



**REFERENCE**

IC/91/31

**INTERNATIONAL CENTRE FOR  
THEORETICAL PHYSICS**

**ON CALCULATIONS OF THE GROUND STATE ENERGY  
IN QUANTUM MECHANICS**

**G.V. Efimov**

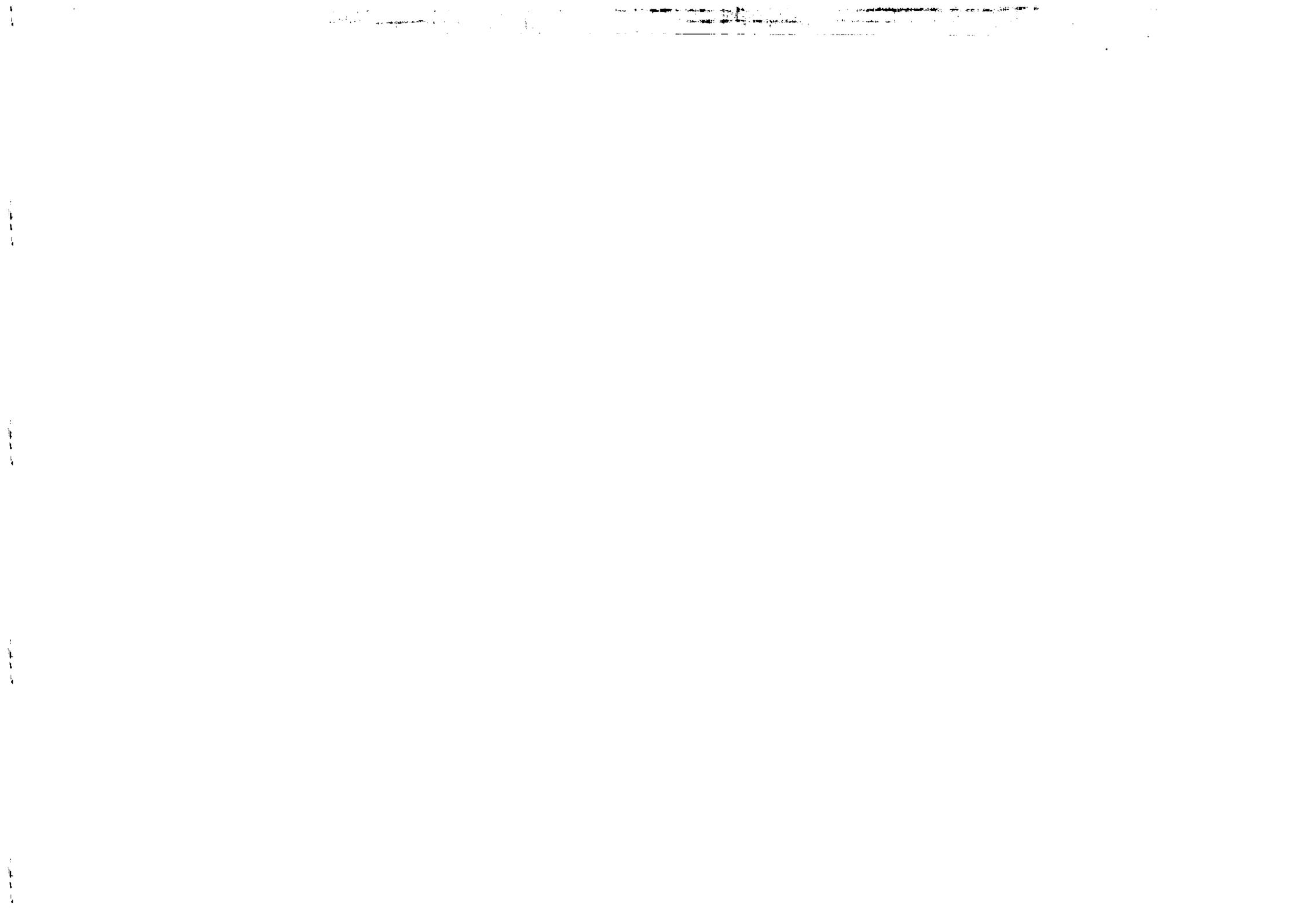


**INTERNATIONAL  
ATOMIC ENERGY  
AGENCY**



**UNITED NATIONS  
EDUCATIONAL,  
SCIENTIFIC  
AND CULTURAL  
ORGANIZATION**

**1991 MIRAMARE - TRIESTE**



International Atomic Energy Agency  
and  
United Nations Educational Scientific and Cultural Organization  
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON CALCULATIONS OF THE GROUND STATE ENERGY  
IN QUANTUM MECHANICS \*

G.V. Efimov \*\*

International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

In nonrelativistic quantum mechanics the Wick-ordering method called the oscillator representation is suggested to calculate the ground-state energy for a wide class of potentials allowing the existence of a bound state. The following examples are considered:

- the orbital excitations of the ground-state in the Coulomb plus linear potential,
- the Schrödinger equation with a "relativistic" kinetic energy  $\sqrt{p^2 + m^2}$ ,
- the Coulomb three-body problem.

MIRAMARE – TRIESTE

February 1991

\* To be submitted for publication.

