 3.3 we find $\Phi_{\alpha\beta,\bar\gamma} = G_{\alpha\beta}\, \theta_{\bar\gamma}$, where $\theta_{\bar\gamma} := -\partial_{\bar\gamma} f$.

Hence, (M,G) is in the class W_1.

To prove the inverse, we shall use Theorem 3.6. In a simply connected neighbuorhood U of a point p in M we find metric $G'_{\alpha\beta} = \exp(f)\, G_{\alpha\beta}$ such that $\theta'_{\bar\gamma} = 0$. By the condition of the considered implication $\Phi_{\alpha\beta,\bar\gamma} - G_{\alpha\beta}\, \theta_{\bar\gamma} = 0$. Applying Proposition 3.4 we obtain

$0 = \Phi'_{\alpha\beta,\bar\gamma} - G'_{\alpha\beta}\theta'_{\bar\gamma} = \Phi'_{\alpha\beta,\bar\gamma}$ i.e. G' is holomorphic. □

4. HOLOMORPHIC SECTIONAL CURVATURES

Let (V,J,g) be a real 2n-dimensional vector space with a complex structure J and an inner product g satisfying the condition (1.1). As in above (V^C,G) denotes the complexification V^C of (V,J) with the complex linear extension G of g satisfying (1,2)

Let R be a complex quadrilinear mapping over V^C with the properties:

(4.1)

(1) $R(X,Y,Z,U) = - R(U,X,Z,U)$;

(2) $R(X,Y,Z,U)+ R(Y,Z,X,U)+R(Z,X,Y,U)= 0$;

(3) $R(X,Y,Z,U) = - R(X,Y,U,Z)$;

(4) $R(X,Y,Z,U) = - R(X,Y,JZ,JU)$;

(5) $R(\bar{X},\bar{Y},\bar{Z},\bar{U}) = \overline{R(X,Y,Z,U)}$

for all X,Y,Z,U in V^C.

On the basis of *(1)* , *(2)* and *(3)* the equality *(4)* is equivalent to the following

(4') $R(JX,Y,Z,U)= R(X,JY,Z,U)= R(X,Y,JZ,U) = R(X,Y,Z,JU)$

for all X,Y,Z,U in V^C.

A tensor R of type (0,4) satisfying (4.1) is said to be an *H-tensor*.

It follows immediately that every H-tensor is completely determined by its restriction onto the eigenspace $V^{1,0}$.

let E^C be a nondegenerate complex 2-plane in V^C , i.e. the restriction of G onto E^C has the maximal rank. If { X,Y } is an arbitrary basis for E^C, the complex sectional curvature of E^C with respect to R is defined in the usual manner:

$$K(E^C) = \frac{R(X,Y,Y,X)}{\pi_1(X,Y,Y,X)},$$

where π_1 is the tensor of type (0,4) defined by

$$\pi_1(X,Y,Z,U) = G(Z,Y)\,G(X,U) - G(Z,X)\,G(Y,U);\ X,Y,Z,U \in V^C.$$

Any complex 2-plane in $V^{1,0}$ is said to be a *holomorphic 2-plane*. A tensor R satisfying properties (4.1) is said to be of *constant holomorphic sectional curvature* c if $K(E^C) = c$ for all nondegenerate holomorphic 2-planes E^C in $V^{1,0}$.

Formally identical to the usual pseudo-Riemannian geometry (e.g. [2]) we have

PROPOSITION 4.1. *Let* R *be a complex quadrilinear mapping over*

$V^{1,0}$ *with the properties (1), (2), (3) of* (4.1) *and* $n \geq 3$. *Then* R *is of constant holomorphic sectional curvature* c *iff*

$$R(X,Y,Z,U) = c\ \pi_1(X,Y,Z,U)\ ;\ X,Y,Z,Y \in V^{1,0}.$$

We define the following tensors of type (0,4) over (V^C,G)

$$\pi_2(X,Y,Z,U) = G(JZ,Y)\ G(X,JU) - G(JZ,X)\ G(Y,JU);$$

$$\pi_3(X,Y,Z,U) = -\ G(Z,Y)\ G(X,JU) + G(Z,X)\ G(Y,JU) - G(X,U)\ G(JZ,Y) + G(Y,U)\ G(JZ,X)$$

for all X,Y,Z,U in V^C.

It is easy to check that $\pi_1 - \pi_2$ and π_3 are H-tensors.

On $V^{1,0}$ we have

$$\pi_1 - \pi_2 = 2\pi_1 \quad , \quad \pi_3 = 2i\pi_1$$

Now Proposition 4.1 implies immediately

PROPOSITION 4.2. *Let* R *be an H-tensor over* (V^C,G) *and* $n \geq 3$. *Then* R *is of constant holomorphic sectional curvature* $c = a + ib$ *iff*

$$R = \frac{1}{2}\left[a(\pi_1 - \pi_2) - b\pi_3 \right]$$

The corresponding notion of a *holomorphic sectional curvature* in the real space (V,J,g) is a *totally real sectional curvature.*

A 2-plane E in (V,J,g) is said to be totally real if E is nondegenerate and E is orthogonal to JE with respect to g.

Any totally real 2-plane E with a basis {x,y} has two sectional curvatures with respect to the real restriction of an H-tensor R onto (V,J,g):

$$K(E) = \frac{R(x,y,y,x)}{\pi_1(x,y,y,x)} \quad ; \quad \tilde{K}(E) = \frac{R(x,y,y,Jx)}{\pi_1(x,y,y,x)}$$

The real restriction of an H-tensor R onto $(\mathbf{V},J,g)$ is said to be of *constant totally real sectional curvatures* υ and $\tilde{\upsilon}$ if

$$K(E) = \upsilon \, ; \tilde{K}(E) = \tilde{\upsilon}$$

for all totally real 2-planes E in $(\mathbf{V},J,g)$.

Taking into account Theorem 4.2 we obtain

PROPOSITION 4.3. *Let* R *be an H-tensor over* $(\mathbf{V}^{\mathbf{C}},G)$ *and* $n \geq 3$ *Then* R *is of constant holomorphic sectional curvature* c *iff the real restriction of* R *onto* $(\mathbf{M},J,g)$ *is of constant totally real sectional curvatures*

$$\upsilon = \frac{1}{2}\,\mathrm{Re}(c); \qquad \tilde{\upsilon} = -\frac{1}{2}\,\mathrm{Im}(c).$$

Now, let (M,G) be an n-dimensional holomorphic Riemannian manifold. The curvature tensor R of the Levi-Civita connection ∇ is defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, $X,Y,Z \in \mathfrak{X}^C M$ .The corresponding tensor of type (0,4) is given by

$$R(X,Y,Z,U) = G(R(X,Y)Z,U) \ , \ X,Y,Z,U \in \mathfrak{X}^C M$$

The properties of ∇ imply in a straightforward way

PROPOSITION 4.4. *The curvature tensor* R *of every holomorphic Riemannian manifold* (M,G) *is an H-tensor.*

This proposition implies immediately that the essential components of R in local holomorphic coordinates are $R_{\alpha\beta\gamma\delta}$

$\left(R_{\overline{\alpha\beta\gamma\delta}} = \overline{R_{\alpha\beta\gamma\delta}}\right)$.

If the holomorphic sectional curvatures $K(E^C,p)$ = const. for all nondegenerate holomorphic 2-planes E^C in $V^{1,0}$ ($V^C = T_p^C M$) and for all points p in M, then (M,G) is said to be of *constant holomorphic sectional curvature.*

The following assertion is the holomorphic analogue of Shur's theorem.

THEOREM 4.5. *Let (M,G) be a connected holomorphic Riemannian manifold of complex dimension $n \geq 3$. If the holomorphic sectional curvatures depend only on the point, them (M,G) is of constant holomorphic sectional curvature.*

Because of completeness we shall sketch the proof.