\*\* Permanent address: Joint Institute for Nuclear Research, Dubna, USSR.

## 1. INTRODUCTION

The Schrödinger equation is the basis of our understanding of the quantum nonrelativistic and sometimes relativistic structure of matter. It provides a correct description of a rich variety of physical phenomena such as atoms and molecules, nuclei and particles, quarks and hadrons. The Schrödinger equation permits to understand how particles interacting by means of a potential can be connected into a bound state with new quantum numbers and a new mass. The Schrödinger equation is so attractive that even in the essentially relativistic problems this equation is used with the relativistic kinetic energy.

In order to have a description of all these phenomena we should be able to solve the Schrödinger equation for any potentials, and for any physical situations: bound states, scattering and so on. There is a sea of literature on the subject: papers, books and textbooks where numerous analytical and numerical methods are worked out for this aim. Now the numerical methods are so powerful that for a large number of physical tasks the problem of solution of the Schrödinger equation does not exist.

Nevertheless the elaboration of analytical methods is continued and, we are convinced, it will be continued. This statement does not require any motivations. As an example, we refer to the papers [1–5]

In this paper we formulate a method of calculation of the bound state energy. This method can be applied practically to any potential which permits the existence of a bound state. This method arises from ideas and methods of quantum field theory (QFT). Namely, in QFT of scalar fields the main contribution into perturbation expansions for any physical amplitudes gives the propagator in the point zero, i.e.  $D(0)$ , which is known as the "tadpole" Feynman diagram. It is the highest divergence in perturbation calculations. In renormalizable models of QFT (for example,  $\varphi^4$ ) this divergence  $D(0)$  is included into renormalization of the masses of particles, vacuum energy and other physical parameters. As a result it leads to the requirement that all quantum field operators should be written in the form of a normal product over the annihilation and creation operators.

One should say that this divergent term  $D(0)$  has a pure quantum nature and plays the main role in forming of the ground state.

Thus the normal form of operators in QFT means that the main quantum contributions into any physical parameters are taken into account.

Now let us come back to the Schrödinger equation in quantum mechanics. In some sense quantum bound states are close to oscillator states; this is especially true for deep levels. We mean that the true wave function can be developed over an oscillator basis with an appropriate frequency and the few lowest orders give an acceptable approximation for the true wave function. Besides,  $D(0)$  is finite for quantum oscillators.

All these facts indicate the following scenario for quantum mechanics. Let a Hamiltonian

is given. Let us pick out then a quantum oscillator from this Hamiltonian and let us write down this oscillator in the representation of creation and annihilation operators. The other part of the Hamiltonian should be written in the form of normal product over these operators in such a way that this interaction Hamiltonian does not contain any quadratic oscillator terms. Then we can hope that the main quantum contributions will be taken into account.

When we begin to realize this idea two problems arise. First, the standard situation is that in most cases potentials have a repulsion for small distances. For example, this situation takes place for orbital excitations, the Coulomb three-body problem, nuclear potentials and so on. Thus the maximum of true wave functions occurs at some distances different from zero. Therefore the direct oscillator representation cannot be a good approach.

In order to solve this problem we go over to spaces of the highest dimensions. Namely the potential with a repulsion in the three-dimension space becomes a potential with an attraction in a  $d$ -dimension space where  $d$  is large enough. Below we shall discuss all technical details.

Second, the asymptotic behaviour of the true wave function at large distances can differ essentially from the oscillator Gaussian asymptotic. It is important for potentials with shallow energy levels. For example, this situation takes place for deuteron. In this case in the radial Schrödinger equation we should go over to a new radial variable in such a way that the true wave function has Gaussian asymptotics in this new radial variable.

The Wick's-ordering method in quantum mechanics is known (see, for example, [6,7]). However, it was not used for the three-dimensional Hamiltonians and the basic point of this approach, namely, the introduction of the indicated representation was not formulated.

The paper is organized as follows. In Section 2 we formulate the oscillator representation (OR) for the one-dimension Schrödinger equation and discuss the quantum meaning of this representation. In Section 3 we introduce OR for the three-dimension Schrödinger equations and obtain general formulas for the bound state energy and its orbital excitations. In Section 4 we apply this method to the Coulomb plus linear potential and the Coulomb, linear and logarithm potentials. Then we discuss the accuracy of OR for these potentials. In Section 5 we discuss the problem of asymptotical behaviour of the wave functions. In Section 6 we apply OR to the Schrödinger equation with "relativistic" kinetic term and find the bound state energy and its orbital excitations for the Coulomb plus linear potential. In Section 7 OR is applied to the Coulomb three-body systems. The idea is based on the special choice of variables and the transition to the highest-dimension space in order to dump the Coulomb repulsion. Then we consider examples: Helium, molecule  $H_2^+$  and a system of two massive ions with large charges and one electron.

In this paper we do not aspire to get the best numerical result for all considered examples. Each of our examples deserve more detailed examination. We are going to study part of them in the future. Here we would like to show how this method makes it possible to obtain suitable agreement with the true results even in the lowest approximation and how to calculate the next corrections.

## 2. THE OSCILLATOR REPRESENTATION IN ONE-DIMENSION SCHRÖDINGER EQUATION

We consider the Schrödinger equation of the one-dimensional motion

$$H\psi = E\psi \quad (2.1)$$

where the Hamiltonian is

$$H(p, q) = H = \frac{p^2}{2} + V(q) \quad (2.2)$$

Here  $q$  is a coordinate ( $-\infty < q < \infty$ ) and  $p = -i\frac{d}{dq}$  is a canonical momentum. They satisfy the canonical commutational relation  $[q, p] = i$ .

We assume that a potential  $V(q)$  can have a bound state. We also assume that the following integral does exist

$$\int_{-\infty}^{\infty} dq e^{-\alpha q^2} |V(q)| < \infty \quad (2.3)$$

for any  $\alpha > 0$ .

Our problem is to calculate the ground state energy of the Schrödinger equation (1.1). First of all, we perform in the Hamiltonian (2.2) the canonical transformation

$$p \rightarrow p, \quad q \rightarrow q + b$$

and rewrite our Hamiltonian in the form

$$H = \frac{1}{2} (p^2 + \omega^2 q^2) + \left[ V(q + b) - \frac{\omega^2}{2} q^2 \right] \quad (2.4)$$

where  $b$  and  $\omega$  are arbitrary parameters.

Let us introduce the representation of the creation and annihilation operators for  $q$  and  $p$

$$\begin{aligned} q &= \frac{1}{\sqrt{\omega}} Q, & p &= \sqrt{\omega} P, \\ Q &= \frac{1}{\sqrt{2}} (a + a^+), & P &= \frac{1}{i\sqrt{2}} (a - a^+), \\ & & [a, a^+] &= 1. \end{aligned} \quad (2.5)$$

The ground or vacuum state  $|0\rangle$  can be introduced by the conditions

$$\langle 0|0\rangle = 1, \quad a|0\rangle = 0.$$

In the representation where the variable  $Q$  is diagonal, one can get

$$\begin{aligned} a &= \frac{1}{\sqrt{2}} \left( Q + \frac{d}{dQ} \right), & a^+ &= \frac{1}{\sqrt{2}} \left( Q - \frac{d}{dQ} \right), \\ |0\rangle &= \frac{1}{\pi^{1/4}} \exp\left(-\frac{1}{2} Q^2\right) = \frac{\omega^{1/2}}{\pi^{1/4}} \exp\left(-\frac{\omega}{2} q^2\right). \end{aligned}$$

Then we have

$$\langle 0 | q q | 0 \rangle = \Delta = \frac{1}{2\omega} \quad (2.6)$$

Let us substitute the representation (2.5) into (2.4) and go over to the Wick-ordered normal product of the operators  $a$  and  $a^+$ . We obtain

$$H = \omega a^+ a + \frac{\omega}{2} + : \left[ \bar{V}(q+b) - \frac{\omega^2}{2}(q^2 + \Delta) \right] : \quad (2.7)$$

where  $: * :$  is a symbol of the normal product and

$$\bar{V}(b, \omega) = \exp\left(\frac{\Delta}{2} \frac{d^2}{db^2}\right) V(b) = \int \frac{dt}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} V(b + t\sqrt{\Delta}) .$$

The Hamiltonian (2.7) can be rewritten in the form

$$H = \omega a^+ a + : \bar{V}_2(q) : + E_0 + : q : \frac{d}{db} \bar{V}(b, \omega) + \frac{1}{2} : q^2 : \left( \frac{d^2}{db^2} \bar{V}(b, \omega) - \omega^2 \right) \quad (2.8)$$

where

$$: \bar{V}_2(q) : = : \left[ \bar{V}(q+b, \omega) - \bar{V}(b, \omega) - \frac{\partial}{\partial b} \bar{V}(b, \omega) q - \frac{1}{2} \frac{\partial^2}{\partial b^2} \bar{V}(b, \omega) q^2 \right] : ,$$

$$E_0 = \frac{\omega}{4} + \bar{V}(b, \omega) .$$

Our idea is that the behaviour of a system near its ground state is similar to an oscillator with a frequency  $\omega$  and the corrections are defined by the interaction potential  $: \bar{V}_2(q) :$ . This potential near an equilibrium point behaves as

$$: \bar{V}_2(q) : = O(q^3) .$$

The contribution of the first order equals zero

$$\langle 0 | : \bar{V}_2(q) : | 0 \rangle = 0 .$$

It means that we should require

$$\frac{\partial}{\partial b} \bar{V}(b, \omega) = 0$$

$$\frac{\partial^2}{\partial b^2} \bar{V}(b, \omega) - \omega^2 = 0 . \quad (2.9)$$

These equations secure our statement and define the parameters  $\omega$  and  $b$ .