By Proposition 4.1 it follows that

$$R_{\alpha\beta\gamma\delta} = k(p)\left(G_{\beta\gamma}G_{\alpha\delta} - G_{\alpha\gamma}G_{\beta\delta}\right)$$

The second Bianchi identity after two contractions implies

$$\partial_{\bar{\alpha}} k = 0 \quad ; \quad (n-2)\,\partial_{\alpha} k = 0$$

Since $n \geq 3$, then k = const. □

Using Proposition 4.2 we obtain

COROLLARY 4.6. *If a holomorphic Riemannian manifold is of constant holomorphic sectional curvature $c = a + ib$, then*

$$R = \frac{1}{2}\left\{ a(\pi_1 - \pi_2) - b\pi_3 \right\}$$

5. CLASSIFICATION OF COMPLEX ANALYTIC RIEMANNIAN MANIFOLDS WITH CONSTANT HOLOMORPHIC SECTIONAL CURVATURE.

In this section we describe briefly the classification theory for the class W_o of complex analytic (holomorphic) Riemannian manifolds with constant holomorphic sectional curvatures, discussed in the previous section.

EXAMPLES.

1. *Complex Euclidian space.* The standard holomorphic metric on $\mathbb{C}^n$ is given by

$$G = \delta_{\alpha\beta} dz^{\alpha} dz^{\beta}.$$

Then $(\mathbb{C}^n, G)$ is a flat complex analytic Riemannian manifold.

Let (M', G) be an $(n+1)$-dimensional complex analytic Riemannian manifold. An n-dimensional complex submanifold M of M' is said to be a *holomorphic hypersurface* of M' if the restriction of G onto M has the maximal rank.

We denote the restriction of G onto M by the same letter. Then (M, G) is a complex analytic (holomorphic) Riemannian manifold.

If η is a local unit holomorphic normal to (M, G) (determined up to a sign), then $\xi = \frac{1}{\sqrt{2}}\left(\eta + \bar{\eta}\right)$ is the real normal to (M, G), corresponding to η, with the properties

$$G(\xi, \xi) = 1, \quad G(\xi, J\xi) = 0 \tag{5.1}$$

With respect to the real normal ξ the complex Gauss and Weingarten formulas are

$$\begin{aligned} \nabla'_Z W &= \nabla_Z W + h_\xi(Z,W)\,\xi - h_\xi(JZ,W)J\xi \quad, \\ \nabla'_Z \xi &= -A_\xi Z; \qquad G(A_\xi Z, W) = h_\xi(Z,W) \quad . \end{aligned} \tag{5.2}$$

for all $Z, W \in \mathfrak{X}^C M$.

The second fundamental form h_ξ corresponding to the real normal ξ is a symmetric tensor with the properties

$$h_\xi(JZ, JW) = -h_\xi(Z, W) \ ; \ h_\xi(\overline{Z}, \overline{W}) = \overline{h_\xi(Z, W)}; \ Z, W \in \mathfrak{X}^C M$$

Hence, the second fundamental form h_ξ is completely determined by its values on holomorphic vectors.

The complex Gauss equation has the form:

$$\begin{aligned} R'(X,Y,Z,U) &= R(X,Y,Z,U) - h_\xi(Z,Y)h_\xi(Y,U) + h_\xi(Z,X)h_\xi(Y,U) \\ &\quad + h_\xi(JZ,Y)h_\xi(X,JY) - h_\xi(JZ,X)h_\xi(Y,JU), \end{aligned} \tag{5.3}$$

where $X, Y, Z, U \in \mathfrak{X}^C M$ and R' , (R) are the curvature tensors on M' (M), respectively.

On the holomorphic distribution formulas (5.2) and (5.3) become

$$\begin{aligned} \nabla'_Z W &= \nabla_Z W + h_\eta(Z,W)\eta \ ; \qquad Z, W \in \mathfrak{X}^{1,0} M \\ \nabla'_Z \eta &= -A_\eta Z \qquad ; \qquad Z \in \mathfrak{X}^{1,0} M \end{aligned} \tag{5.4}$$

$$R'(X,Y,Z,U) = R(X,Y,Z,U) - h_\eta(Z,Y)h_\eta(X,U) + h_\eta(Z,X)h_\eta(Y,U) \tag{5.5}$$

where $X, Y, Z, U \in \mathfrak{X}^{1,0} M$; $A_\eta Z = \sqrt{2}\, A_\xi Z$; $h_\eta(Z,W) = \sqrt{2}\, h_\xi(Z,W)$ for all Z, W in $\mathfrak{X}^{1,0} M$.

2. *Complex sphere.* Let $(\mathbb{C}^{n+1}, G)$ be a complex Euclidian space and $CS^n(r) = \left\{ (z^1 \dots z^{n+1}) \in \mathbb{C}^{n+1} \ \middle| \ \sum_{\alpha=1}^{n+1} (z^\alpha)^2 = r^2, \ r \in \mathbb{C}\setminus\{0\} \right\}$ be the usual complex n-sphere $(n \geq 2)$. The restriction of G to CS^n

has the maximal rank. Hence, $\left\{S^n(r),G\right\}$ is holomorphic hypersurface of $(\mathbb{C}^{n+1},G)$.

Proposition 5.1. *Every complex sphere* $CS^n(r)$ *is a connected, simply connected and complete complex analytic Riemannian manifold of constant holomorphic sectional curvature* $K = \dfrac{1}{r^2}$.

Proof. Identifying any point $z = (z^1 \ldots z^{n+1}) \in \mathbb{C}^{n+1}$ with the holomorphic position vector Z, it follows in a standard way that $\eta = \dfrac{1}{r} Z$, $A_\eta = \dfrac{1}{r}$ id. Then, (5.5) implies that

$$(5.6) \qquad R(X,Y,Z,U) = \frac{1}{r^2}\left\{ G(Z,Y)G(X,U) - G(Z,X)G(Y,U) \right\}$$

for holomorphic tangent vectors X,Y,Z,U. Hence, $\left\{S^n(r),G\right\}$ is of constant holomorphic sectional curvature $K = 1/r^2$.

Since $CS^n(1)$ is diffeomorphic to the tangent bundle TS^n of the unit sphere $S^n(1)$ in $\mathbb{R}^{n+1}$ we have that $CS^n(1)$ is connected and simply connected. Any complex sphere $CS^n(r)$ is diffeomerphic to $CS^n(1)$. Thus, $CS^n(r)$ is connected and simply connected.

To prove the completeness of the underlying real connection of ∇ we shall find the differential equation, describing the real geodesics on $CS^n(r)$.

Let c: $Z = Z(t)$, where Z is the position vector in $\mathbb{R}^{2n+2}$, be a real geodesic on $CS^n(r)$ of ∇ with an affine parameter, i.e.

$\nabla_{\dot{Z}} \dot{Z} = 0$, $\dot{Z}$ being the real tangent vector field on c. Using

that for holomorphic vectors $A_\xi = \dfrac{1}{\sqrt{2}} \dfrac{1}{r^2}$ id, we obtain

$$A_\xi X = \frac{a}{\sqrt{2}} X + \frac{b}{\sqrt{2}} JX; \quad \frac{1}{r} = a + ib$$

for real vectors X.

From (5.2) it follows that

$$\nabla'_{\dot Z}\dot Z = \frac{1}{\sqrt{2}}\left\{(a^2 - b^2)\, g(\dot c,\dot c) + 2ab\, \tilde g(\dot c,\dot c)\right\} Z$$
$$+ \frac{1}{\sqrt{2}}\left\{-(a^2 - b^2)\, \tilde g(\dot c,\dot c) + 2ab\, g(\dot c,\dot c)\right\} JZ$$

The conditions $\nabla_{\dot Z}\dot Z = 0$ and $\nabla J = \nabla G = 0$ imply that $g(\dot Z\ \dot Z) = k$ $=$ const. , $\tilde g\ (Z,Z) = \tilde k =$ const. on c. Then the real geodesics of ∇ on $CS^n(r)$ are described by the following differential equation :

(5.7) $\qquad \ddot Z = c\ Z + d\ JZ$

where $c = \frac{1}{\sqrt{2}}\left\{(a^2 - b^2)k + 2ab\ \tilde k\right\}$,$d = \frac{1}{\sqrt{2}}\left\{-(a^2 - b^2)\tilde k + 2ab\ k\right\}$.