We should like to make two remarks. First, Eqs.(2.9) define minimum of the energy  $E_0 = E_0(\omega, b)$  in (2.8):

$$\frac{\partial}{\partial b} E_0(\omega, b) = \frac{\partial}{\partial b} \bar{V}(b, \omega) = 0 ,$$

$$\frac{\partial}{\partial \omega} E_0(\omega, b) = -\frac{1}{4\omega^2} \frac{\partial^2}{\partial b^2} \bar{V}(b, \omega) + \frac{1}{4} = 0 .$$

Second, the function  $E_0(\omega, b)$ , where  $\omega$  and  $b$  are found from (2.9), is the upper variational estimation for the ground-state energy

$$E < E_0(\omega, b) .$$

Thus this method is founded on an oscillator basis and gives in its lowest approximation a variational estimation for the ground state energy. Now we would like to remark that there exist well formulated approaches which have used the oscillator basis for variational calculations in atom and nuclear physics (see, for example, [8-10]).

Finally we have

$$H = H_0 + H_I + E_0 ,$$

$$H_0 = \omega a^+ a ,$$

$$H_I = : \bar{V}_2(q) : ,$$

$$E_0 = \frac{\omega}{4} + \bar{V}(b, \omega) . \quad (2.10)$$

The ground-state energy of the initial problem (2.1-2) is

$$E = E_0 + E_{\text{int}} \quad (2.11)$$

where  $E_{\text{int}}$  should be found by the perturbation method from the equation

$$(H_0 + H_I)\Phi = E_{\text{int}} \Phi .$$

This energy  $E_{\text{int}}$  can be calculated by using, for example, the methods of quantum statistics (see, for example, [11]).

$$E_{\text{int}} = -\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \ln Z_\beta \quad (2.12)$$

$$Z_\beta = \langle 0 | T \exp \left\{ -\int_0^\beta d\tau H_I(q(\tau)) \right\} | 0 \rangle =$$

$$= \exp \left\{ \frac{1}{2} \int \int_0^\beta d\tau_1 d\tau_2 D(\tau_1 - \tau_2) \frac{\delta^2}{\delta q(\tau_1) \delta q(\tau_2)} \right\} \exp \left\{ -\int_0^\beta d\tau H_I(q(\tau)) \right\} \Big|_{q(\tau)=0}$$

Here  $T$  is the "time" Wick-ordering of the operators  $q(\tau)$  over the index  $\tau$ :

$$q(\sigma) = e^{H_0\sigma} q e^{-H_0\sigma} = \frac{1}{\sqrt{2}} (a e^{-\omega\sigma} + a^+ e^{\omega\sigma}) ,$$

$$D(\tau_1 - \tau_2) = \langle 0 | T(q(\tau_1)q(\tau_2)) | 0 \rangle = \frac{1}{2\omega} e^{-\omega|\tau_1 - \tau_2|} . \quad (2.13)$$

We should calculate  $Z_\beta$  by the perturbation method. Each term of the perturbation series can be represented in the form of the Feynman diagrams in which the function  $D(\tau_1 - \tau_2)$  is a propagator. It should be accentuated that the "tadpole"-type diagrams which are proportional to

$D(0)$  are absent in the perturbation terms since the interaction Hamiltonian is chosen in the normal form.

In the language of the Feynman diagrams, the oscillator representation means the following. Let us go back to (2.4) and choose the parameters  $b$  and  $\omega$  from the classical equations

$$V'(b) = 0, \quad V''(b) = \omega^2$$

which define the location and frequency of a classical oscillator in the equilibrium point of the classical potential. We obtain the interaction Hamiltonian in the form  $V_2(q)$ . The perturbation series with this Hamiltonian contains so-called "cactus"-type diagrams, which are proportional to  $D(0)$ . Then, in order to go to the oscillator representation we should sum in this perturbation series all these cactus diagrams. Eqs.(2.9) do this summation effectively.

Thus the summation of all terms  $D(0)$  means taking into account of all main quantum contributions. It is the principal reason why the oscillator representation gives good results for the ground state energy.

In this paper we shall not give examples which show effective applications of this method for the one-dimension Schrödinger equations. It was done in [19]. Now we go over to the three-dimension quantum mechanics problems.

### 3. THE OSCILLATOR REPRESENTATION IN THE SPACE $\mathbf{R}^3$

Let us consider the Schrödinger equation in the space  $\mathbf{R}^3$

$$\left( \frac{\vec{p}^2}{2} + V(r) \right) \psi = E\psi, \quad (3.1)$$

$$\vec{p}^2 = -\Delta_3 = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}.$$

The potential  $V(r)$  is supposed to have the rotational symmetry and to be an attractive one, i.e.  $V'(r) > 0$ . Besides it satisfies the condition

$$\int d\vec{r} e^{-\alpha r^2} |V(r)| < \infty \quad (3.2)$$

for any positive  $\alpha$ .

The energy levels have two quantum numbers  $(n, \ell)$ :

$$H \psi_{n\ell}(\tau) = E_{n\ell} \psi_{n\ell}(\tau)$$

where  $n$  is a radial and  $\ell$  is an orbital quantum number. The method formulated in this section gives good results for  $E_{0\ell}$  and  $E_{1\ell}$ . For  $n > 1$  we should use a modified oscillator representation (see Section 5) or use the quasiclassical approach (see, for example, [1]).