The solution of the linear differential equation (5.7) with constant coeficients c and d are defined for all values of the affine parameter. Hence, the real Levi-Civita connection on $CS^n(r)$ is complete. □

From this theorem and formula (5.6) we obtain

Corollary 5.2. *Every complex sphere* $\left[CS^n(r), G\right]$ $(n \geq 2)$ *is a symmetric complex analytic Riemannian manifold.*

To prove the classification theorem we need the following basic results in geometry of linear connections.

Theorem I [e.g. 4 , p. 261]. *Let* **M** *and* **M'** *be differentiable* *manifolds with linear connections. Let* T, R *and* ∇ *(resp.* T′, R′ *and* ∇*) be the torsion , the curvature and the covariant differentiation on* **M** *(resp. on* **M'***). Assume* $\nabla T = \nabla R = 0$, $\nabla' T = \nabla' R = 0$ *. If* F *is a linear isomorphism of* T_xM *onto* T_yM' *and maps the tensors* T_x *and* R_x *at* x *into the tensors* T'_y *and* R'_y *at* y *respectively, then there is an affine isomorphism f of a neighbourhood U of x onto a neighbourhood V of y such that* f(x) = y *and that the differential of f at* **x** *coincides with F.*

Theorem II. [e.g. 4, p. 265]. *In Theorem I if* **M** *and* **M'** *are, moreover, connected, simply connected and complete then there exists a unique affine isomorphism f of* **M** *onto* **M'** *such that* f(x) = y *and that the differential of f at x coincides with F.*

Using Theorem I we shall prove the following

Theorem 5.3. *Any two complex analytic Riemannian manifolds* (*M*,G) *and* (*M'*,G′) *of constant holomorphic sectional curvature* **k** *are locally holomorphical isometric to each other.*

Proof. Let (**M**,J,g) and (**M'**,J′,g′) be the realizations of (*M*,G) and (*M'*,G′), respectively and R (resp. R′) be the curvature tensor of the Levi-Civita connection ∇ (resp.∇'). From Corollary 4.6 it follows that $\nabla R = \nabla' R' = 0$.

If $F : T_xM \longrightarrow T_yM'$ is a linear isomorphism preserving the complex structure J_x and the metric g_x, then Corollary 4.6 implies that F maps R_x into R'_y Taking into account that ∇ and $\nabla'$are torsion-free, from Theorem I we obtain that there is a local affine isomorphism f at a neighbourhood U of x onto a neighbourhood V of y such that f(x) = y and the differential of f at x coincides with F. If x´ is a point in U and c is a curve in U from x to x´, we set f(c) = c´ and y´ = f(x´). Since the parallel displacement along c´ corresponds to that along c under f and since the complex structures and the metrics on M and M' are parallel, the affine isomorphism f is a holomorphic isometry. □

Combining Theorem 5.3 and Proposition 5.1, we obtain

Theorem 5.4. (Classification theorem. Local version). *Any complex analytic Riemannian manifold of constant holomorphic sectional curvature* k *is locally holomorphical isometric to a complex sphere if* $k \neq 0$ *or to* $\mathbb{C}^n$ *if* $k = 0$.

Using Theorem II, by similar arguments as in the proof of Theorem 5.3 , we obtain a global version of Theorem 5.3.

Theorem 5.5. *Any two connected, simply connected , complete complex analytic Riemannian manifolds of constant holomorphic sectional curvature* k *are holomorphically isometric to each other*

From Theorem 5.5 and Proposition 5.1 we obtain

Theorem 5.6.(Classification theorem. Global version).. *Any connected, simply connected and complete complex analytic Riemannian manifold of constant holomorphic sectional curvature* k *is holomorphically isometric to a complex sphere if* $k \neq 0$ *or to* $\mathbb{C}^n$ *if* $k = 0$..

6. CHARACTERISTIC CURVATURE TENSORS.

In this section we establish the properties of the characteristic curvature tensor (i.e. the curvature tensor of the characteristic connection) on a complex Riemannian manifold. We introduce the space of all tensors having the same algebraic properties as the characteristic curvature tensor and give an invariant decomposition of this space.

Let (M,G) be a complex Riemannian manifold with characteristic connection D and characteristic curvature tensor K. From the definition of D we have :

$$(6.1)\quad \begin{aligned} &1)\ K(X,Y,Z,U) = - K(Y,X,Z,U)\ ;\\ &2)\ K(X,Y)Z + K(Y,Z)X + K(Z,X)Y = 0;\\ &3)\ K(X,Y,Z,U) = - K(X,Y,JZ,JU);\\ &4)\ K(\overline{X},\overline{Y},\overline{Z},\overline{U}) = \overline{K(X,Y,Z,U)} \end{aligned}$$

for all $X,Y,Z,U \in \mathfrak{X}^{C}M$

Using the properties (2.7) of D we obtain

Proposition 6.1. *Let* K_{ABCD} *be the components of the characteristic curvature tensor with respect to a holomorphic coordinate system. Then*

$$1)\ K_{\alpha\beta\gamma\delta} = G_{\sigma\delta}\left\{ \partial_{\alpha}\Gamma^{\sigma}_{\beta\gamma} - \partial_{\beta}\Gamma^{\sigma}_{\alpha\gamma} + \Gamma^{\lambda}_{\beta\gamma}\Gamma^{\sigma}_{\alpha\lambda} - \Gamma^{\lambda}_{\alpha\gamma}\Gamma^{\sigma}_{\beta\lambda} \right\}$$

$$2)\ K_{\overline{\alpha}\beta\gamma\delta} = \partial_{\overline{\alpha}}\,\Gamma^{\sigma}_{\beta\gamma}\, G_{\sigma\delta},$$

$$3)\ K_{\overline{\alpha\beta}\gamma\delta} = 0$$

Proposition 6.2. *The characteristic curvature tensor K of every complex Riemannian manifold satisfies the following identities:*

$$(6.2)\quad K(X,Y,Z,U) - K(JX,JY,Z,U) + K(JX,Y,JZ,U) + K(X,JY,JZ,U) = 0$$

$$(6.3)\quad K(X,Y,Z,U) - K(JX,JY,Z,U) + K(X,Y,U,Z) - K(JX,JY,U,Z) = 0$$

for all $X,Y,Z,U \in \mathfrak{X}^{C}M$.

Proof. The third equality of Proposition 6.1 is equivalent to $K(\overline{X},\overline{Y},Z,U)=0$ for all holomorphic vectors X,Y,Z,U. This identity implies (6.2). Using $D_\alpha G_{\beta\gamma} = 0$ we have $K_{\alpha\beta\gamma\delta} + K_{\alpha\beta\delta\gamma} = 0$ which is equivalent to $K(X,Y,Z,U) + K(X,Y,U,Z) = 0$ for all holomorphic vectors X,Y,Z,U. The last identity and (6.2) imply (6.3). □

Now, let (V^C,G) be the complexification of a real 2n-dimensional vector space (V,J,g) with a complex structure J and an inner product g satisfying the condition (1.1).

We consider the vector space $\mathfrak{R}$ of complex tensors of type (0,4) over V^C satisfying the conditions (6.1), (6.2) , (6.3). This space has a natural inner product induced from that on V^C.

$$\langle K,\tilde{K}\rangle = \sum_{A,B,C,D} K(E_A,E_B,E_C,E_D)\tilde{K}(E_A,E_B,E_C,E_D),$$

where $K,K' \in \mathfrak{R}$ and $\{ E_A \}$ is an arbitrary orthogonal basis of V^C.

Let $A \in O(n,\mathbb{C})$ and $Z = Z^+ + Z^-$ ($Z^+ \in V^{1,0}$, $Z^- \in V^{0,1}$) be an arbitrary vector in V^C. Then the natural representation λ of $O(n,\mathbb{C})$ in V^C is determined by

$$\lambda(A)\, Z = A\, Z^+ + \overline{A}\, Z^-$$

The restriction of λ on the real space V is exactly the real representation r of $O(n,\mathbb{C})$: $r(O(n,\mathbb{C})) = r(GL(n,\mathbb{C})) \cap O(n,\mathbb{C})$.

There is a natural induced representation $\tilde{\lambda}$ of $O(n,\mathbb{C})$ in $\mathfrak{K}$ given by

$$\left(\tilde{\lambda}(A)(K)\right)(X,Y,Z,U) = K\left[\lambda(A^{-1})X,\lambda(A^{-1})Y,\lambda(A^{-1}Z\ ,\lambda(A^{-1})U\right]$$

for all X, Y, Z, $U \in V^C$, $K \in \mathfrak{K}$ and $A \in O(n,\mathbb{C})$.

We have

$$\langle\tilde{\lambda}(A)K',\tilde{\lambda}(A)K''\rangle=\langle K',K''\rangle,\ A \in O(n,\mathbb{C}),K',K''\in \mathfrak{K}$$

We define the following operator $\mathfrak{I} : \mathfrak{K} \longrightarrow \mathfrak{K}$

$$\mathfrak{I}(K)\ (X,Y,Z,U) = -\ K(JX,JY,Z,U),\ X,Y,Z,U \in V^C$$

It is easy to check

Lemma 6.3. *The operator* $\mathfrak{I}$ *is an involutive isometry.*

From this lemma we obtain two orthogonal invariant classes:

$$\mathfrak{K}_1 = \left\{ K \in \mathfrak{K}\ |\ \mathfrak{I}(K) = K \right\} =$$

$$\left\{ K \in \mathfrak{K}\ |\ K(X,Y,Z,U)= -K(JX,JY,Z,U)\right\};$$

$$\mathfrak{K}_2 = \left\{ K \in \mathfrak{K}\ |\ \mathfrak{I}(K) = -\ K \right\} =$$

$$\left\{ K \in \mathfrak{K}\ |\ K(X,Y,Z,U) = K(JX,JY,Z,U) \right\}.$$

Using (6.2) and (6.3) we obtain the following characterization of the class $\mathfrak{K}_1$:

Lemma 6.4. *A tensor* $K \in \mathfrak{K}$ *is in the class* $\mathfrak{K}_1$ *iff* K *is an* H-*tensor.*

Let $\left\{ Z_\alpha, Z_{\bar\alpha} \right\}$ be a natural basis of V^C, $Z_\alpha \in V^{1,0}$. From Lemma 6.4 it follows that every tensor $K \in \mathfrak{K}_1$ has the only essential components $K_{\alpha\beta\gamma\delta}$ $(K_{\overline{\alpha\beta\gamma\delta}})$ with respect to $\left\{ Z_\alpha, Z_{\bar\alpha} \right\}$ and

(6.4) $$K_{\alpha\beta\gamma\delta} = - K_{\beta\alpha\gamma\delta} = - K_{\alpha\beta\delta\gamma} = - K_{\beta\gamma\alpha\delta} - K_{\gamma\alpha\beta\delta}$$

Conversely, the components $K_{\alpha\beta\gamma\delta}$ satisfying (6.4) determine uniquely the tensor $K \in \mathfrak{K}_1$.

From (6.1) it follows that every tensor $K \in \mathfrak{K}_1^{\perp}$ has the only essential components $K_{\bar\alpha\beta\gamma\delta}$, $\left\{ K_{\alpha\overline{\beta\gamma\delta}} \right\}$ with respect to $\left\{ Z_\alpha, Z_{\bar\alpha} \right\}$ and

(6.5) $$K_{\bar\alpha\beta\gamma\delta} = - K_{\beta\bar\alpha\gamma\delta} = K_{\bar\alpha\gamma\beta\delta}$$

Conversely, the components $K_{\bar\alpha\beta\gamma\delta}$, satisfying (6.5) determine uniquely the tensor $K \in \mathfrak{K}_1^{\perp}$.

Further we define the classes

$$\mathfrak{K}_2 = \left\{ K \in \mathfrak{K}_1^{\perp} \mid K_{\bar\alpha\beta\gamma\delta} = - K_{\bar\alpha\beta\delta\gamma} \right\}$$

$$\mathfrak{K}_3 = \left\{ K \in \mathfrak{K}_1^{\perp} \mid K_{\bar\alpha\lambda\mu\nu} + K_{\bar\alpha\mu\nu\lambda} + K_{\bar\alpha\nu\lambda\mu} = 0 \right\}$$

where the components $K_{\bar\alpha\beta\gamma\delta}$ of a tensor K are taken with respect to a natural basis $\left\{ Z_\alpha, Z_{\bar\alpha} \right\}$ of V^C.

Proposition 6.5. (Partial decomposition). *The spaces* $\mathfrak{K}_1, \mathfrak{K}_2$ *and* $\mathfrak{K}_3$ *are* mutually *orthogonal and invariant under the action of* $O(n,\mathbb{C})$ *and*

$$\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2 \oplus \mathfrak{K}_3$$

Proof. We define the operator $\mathfrak{I}_1 : \mathfrak{K}_1^{\perp} \longrightarrow \mathfrak{K}_1^{\perp}$

$$\mathfrak{I}_1(K)\ (X,Y,Z,U) = \frac{1}{3}\Big\{ - K(X,Y,Z,U) + 2\,K(X,Y,U,Z) +$$

$$K(X,Z,U,Y) - K(Y,Z,U,X) - K(X,JZ,JU,Y) + K(Y,JZ,JU,X) \Big\},$$

$K \in \mathfrak{R}_1^{\perp}$, $X,Y,Z,U \in V^C$.

By a direct verification it follows that $\mathfrak{I}_1$ is an involutive isometry and

$$\mathfrak{R}_2 = \Big\{K \in \mathfrak{R}_1^{\perp} \mid \mathfrak{I}_1(K) = K\Big\}, \quad \mathfrak{R}_3 = \Big\{K \in \mathfrak{R}_1^{\perp} \mid \mathfrak{I}_1(K) = -K\Big\}.$$

which proves the statement. □

There are three different Ricci-type contractions of an arbitrary tensor $K \in \mathfrak{R}$:

$$r_{\alpha\beta} = G^{\lambda\mu} K_{\lambda\alpha\beta\mu}\ ,\ s_{\overline{\alpha\beta}} = G^{\lambda\mu} K_{\overline{\alpha\beta}\lambda\mu}\ ,\ q_{\overline{\alpha\beta}} = G^{\lambda\mu} K_{\overline{\alpha}\lambda\mu\overline{\beta}}$$

There is one scalar curvature of $K \in \mathfrak{R}$: $\tau = G^{\lambda\mu} r_{\lambda\mu}$

For the basic classes $\mathfrak{R}_1$ $\mathfrak{R}_2$ and $\mathfrak{R}_3$ we have more precisely:

$$\mathfrak{R}_1\ :\ s_{\overline{\alpha\beta}} = q_{\overline{\alpha\beta}} = 0;$$

$$\mathfrak{R}_2\ :\ r_{\alpha\beta} = 0\ ,\ s_{\overline{\alpha\beta}} = q_{\overline{\alpha\beta}}$$

$$\mathfrak{R}_3\ :\ r_{\alpha\beta} = 0\ ,\ q_{\overline{\alpha\beta}} = -\,2s_{\overline{\alpha\beta}}$$