Our aim is to calculate  $E_{0\ell}$ , i.e., the ground state energy and its  $\ell$ -orbital excitations. Excluding the angle variables one can get the standard equation

$$\left[ -\frac{1}{2} \frac{1}{r} \left( \frac{d}{dr} \right)^2 r + \frac{\ell(\ell+1)}{2r^2} + V(r) \right] \psi(r) = E\psi(r). \quad (3.3)$$

The total potential in (3.3) contains the orbital repulsive part and, therefore, the wave function  $\psi(r)$  behaves as  $\psi(r) \simeq O(r^\ell)$  for  $r \rightarrow 0$ . That is the reason why we cannot apply our oscillator approximation directly to Eq.(3.3).

The idea is the following. Let us consider the Schrödinger equation in the space  $\mathbf{R}^d$

$$\left[ -\frac{1}{2} \Delta_d + V(r) \right] \psi = E\psi. \quad (3.4)$$

Here  $x \in \mathbf{R}^d$ , i.e.  $x = (x_1, \dots, x_d)$ ,

$$\Delta_d = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}, \quad r = \sqrt{x_1^2 + \dots + x_d^2}.$$

The operation of the Laplacian  $\Delta_d$  on a spherically symmetrical function in  $\mathbf{R}^d$  leads to

$$-\Delta_d R(r) = \left[ -\frac{1}{r^{\frac{d-1}{2}}} \left( \frac{d^2}{dr^2} \right) r^{\frac{d-1}{2}} - \frac{L_d(L_d+1)}{r^2} \right] R(r), \quad (3.5)$$

$$L_d = \frac{d-3}{2}.$$

Let us come back to Eq.(3.3) and introduce a new function

$$\psi(r) = r^\ell R(r),$$

where

$$\ell = \frac{d-3}{2} \quad \text{or} \quad d = 2\ell + 3. \quad (3.6)$$

Then Eq.(3.3) becomes the form

$$\left[ -\frac{1}{2} \cdot \frac{1}{r^{\frac{d-1}{2}}} \left( \frac{d^2}{dr^2} \right) r^{\frac{d-1}{2}} + \frac{L_d(L_d+1)}{2r^2} + V(r) \right] R(r) = E R(r). \quad (3.7)$$

We can see that Eq.(3.7) coincides with Eq.(3.4) when  $d = 2\ell + 3$  and  $\psi = \psi(r)$ .

Thus the calculation of an  $\ell$ -orbital excitation of the ground state in the  $\mathbf{R}^3$  is equivalent to the calculation of the ground state energy in the space  $\mathbf{R}^d$ , where  $d = 2\ell + 3$ .

Moreover, if our potential  $V(r)$  in (3.1) contains a repulsive part at small distances  $r$  and, therefore the wave function of the ground state does not have its maximum at  $r = 0$ , then we can consider the equivalent equation

$$\left[ -\frac{1}{2} \Delta_d - \frac{L_d(L_d+1)}{2r^2} + V(r) \right] \psi(r) = E \psi(r) \quad (3.8)$$

in the space  $\mathbb{R}^d$  for an arbitrary  $d > 3$ . As compared with (3.1) Eq.(3.8) contains the attractive term and this term can compensate the repulsive behaviour of the initial potential  $V(r)$  at small distances. From this point of view the dimension  $d$  can be considered as an additional variational parameter.

Now let us formulate the oscillator representation for the Hamiltonian

$$H = -\frac{1}{2} \Delta_d + V(r) = \frac{1}{2} p^2 + V(r) \quad (3.9)$$

in the space  $\mathbb{R}^d$ . The Hamiltonian  $H$  can be rewritten in the form

$$H = \frac{1}{2} (p^2 + \omega^2 r^2) + \left( V(r) - \frac{\omega^2}{2} r^2 \right) \quad (3.10)$$

where  $\omega$  is an oscillator frequency. The oscillator canonical variables are

$$\begin{aligned} x_i &= \frac{1}{\sqrt{\omega}} Q_i, & Q_i &= \frac{a_i + a_i^\dagger}{\sqrt{2}}, \\ p_i &= \sqrt{\omega} P_i, & P_i &= \frac{a_i - a_i^\dagger}{i\sqrt{2}}, \end{aligned} \quad (i = 1, \dots, d) \quad (3.11)$$

$$[a_i, a_j^\dagger] = \delta_{ij}.$$

The ground or vacuum state  $|0\rangle$  is supposed to exist and satisfies the conditions:

$$\langle 0|0\rangle = 1, \quad a_j|0\rangle = 0 \quad (j = 1, \dots, d).$$

Let us substitute the representation (3.11) into (3.10) and go over to the normal product of the operators  $a_i^\dagger$  and  $a_i$ . One can get

$$\begin{aligned} \frac{1}{2} (p^2 + \omega^2 r^2) &= \omega a_i^\dagger a_i + \frac{d}{2} \omega, \\ V(r) - \frac{\omega^2}{2} r^2 &= \int \left( \frac{d\rho}{2\pi} \right)^d \tilde{V}_d(u^2) e^{-\frac{\rho^2}{2\omega}} : e^{i\tilde{u}\rho} : - \frac{\omega^2}{2} \left( : r^2 : + \frac{d}{2\omega} \right), \end{aligned}$$

where

$$\tilde{V}_d(u^2) = \int (d\rho)^d V(\rho) e^{i\tilde{u}\rho}. \quad (3.12)$$

Now we require that interaction part of our Hamiltonian should not contain the term with  $: r^2 :$  because it is postulated that this term is included into the oscillator part completely. This requirement gives the equation on the frequency  $\omega$ :

$$\omega^2 + \int \left( \frac{d\rho}{2\pi} \right)^d \tilde{V}_d(u^2) e^{-\frac{\rho^2}{2\omega}} \frac{u^2}{d} = 0. \quad (3.13)$$