Following the well known scheme of [7,11] we have

$$\mathfrak{R}_1 = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{13}\ ,$$

where

$$\mathfrak{R}_{11} = \Big\{ K \in \mathfrak{R}_1 \mid K_{\alpha\beta\gamma\delta} = \frac{1}{n(n-1)} \Big(G_{\beta\gamma} G_{\alpha\delta} - G_{\alpha\gamma} G_{\beta\delta} \Big)\Big\}\ ,$$

$$\mathfrak{R}_{12} = \Big\{ K \in \mathfrak{R}_1 \mid K_{\alpha\beta\gamma\delta} = \frac{1}{n-2} \Big\{ r_{\beta\gamma} G_{\alpha\delta} - r_{\alpha\gamma} G_{\beta\delta} +$$

$$G_{\beta\gamma} r_{\alpha\delta} - G_{\alpha\gamma} r_{\beta\delta} \Big\}\Big\},$$

$$\mathfrak{R}_{13} = \Big\{ K \in \mathfrak{R}_1 \mid r_{\alpha\beta} = 0 \Big\}.$$

are mutually orthogonal subspaces of $\mathfrak{K}_1$ invariant under the action of $O(n,\mathbb{C})$.

Further we define the subspaces of $\mathfrak{K}_2$ and $\mathfrak{K}_3$:

$$\mathfrak{K}_{21} = \left\{K \in \mathfrak{K}_2 \mid K_{\bar{\alpha}\beta\gamma\delta} = \frac{1}{n+2}\left(s_{\bar{\alpha}\beta}\, G_{\gamma\delta} + s_{\bar{\alpha}\gamma}\, G_{\beta\delta} + s_{\bar{\alpha}\delta}\, G_{\beta\gamma}\right)\right\},$$

$$\mathfrak{K}_{22} = \left\{ K \in \mathfrak{K}_2 \mid s_{\bar{\alpha}\beta} = 0 \right\},$$

$$\mathfrak{K}_{31} = \left\{K \in \mathfrak{K}_3 \mid K_{\bar{\alpha}\beta\gamma\delta} = \frac{1}{n-1}\left(s_{\bar{\alpha}\beta} G_{\gamma\delta} + s_{\bar{\alpha}\gamma}\, G_{\beta\delta} - 2 s_{\bar{\alpha}\delta} G_{\beta\gamma}\right)\right\},$$

$$\mathfrak{K}_{32} = \left\{ K \in \mathfrak{K}_3 \mid s_{\bar{\alpha}\beta} = 0 \right\}.$$

Now we obtain at once

Theorem 6.6 (Complete decomposition). *The seven subspaces* $\mathfrak{K}_{11},\ldots,\mathfrak{K}_{32}$ *are mutually orthogonal and invariant under the action of* $O(n,\mathbb{C})$ *so that*

$$\mathfrak{K}_1 = \mathfrak{K}_{11} \oplus \mathfrak{K}_{12} \oplus \mathfrak{K}_{13}, \quad \mathfrak{K}_2 = \mathfrak{K}_{21} \oplus \mathfrak{K}_{22}, \quad \mathfrak{K}_3 = \mathfrak{K}_{31} \oplus \mathfrak{K}_{32};$$

$$\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_2 \oplus \mathfrak{K}_3$$

Let $K \in \mathfrak{K}$. For an arbitrary nondegenerate holomorphic 2-plane $E^c = \operatorname{span}\{X,Y\}$ in $V^{1,0}$ we define a *holomorphic sectional curvature* with respect to K by

$$H(E^c) = \frac{K(X,Y,Y,X)}{G(X,X)\,G(Y,Y) - G^2(X,Y)}$$

Using Proposition 4.2 and Lemma 6.4 we obtain

Proposition 6.7. *A tensor* $K \in \mathfrak{K}$ *is of constant holomorphic sectional curvature* $c = a + ib$ *iff*

$$K(X,Y,Z,U) + K(JX,JY,Z,U) = a\left(\pi_1(X,Y,Z,U) - \pi_2(X,Y,Z,U)\right)$$

$$- b\, \pi_3(X,Y,Z,U), \qquad X,Y,Z,U \in V^C.$$

For a tensor $K \in \mathfrak{R}$, the projection W of K on the space $\mathfrak{R}_{13}$(the Weyl component) is determined by

$$W_{\alpha\beta\gamma\delta} = K_{\alpha\beta\gamma\delta} - \frac{1}{n-2}\left(r_{\beta\gamma}G_{\alpha\delta} - r_{\alpha\gamma}G_{\beta\delta} + r_{\alpha\delta}G_{\beta\gamma} - r_{\alpha\gamma}G_{\beta\delta}\right) + \tag{6.6}$$
$$+ \frac{1}{2(n-1)(n-2)}\left(G_{\beta\gamma}\, G_{\beta\gamma} - G_{\beta\gamma}G_{\beta\gamma} \right)$$

The tensor W is the Weyl tensor associated with the tensor $K \in \mathfrak{R}$.

For a tensor $K \in \mathfrak{R}_2$ the traceless component B is determined by the projection of K on the space $\mathfrak{R}_{22} \oplus \mathfrak{R}_{32}$ and

$$B_{\overline{\alpha\beta\gamma\delta}} = K_{\overline{\alpha\beta\gamma\delta}} - \frac{1}{(n-1)(n+2)} \left\{ \left(n\, s_{\overline{\alpha\beta}} - q_{\overline{\alpha\beta}}\right) G_{\gamma\delta} + \right.$$
$$\left. \left(n\, s_{\overline{\alpha\gamma}} - q_{\overline{\alpha\gamma}}\right) G_{\beta\delta} + \left(-2\, s_{\overline{\alpha\delta}} + (n+1)\, q_{\overline{\alpha\delta}}\right) G_{\beta\gamma} \right\}$$

7. COMPLEX RIEMANNIAN MANIFOLDS OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURES.

Let (M,G) be a complex Riemannian manifold with characteristic connection D and characteristic curvature tensor K.

Definition. The manifold (M,G) is said to be a $\mathfrak{R}_1$*-manifold* if the tensor K is in the space $\mathfrak{R}_1$ at every point $p \in M$, $V^C = T^C_pM$.

From Lemma 6.4 it follows that (M,G) is a $\mathfrak{R}_1$-manifold iff K is an H-tensor.

The following proposition characterizes $\mathfrak{R}_1$-manifold in terms of the characteristic connection.

Proposition 7.1. *A complex Riemannian manifold* (M,G) *is a* $\mathfrak{R}_1$*-manifold iff the characteristic connection is holomorphic.*

Proof. From Proposition 6.1 and Lemma 6.4 it follows that (M,G) is a $\mathfrak{R}_1$-manifold iff $K^{\lambda}_{\bar{\alpha}\beta\gamma} = \partial_{\bar{\alpha}} D^{\lambda}_{\beta\gamma} = 0$ i.e. the characteristic connection D is holomorphic. □

In order to characterize complex Riemannian manifold with flat characteristic connection we give the following definition:

Definition. A complex Riemannian manifold is said to be *anti-holomorphic* if every point has an open neighbourhood, parameterized by holomorphic coordinates such that the components $G_{\beta\gamma}$ of the metric G are anti-holomorphic functions.

Theorem 7.2. *Let* (M,G) *be a complex Riemannian manifold. The following conditions are equivalent:*

1) (M,G) is an anti- holomorphic manifold;

2) The characteristic connection D is flat.

Proof. Let (M,G) be an anti holomorphic manifold with local holomorphic parameterization such that $\partial_{\alpha} G_{\beta\gamma} = 0$. Then the components of D are zero and hence $K = 0$.