Using these formulas we can rewrite the Hamiltonian (3.10) in the form:

$$\begin{aligned} H &= H_0 + H_I + E_0, \\ H_0 &= \omega a_i^\dagger a_i, \\ H_I &= \int \left( \frac{d\rho}{2\pi} \right)^d \tilde{V}_d(u^2) e^{-\frac{\rho^2}{2\omega}} : e^{i\tilde{u}\rho} - 1 + \frac{u^2 r^2}{2d} : = \\ &= \int \left( \frac{d\rho}{\sqrt{\pi}} \right)^d e^{-\rho^2} V\left(\frac{\rho}{\sqrt{\omega}}\right) : e^{2\rho Q - Q^2} - 1 + Q^2 \left(1 - \frac{2\rho^2}{d}\right) : , \\ E_0 &= \frac{d\omega}{4} + \int \left( \frac{d\rho}{2\pi} \right)^d \tilde{V}_d(u^2) e^{-\frac{\rho^2}{2\omega}}. \end{aligned} \quad (3.14)$$

One can see that Eq.(3.13) defines the minimum of  $E_0$  in (3.14), i.e.

$$\begin{aligned} E_0 &= \min_{\omega} \left\{ \frac{d\omega}{4} + \int \left( \frac{d\rho}{2\pi} \right)^d \tilde{V}_d(u^2) e^{-\frac{\rho^2}{2\omega}} \right\} = \\ &= \min_{\omega} \left\{ \frac{d\omega}{4} + \int \left( \frac{d\rho}{\sqrt{\pi}} \right)^d e^{-\rho^2} V\left(\frac{\rho}{\sqrt{\omega}}\right) \right\} = \\ &= \min_{\omega} \left\{ \frac{D\omega}{2} + \int_0^\infty \frac{du u^{D-1} e^{-u}}{\Gamma(D)} V\left(\sqrt{\frac{u}{\omega}}\right) \right\}. \end{aligned} \quad (3.15)$$

where  $D = \frac{d}{2}$ .

Eqs.(3.13) and (3.15) can be rewritten in the form

$$\begin{aligned} \omega &= 2 \int_0^\infty \frac{du u^D e^{-u}}{\Gamma(D+1)} \cdot \frac{d}{du} V\left(\sqrt{\frac{u}{\omega}}\right), \\ E_0 &= \int_0^\infty \frac{du u^{D-1} e^{-u}}{\Gamma(D)} \cdot \frac{d}{du} \left[ u V\left(\sqrt{\frac{u}{\omega}}\right) \right]. \end{aligned} \quad (3.16)$$

Now let us calculate the second correction to the ground state energy. The first correction equals to zero:

$$E_1 = \langle 0|H_I|0\rangle = 0.$$

The second correction is defined by the standard formula

$$\begin{aligned} E_2 &= -\langle 0|H_I \frac{1}{H_0} H_I|0\rangle = -\frac{1}{\omega} \int \left( \frac{du_1}{2\pi} \right)^d \int \left( \frac{du_2}{2\pi} \right)^d \tilde{V}_d(u_1^2) \tilde{V}_d(u_2^2) e^{-\frac{u_1^2 + u_2^2}{2\omega}} \times \\ &\times \int_0^1 \frac{ds}{s} \left[ \cosh \frac{(u_1 u_2) s}{2\omega} - 1 - \frac{(u_1 u_2)^2}{8\omega^2} s^2 \right] = -\omega \sum_{n=2}^\infty \frac{\Gamma(D) C_n^2}{2\pi \cdot n! \Gamma(D+n)}, \\ C_n &= \frac{(4\omega)^D}{\omega} \int \left( \frac{dq}{2\pi} \right)^d \tilde{V}_d(4\omega q^2) (q^2)^n e^{-q^2} = \\ &= \frac{1}{\omega} \left( \frac{\omega}{\pi} \right)^D \int_0^\infty \frac{du u^{D+n-1} e^{-u}}{\Gamma(D)} \tilde{V}_d(4\omega u). \end{aligned} \quad (3.17)$$

All these formulas (3.14)–(3.17) give the desirable representation of the initial Hamiltonian (3.9) in the oscillator representation.

The next radial excitations in the oscillator representation can be constructed in the following way

$$|n\rangle = N(a_j^\dagger a_j^\dagger)^n |0\rangle, \quad N^2 = \frac{\Gamma(D)}{2^{2n} \Gamma(D+n)}, \quad D = \frac{d}{2}. \quad (3.18)$$

The proper energy value of this state is

$$H_0 |n\rangle = \omega(a_j^\dagger a_j) |n\rangle = 2n\omega |n\rangle, \quad (3.19)$$

i.e., we have the equidistant oscillator spectrum.

We want to stress that we can trust to the calculations of radial excitations in the lowest perturbation orders of the oscillator representation only for  $n = 0$  and, may be, for  $n = 1$  (see Section 5).

In the conclusion of this section we want to do the next remark. In our approach we use the conception of a space  $\mathbb{R}^d$  and use the algebra of the creation and annihilation operators. What does it mean from the point of view of the solution of our initial Schrödinger equation (3.3)? We claim that the solution of this equation is equivalent to the solution of Eq.(3.4) in the space  $\mathbb{R}^d$  for the ground state. Mathematically it means that the wave function of the ground state and its orbital excitations depend only on the radial variable  $r^2 = u$ :

$$\psi_0 = \psi_0(u) = |0\rangle = e^{-\frac{u}{2}} = e^{-\frac{r^2}{2}}.$$

Here we have omitted the normalization factor for simplicity. All radial excitations can be written in the form (3.18):

$$\begin{aligned} \psi_n = |n\rangle &= N_n (a_j^\dagger a_j^\dagger)^n |0\rangle = \\ &= N_n \left[ \frac{q}{u^{D-1}} e^{\frac{u}{2}} \frac{d}{du} u^D \frac{d}{du} e^{-\frac{u}{2}} \right]^n e^{-\frac{u}{2}} = P_n^{(D)}(u) e^{-\frac{u}{2}} \end{aligned}$$

where  $P_n^{(D)}(u)$  is a polynomial of the  $n$ -th order. The orthogonal condition is

$$\langle n|m\rangle = \int_0^\infty \frac{du u^{D-1} e^{-u}}{\Gamma(D)} P_n^{(D)}(u) P_m^{(D)}(u) = \delta_{nm}.$$

One can check easily that this condition is fulfilled for any positive  $D$ .

Thus one can say that we look for the solution of Eq.(3.3) in the class of polynomials which are orthogonal on the interval  $0 < u < \infty$  with the weight

$$\rho(u) = \frac{u^{D-1} e^{-u}}{\Gamma(D)}.$$

The dimension  $D$  of our space  $\mathbb{R}^d$  ( $D = \frac{d}{2}$ ) can be considered as an additional variational parameter and this parameter can be any positive number.

At the same time the algebra of operators  $a_j$  and  $a_j^\dagger$  gives the simple formalism to carry out all mathematical calculations.

## 4. EXAMPLES

We consider a few examples of application of OR to some well-known potentials. For all considered potentials we obtain the explicit formulas for  $E_{0\ell}^{(0)}$  in the lowest approximation and evaluate the accuracy as the relation

$$\delta_\ell = \left| \frac{E_{0\ell}^{(2)}}{E_{0\ell}^{(0)}} \right|.$$

It is turned out that the accuracy improves for large  $\ell$ .

### 4.1 The Coulomb plus linear potential

Here we consider the potential

$$V(r) = -\frac{\alpha}{r} + hr. \quad (4.1)$$

This potential is applied for the description of quarkonium (see, for example, [19]). It is convenient to go to the atom units

$$E = \frac{\alpha^2}{2} \varepsilon, \quad \vec{r} = \frac{\sqrt{2}}{\alpha} \vec{\rho} \quad (4.2)$$

so that the Schrödinger equation looks as

$$\left[ -\frac{1}{2} \Delta_3 + \sqrt{2} \left( -\frac{1}{\rho} + \lambda \rho \right) \right] \psi = \varepsilon \psi \quad (4.3)$$

where  $\lambda = \frac{2h}{\alpha^2}$ .

The formula (3.15) gives

$$\begin{aligned} \varepsilon_0 &= \max_\omega \left\{ \frac{d\omega}{4} - \sqrt{2}\omega \frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} + \lambda \sqrt{\frac{2}{\omega}} \frac{\Gamma(\frac{d+1}{2})}{\Gamma(\frac{d}{2})} \right\} = \\ &= \frac{(\ell + \frac{3}{2})}{2} \max_\omega \left\{ \omega - 2\sqrt{2}\omega \frac{\Gamma(\ell+1)}{\Gamma(\ell + \frac{3}{2})} + 2\lambda \sqrt{\frac{2}{\omega}} \frac{\Gamma(\ell+2)}{\Gamma(\ell + \frac{3}{2})} \right\}. \end{aligned}$$

The equation on  $\omega$  is

$$\omega = \frac{\Gamma(\ell+1)}{\Gamma(\ell + \frac{3}{2})} \sqrt{2\omega} + \lambda \sqrt{\frac{2}{\omega}} \frac{\Gamma(\ell+2)}{\Gamma(\ell + \frac{3}{2})} \quad (4.4)$$

and

$$\varepsilon_0 = \frac{1}{\sqrt{2}} \left\{ -\sqrt{\omega} \frac{\Gamma(\ell+1)}{\Gamma(\ell + \frac{3}{2})} + \frac{3\lambda}{\sqrt{2\omega}} \frac{\Gamma(\ell+2)}{\Gamma(\ell + \frac{3}{2})} \right\} \quad (4.5)$$

where  $\omega$  satisfies (4.4).