To prove the inverse, we need the following

Lemma. *Let* (M,G) *be a complex Riemannian manifold with flat characteristic connection D. Then there exist local holomorphic coordinates such that the components of D are zero.*

Proof of the lemma: Let $D^{\lambda}_{\beta\gamma}$ be the components of D in a holomorphic coordinate system $(w^1, \ldots, w^n)$. To find a holomorphic coordinate system $(z^1, \ldots, z^n)$ in which the components of D are zero, we have to solve the system

$$(7.1) \qquad \frac{\partial^2 z^\sigma}{\partial w^\lambda \, \partial w^\mu} + D^\sigma_{\beta\gamma} \frac{\partial z^\beta \, \partial z^\gamma}{\partial w^\lambda \, \partial w^\mu} = 0$$

The condition $K^\lambda_{\bar{\alpha}\beta\gamma} = 0$ implies $D^\lambda_{\beta\gamma}$ are holomorphic functions. The integrability condition for the system (7.1) is $K^\lambda_{\alpha\beta\gamma} = 0$, which proves the assertion.

To complete the proof of the theorem, let $(z^1, \ldots, z^n)$ be a local holomorphic coordinate system as in the Lemma. Then the condition $D_\alpha G_{\beta\gamma} = 0$ reduces to $\partial_\alpha G_{\beta\gamma} = 0$.

This completes the proof of the theorem. □

Next manifolds we shall consider are described by the following

Definition. A complex Riemannian manifold (M,G) is said to be of *pointwise constant holomorphic characteristic curvature* if $K(E^C;p) = c(p)$ for all nondegenerate holomorphic 2-planes E^C in $V^{1,0} = T^{1,0}_p(M)$.

In this definition sectional curvatures $K(E^C,p)$ are taken with regard to the characteristic curvature tensor K of the manifold.

Applying Proposition 6.7, we obtain

Proposition 7.3. *A complex Riemannian manifold* (M,G) *is of constant holomorphic characteristic curvature iff in local holomorphic coordinates*

$$K_{\alpha\beta\gamma\delta} = c(p)\left[G_{\beta\gamma}G_{\alpha\delta} - G_{\alpha\gamma}G_{\beta\delta}\right]$$

where $c(p) = \frac{\tau}{n(n-1)}$.

For an arbitrary complex Riemannian manifold (M,G) the equality $D_\alpha G_{\beta\gamma} = 0$ implies.

Lemma 7.4. *The local components of the characteristic curvature tensor K satisfy the following identities:*

1) $D_\alpha K^\mu_{\beta\gamma\lambda} + D_\beta K^\mu_{\gamma\alpha\lambda} + D_\gamma K^\mu_{\alpha\beta\lambda} = 0$ (Bianchi identity) ;

2) $D_\alpha K^\alpha_{\beta\gamma\lambda} = D_\beta r_{\gamma\lambda} - D_\gamma r_{\beta\lambda}$;

3) $\partial_\alpha \tau = 2 D_\sigma r^\sigma_\alpha$.

For a $\mathfrak{R}_1$- manifold it follows that $\partial_{\bar\gamma} r_{\alpha\beta} = 0$ i.e. $r_{\alpha\beta}$ are holomorphic functions.

Proposition 7.5 (Shur's type theorem). *Let* (M,G) *be of pointwise constant holomorphic characteristic curvature* $c(p)$. *If* $\dim M = n \geq 3$, *then* $c(p)$ *is an anti- holomorphic function.*

Proof. Applying Proposition 7.3 and Lemma 7.4 we obtain

$$(n-2)\ \partial_\alpha c = 0$$

Hence, by $n \geq 3$, $\partial_\alpha c = 0$. □

Proposition 7.6. *Let* (M,G) *be a* $\mathfrak{R}_1$*-manifold of pointwise constant characteristic curvature* $c(p)$ *and* $\dim M = n \geq 2$. *Then in local holomorphic coordinates*

$$G_{\alpha\beta}\ \partial_{\bar\gamma} c + c\ \partial_{\bar\gamma} G_{\alpha\beta} = 0$$

Proof. Applying Proposition 7.3 and Lemma 7.4, we obtain

(7.2) $$r_{\alpha\beta} = (n-1)\, c(p)\, G_{\alpha\beta}$$

By the conditions of the proposition the functions $r_{\alpha\beta}$ are analytic, i.e. $\partial_{\bar\gamma} r_{\alpha\beta} = 0$. Then, (7.2) implies the assertion. □

Now we shall prove the main result in this section.

Theorem 7.7. (Classification theorem). *Let* (M,G) (dim $M = n \geq 3$) *be a* $\mathfrak{R}_1$*-manifold of pointwise constant holomorphic characteristic curvature* $c(p)$ *which is not identically zero. Then*

i) If c *is constant, then* (M,G) *is a holomorphic Riemannian manifold locally holomorphical isometric to a complex sphere;*

ii) If c *is not constant, then* (M,G) *is locally conformal equivalent to the unit complex sphere.*

Proof. *i)* Applying Proposition 7.6 , we obtain $\partial_{\bar\gamma} G_{\alpha\beta} = 0$ i.e. (M,G) is a holomorphic Riemannian manifold of constant holomorphic sectional curvature. The assertion follows from Theorem 5.4.

ii) Since $c(p) \neq 0$ we can consider the metric G' determined by $G'_{\alpha\beta} = c\, G_{\alpha\beta}$ We claim that G' is a holomorphic metric of constant curvature 1. Using the defining equality for G' we compute

$$\partial_{\bar\gamma} G'_{\alpha\beta} = \partial_{\bar\gamma} c\; G_{\alpha\beta} + c\; \partial_{\bar\gamma} G_{\alpha\beta}$$

Applying Proposition 7.6, we obtain G' is a holomorphic metric. If $\Gamma^{\gamma}_{\alpha\beta}$ are the local components of the Levi-Civita connection of G'

we find $\Gamma^{\gamma}_{\alpha\beta} = D^{\gamma}_{\alpha\beta}$. Hence, $R^{\lambda}_{\alpha\beta\gamma} = K^{\lambda}_{\alpha\beta\gamma}$, where $R^{\lambda}_{\alpha\beta\gamma}$ are the local components of the curvature tensor of G'. From Proposition 7.3 it follows that

$$R^{\lambda}_{\alpha\beta\gamma} = G'_{\beta\gamma}\delta^{\lambda}_{\alpha} - G'_{\alpha\gamma}\delta^{\lambda}_{\beta}$$

Hence, (M,G) is of constant holomorphic sectional curvature 1. Now the statement follows by applying Theorem 5.4. □

The above theorem does not include the case $c(p) = 0$, i.e. the flat case : $K = 0$.

EXAMPLE 7.1. Complex Riemannian manifold with flat characteristic connection.

Let $A\colon a_{\alpha}z^{\alpha} + a = 0$ be a holomorphic hyperplane in $\mathbb{C}^n$. We set $F = \mathbb{C}^n \setminus \{A\}$ and consider the metric G , defined by

$$G_{\alpha\beta} = \left[a_{\bar{\gamma}} \bar{z}^{\gamma} + \bar{a} \right] \delta_{\alpha\beta} ,$$

where $\delta_{\alpha\beta}$ is the complex Euclidian metric on F. Then (F,G) is a complex Riemannian manifold such that :

$F \in W_1$, but $F \notin W_0$;

(7.2) $K = 0$

$D\Phi = 0$, i.e. the fundamental tensor Φ is parallel with respect to D.

The next proposition describes all complex Riemannian manifolds satisfying the conditions (7.2).

Theorem 7.8. *Let* (M,G) *be a complex Riemannian manifold in the class* W_1 *with* $K = 0$ *and parallel fundamental tensor* Φ , *i.e.* $D\Phi = 0$. *Then* (M,G) *is locally holomorphic isometric to the manifold* (F,G) *in Example* 7.1.

Proof: We choose holomorphic coordinates $z^1,\dots,z^n$ as in Theorem 7.2 such that $\partial_\alpha G_{\beta\gamma} = 0$ and $D^\gamma_{\alpha\beta} = 0$. By Theorem 3.7 there exists locally a function $f \neq 0$, such that $f^{-1}G_{\beta\gamma} = H_{\beta\gamma}$ are local components of a holomorphic metric. From the condition $\partial_\alpha G_{\beta\gamma} = 0$ we obtain

(7.3) $\qquad \partial_\alpha f\, H_{\beta\gamma} + f\, \partial_\alpha H_{\beta\gamma} = 0;$

and after contraction

(7.4) $\qquad \partial_\alpha \log \left[f\, H^{1/n} \right] = 0$

where $H = \det (H_{\beta\gamma})$.

We denote the antiholomorphic function $f.\ H^{1/n} = B(\bar z^1,\dots,\bar z^n)$. Since $\Phi_{\alpha\beta\bar\gamma} = \partial_{\bar\gamma} f\, H_{\alpha\beta}$ and $D^\gamma_{\alpha\beta} = 0$, the condition $D\Phi = 0$ implies

$$\partial_{\bar\delta}\, \partial_{\bar\gamma}\, f = 0 \quad \text{and} \quad \partial_{\bar\delta}\, \partial_{\bar\gamma}\, B = 0$$

Hence B is a linear antiholomorphic function : $B = a_{\bar\alpha}\, \bar z^\alpha + a$

Taking into account (7.3) and (7.4) we find we find $\partial_\alpha \left[H^{-1/n} H_{\beta\gamma}\right] = 0$. Thus, we obtained that $H^{-1/n}H_{\beta\gamma}$ are constants, i.e. $H^{-1/n}H_{\beta\gamma}$ are components of the standard flat holomorphic Euclidian metric in dhe considered domain in $\mathbb{C}^n$. □

Corollary 7.9. *If by the conditions of Theorem 7.8 (M,G) is connected, simply connected and complete , then (M,G) is holomorpically isometric to the manifold (F,G) in Example 7.1.*

8. CONFORMAL TRANSFORMATIONS OF COMPLEX RIEMANNIAN MANIFOLDS.

In this section we study conformal transformations of the characteristic connection and conformal tensor invariants. We show that the conformal geometry on holomorphic Riemannian manifold can be naturally developed as the characteristic conformal geometry in the class W_1 , i.e. the class of complex Riemannian manifold locally conformal to holomorphic Riemannian manifolds.

Let (M,G) be a complex Riemannian manifold and (M,G') be locally conformal to (M,G). For convenience we write

$$G'_{\alpha\beta} = \exp(2u)\ G_{\alpha\beta} \tag{8.1}$$

with regard to a local holomorphic coordinate system.

If $D^{\gamma}_{\alpha\beta}$ and $D'^{\gamma}_{\alpha\beta}$ are the components of the characteristic connections D and D' of G and G', respectively, then (2.7) implies

$$D'^{\gamma}_{\alpha\beta} = D^{\gamma}_{\alpha\beta} + u_{\alpha}\ \delta^{\gamma}_{\beta} + u_{\beta}\ \delta^{\gamma}_{\alpha} - G_{\alpha\beta}\ u^{\gamma} \tag{8.2}$$

where $u_{\alpha} = \partial_{\alpha} u$

From (8.2) it follows immediately:

A conformal transformations (8.1) preserves the characteristic connection D iff u is anti-holomorphic function, i.e. $\partial_{\alpha} u = 0$. We call such a conformal transformation *trivial*.

Further we have

Proposition 8.1. *Let (M,G) be a complex Riemannian manifold with holomorphic characteristic connection D. If (M,G) admits a nontrivial conformal metric $G'_{\alpha\beta} = \exp(2u)\ G_{\alpha\beta}$ with holomorphic characteristic connection D', then:*

i) (M,G) is in the class W_1 ;

ii) the function u satisfies the condition $\partial\bar{\partial}\, u = 0$

Proof. Let the characteristic connection D' of

$G'_{\alpha\beta} = \exp(2u)\, G_{\alpha\beta}$ be holomorphic. From (8.2) we have

$$(8.3) \qquad \partial_{\bar\alpha}\, \partial_\beta\, u\, \delta^\lambda_\gamma + \partial_{\bar\alpha}\, \partial_\gamma\, u\, \delta^\lambda_\beta - \partial_{\bar\alpha}\, (G_{\beta\gamma}\, u^\lambda) = 0$$

Contracting , we find $n\, \partial_{\bar\alpha}\, \partial_\beta\, u = 0$, i.e. $\partial\bar\partial u = 0$.

Further, (8.3) implies

$$(8.4) \qquad \partial_{\bar\alpha} G_{\beta\gamma}\, u^\lambda - G_{\beta\gamma}\, \partial_{\bar\alpha}\, u^\lambda = 0$$

A new contraction in (8.4) leads to $\partial_{\bar\alpha}\, u^\lambda = - \theta_{\bar\alpha}\, u^\lambda$.
Substituting this expression into (8.4), we obtain

$$\left(\partial_{\bar\alpha} G_{\beta\gamma} - \theta_{\bar\alpha}\, G_{\beta\gamma} \right) v^\lambda = 0$$

Hence, (M,G) is in the class W_1 .

Using Proposition 8.1 we shall give the following characterization for the class $W_1 \cap \mathfrak{R}_1$, i.e. the class of locally conformal holomorphic Riemannian manifolds with holomorphic characteristic connection:

Proposition 8.2. *Let (M,G) be a W_1-manifold. Then the following conditions are equivalent:*

i) The characteristic connection D *is holomorphic;*

ii) The Lee form θ is closed, i.e. $d\theta = 0$.

Proof. Since (M,G) is a W_1-manifold, then there exists locally a function u such that $\exp(-2u)\, G_{\alpha\beta}$ is a holomorphic metric. Then

$$\theta_{\bar\alpha} = G^{\beta\gamma}\, \partial_{\bar\alpha} G_{\beta\gamma} = \frac{2}{n}\, \partial_{\bar\iota}\, u\ .$$

By Proposition 8.1 $D^\gamma_{\alpha\beta}$ are holomorphic functions iff $\partial\bar\partial u = 0$

Hence, the characteristic connection D is holomorphic iff $\bar{\partial}\theta = 0$. i.e. $\bar{\partial}\,\theta = 0$. . The last equality is equivalent to $d\theta = 0$ because of Proposition 3.1, which proves the assertion. □

Now we shall prove an explicit formula for the second Ricci tensor $s_{\alpha\bar{\beta}}$ in local coordinates

Lemma 8.3. *For every complex Riemannian manifold in a local holomorphic coordinate system*

$$s_{\alpha\bar{\beta}} = \frac{1}{2}\partial_{\bar{\alpha}}\,\partial_{\beta}\log G$$

where $G = \det\,(G_{\alpha\beta})$.

Proof. By the definition of $s_{\alpha\bar{\beta}}$ we have

$$s_{\alpha\bar{\beta}} = K^{\lambda}_{\alpha\bar{\beta}\lambda} = \partial_{\bar{\alpha}}\,D^{\lambda}_{\beta\lambda} = \frac{1}{2}\,\partial_{\bar{\alpha}}\left(G^{\lambda\mu}\,\partial_{\beta}\,G_{\lambda\mu}\right) = \frac{1}{2}\,\partial_{\bar{\alpha}}\,\partial_{\beta}\log G \quad \square$$

Proposition 8.4. *Every complex Riemannian manifold is locally conformal to a complex Riemannian manifold with* $s_{\alpha\bar{\beta}} = 0$.

Proof. Applying Lemma 8.3, (3.1) and (3.2) we find

$$s_{\alpha\bar{\beta}} = \frac{1}{2}\,\partial_{\bar{\alpha}}\,\partial_{\beta}\log G = \frac{1}{2}\,\partial_{\beta}\,\partial_{\bar{\alpha}}\log G = \frac{1}{2}\,\partial_{\beta}\,\theta_{\bar{\alpha}}$$

By Proposition 3.6 there exists locally a metric $G'_{\alpha\beta}$ such that $\theta'_{\bar{\alpha}} = 0$. . Hence the metric G' is with zero tensor $s'_{\alpha\bar{\beta}}$. □

Let G and G' be two complex Riemannian metrics related as in (8.1) and K , K' be their corresponding characteristic curvature tensors. Using (8.2) we compute

$$(8.5) \qquad \exp(-2u)\, K'_{\alpha\beta\gamma\delta} = K_{\alpha\beta\gamma\delta} - \Big\{ L_{\beta\gamma}\, G_{\alpha\delta} - L_{\alpha\gamma}\, G_{\beta\delta} + L_{\alpha\delta}\, G_{\beta\gamma} - L_{\beta\delta}\, G_{\alpha\gamma} \Big\}$$

$$(8.6) \qquad \exp(-2u)\, K'_{\bar{\alpha}\beta\gamma\delta} = K_{\bar{\alpha}\beta\gamma\delta} + \Big\{ \partial_{\bar{\alpha}} u_{\beta}\, G_{\gamma\delta} + \partial_{\bar{\alpha}} u_{\gamma}\, G_{\beta\delta} - \partial_{\bar{\alpha}} u_{\delta}\, G_{\beta\gamma} \quad \partial_{\bar{\alpha}} G_{\beta\gamma}\, u^{\delta} + \partial_{\bar{\alpha}} G_{\lambda\delta}\, u^{\lambda}\, G_{\beta\gamma} \Big\}$$

where $$L_{\alpha\beta} = D_{\alpha} u_{\beta} - u_{\alpha} u_{\beta} + \frac{1}{2} u_{\lambda} u^{\lambda} G_{\alpha\beta}$$

Further we call the Weyl tensor (6.6) associated with the characteristic curvature tensor K *the characteristic conformal tensor.*

By contracting in (8.5) with $G^{\alpha\delta}$ and next in the resulting relation, with $G^{\beta\gamma}$, we obtain in a standart way

Proposition 8.5. *The characteristic conformal tensor* $W^{\delta}_{\alpha\beta\gamma}$ *is a conformal invariant.*

Now let (M,G) be a $W_{1}\bot$-manifold and $G'_{\alpha\beta} = \exp(2u)\, G_{\alpha\beta}$ be a holomorphic Riemannian metric in a holomorphic coordinate system. Then (8.6) implies

$$K_{\bar{\alpha}\beta\gamma\delta} + \partial_{\bar{\alpha}} u_{\beta}\, G_{\gamma\delta} + \partial_{\bar{\alpha}} u_{\gamma}\, G_{\beta\delta} - \partial_{\bar{\alpha}} u_{\delta}\, G_{\beta\gamma} = 0$$

After a contraction with $G^{\gamma\delta}$ we obtain

Proposition 8.6. *For every* W_{1}*-manifold the tensor* $K_{\bar{\alpha}\beta\gamma\delta}$ *has the form*

$$K_{\bar{\alpha}\beta\gamma\delta} = \frac{1}{n} \Big\{ s_{\bar{\alpha}\beta}\, G_{\gamma\delta} + s_{\bar{\alpha}\gamma}\, G_{\beta\delta} - s_{\bar{\alpha}\delta}\, G_{\beta\gamma} \Big\}$$

Now we shall prove the main result of this section.

Theorem 8.7. *Let (M,G) be a complex Riemannian manifold with $\dim M = n \geq 4$. Then the following conditions are equivalent;*

i) (M,G) is locally conformal to a complex Euclidian space;

ii) The conformal fundamental tensor and the characteristic conformal tensor of (M,G) are zero.

Proof. The implication i) ⟶ ii) follows immediately from Theorem 3.7 and (8.5).

To prove the inverse implication we apply Theorem 3.7 and consider a holomorphic metric $G'_{\alpha\beta} = \exp(2h)\, G_{\alpha\beta}$ in a holomorphic coordinate neighbourhood U of a point $p \in M$. Let ∇', K', r', τ' be the Levi-Civita connection, the curvature tensor, the Ricci tensor and the scalar curvature of the metric G' , respectively. Under the hypothesis it follows from Proposition 8.2 that the conformal tensor $W'_{\alpha\beta\gamma\delta}$ associated with K' is zero.

In order to find a flat holomorphic metric $\exp(2u)\, G'_{\alpha\beta}$ we follow the standart scheme in the Riemannian geometry and consider the system

$$\text{(8.7)} \qquad \nabla'_{\alpha} u^{\beta} - u_{\alpha}u_{\beta} + \frac{1}{2} u_{\lambda}u^{\lambda} G'_{\alpha\beta} = \frac{1}{n-2} r'_{\alpha\beta} - \frac{\tau'}{2(n-1)(n-2)} G'_{\alpha\beta}$$

In this system $G'_{\alpha\beta}$, $r'_{\alpha\beta}$, τ' and the local components of ∇' are holomorphic functions. Under the assumption $n \geq 4$ and the condition $W'_{\alpha\beta\gamma\delta} = 0$ the system (8.7) is completely integrable.

Let u be a local holomorphic solution of (8.7) (in a refined neighbourhood U' of p if necessary). Then $\exp\left[2(h+u)\right] G_{\alpha\beta}$ is a flat holomorphic metric in U'. The final conclusion follows now from Theorem 5.1. □

For complex analytic Riemannian 4-manifold see comments in [9, ˋ˙].

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REFERENCES

1. Ganchev,G., Gribachev,K.,and Mihova,V.,'B-connections and their conformal invariants on conformally Kaehler manifold with B-metric ', *Publ. l'Inst. Math. N.S.* **42** (1987), 107 - 121.

2. Graves,L.,and Nomizu,K., 'On sectional curvatures of indefinite metrics', *Math. Ann.* 232 (1978) , 267 - 272.

3. Issenberg,J., Yasskin,P., and Green,P.S., 'Non-self-dual gauge fields', *Phys. Lett. B* 78 (1978), 462 - 464.

4. Kobayashi S.,and Nomizu K., *Foundations of Differential Geometry* , Intersience , New York , vol. I 1963, vol. II , 1969 .

5. LeBrun,C., ' Spaces of complex null geodesics in complex-Riemannian geometry', *Trans. AMS* **278** 1 (1983), 209 - 231.

6. Manin Yu., *Gauge Field Theory and complex Geometry* , Springer - Verlag , Berlin , Heidelberg , New York , 1988 (translated from the Russian).

7. Nomizu,K., ' On the decomposition of generalized curvature tensor fields' , *Diff. Geometry (in honor of K. Yano)* , *Kinokuniya, Tokyo*, 1972 , 333 - 345 .

8. Penrose,R., 'Non linear gravitons and curved twistor theory', *Gen. Relativity and Gravitation* 7 (1976), 31 - 52.

9. Penrose R.,and Rindler W., *Spinors and space-time*., vol. II , Cambridge , London , New York , 1986 .

10. Penrose,R., and Word,R.S., 'Twistors for flat and curved space-time', *Gen. Relativity and Gravitation , One Hundred years after the Birth of A.Einstein ed. A. Held , Plenum , New York* , 1980 .

11. Synger,I.M., and Thorpe,J.A., 'The curvature of 4-dimensional Einstein spaces' , *Global analysis (papers in honor of K.Kodaira) , University of Tokyo Press , Tokyo* , 1969 355-365

12 Witten,E., 'An interpretation of classical Yang-Mils theory', *Phys. Lett. B 77* (1978), 394 - 397.