The solution of Eq.(4.4) can be written in the explicit form

$$\begin{aligned} \frac{1}{\sqrt{\omega}} &= \frac{F(\lambda, \ell)}{\sqrt{2} \lambda^{1/3}}, \\ F(\lambda, \ell) &= \left( \frac{\Gamma(\ell + \frac{5}{2})}{\Gamma(\ell + 2)} \right)^{1/3} \left[ R_+^{1/3} - R_-^{1/3} \right], \\ R_{\pm} &= \sqrt{1 + G(\lambda, \ell)} \pm 1, \\ G(\lambda, \ell) &= \frac{1}{\lambda} \left( \frac{2\Gamma(\ell + 1)}{3\Gamma(\ell + \frac{5}{2})} \right)^3 \frac{\Gamma(\ell + \frac{3}{2})}{\Gamma(\ell + 2)}. \end{aligned} \quad (4.6)$$

Substituting this solution into (4.5) one obtains

$$\varepsilon_0 = -\frac{\lambda^{1/3}}{F(\lambda, \ell)} \cdot \frac{\Gamma(\ell + 1)}{\Gamma(\ell + \frac{3}{2})} + \frac{3\lambda^{2/3} F(\lambda, \ell) \Gamma(\ell + 2)}{2\Gamma(\ell + \frac{3}{2})}. \quad (4.7)$$

The second perturbation correction can be calculated according to the formulas (3.17).

We get

$$\begin{aligned} \overline{W}_d(u^2) &= (4\pi)^{\frac{d}{2}} \sqrt{\frac{2}{\pi}} \left[ -\frac{\Gamma(\frac{d-1}{2})}{2(u^2)^{\frac{d-1}{2}}} - \lambda \frac{\Gamma(\frac{d+1}{2})}{(u^2)^{\frac{d+1}{2}}} \right], \\ C_n &= -\sqrt{\frac{2}{\pi\omega}} \cdot \frac{[\Gamma(\ell + 1)\Gamma(n + \frac{1}{2}) + \frac{\lambda^{1/3}}{4} F^2(\lambda, \ell) \Gamma(n - \frac{1}{2})]}{\Gamma(\ell + \frac{3}{2})} \end{aligned}$$

and

$$\begin{aligned} \varepsilon_2 &= -\frac{1}{\pi} \cdot \frac{\Gamma^2(\ell + 1)}{\Gamma(\ell + \frac{3}{2})} \cdot \sum_{n=2}^{\infty} \frac{[\Gamma(n + \frac{1}{2}) + \frac{\lambda^{1/3}}{4} (\ell + 1) F^2(\lambda, \ell) \Gamma(n - \frac{1}{2})]^2}{n \cdot n! \Gamma(n + \ell + \frac{3}{2})}, \\ \delta(\lambda, \ell) &= \left| \frac{\varepsilon_2(\lambda, \ell)}{\varepsilon_0(\lambda, \ell)} \right|. \end{aligned} \quad (4.8)$$

Here we shall not give the detailed numerical values of  $\delta(\lambda, \ell)$  for different  $\lambda$  and  $\ell$ . The general situation is characterized by Table 1 and consists in that for large  $\lambda$  we have practically the linear potential and for small  $\lambda$  the situation is close to the Coulomb potential for small orbital momenta  $\ell$ .

#### 4.2 The Coulomb potential

Here we give the results for the pure Coulomb potential

$$V(r) = -\frac{\alpha^2}{r}.$$

The formulas (4.5)–(4.8) give

$$E_{0\ell} = -\frac{\alpha^2}{2(\ell + 1)^2} C_{\ell}.$$

Here the principal quantum number

$$n = n_r + \ell + 1 = \ell + 1$$

since

$$\begin{aligned} n_r &= 0, \\ C_{\ell} &= \frac{\Gamma^2(\ell + 2)}{\Gamma(\ell + \frac{3}{2}) \Gamma(\ell + \frac{5}{2})} (1 + \delta_{\ell}) = C_{\ell}^{(0)} + C_{\ell}^{(2)}, \\ \delta_{\ell} &= \frac{1}{\pi} \sum_{n=2}^{\infty} \frac{\Gamma^2(n + \frac{1}{2}) \Gamma(\ell + \frac{5}{2})}{n \cdot n! \Gamma(n + \ell + \frac{3}{2})}. \end{aligned}$$

The numerical results are done in Table 2. We remind that the exact value is

$$C_{\ell} = 1.$$

#### 4.3 The linear potential

Now we represent the results for the pure linear potential

$$V(r) = hr.$$

In this case we have

$$\begin{aligned} E_{0\ell} &= \frac{3}{2} \left( \ell + \frac{3}{2} \right) \left[ \frac{h\Gamma(\ell + 2)}{\Gamma(\ell + \frac{5}{2})} \right]^{2/3} (1 - \delta_{\ell}) \\ \delta_{\ell} &= \frac{1}{12\pi} \sum_{n=2}^{\infty} \frac{\Gamma^2(n - \frac{1}{2}) \Gamma(\ell + \frac{5}{2})}{n \cdot n! \Gamma(n + \ell + \frac{3}{2})}. \end{aligned}$$

The numerical values of  $\delta_{\ell}$  are given in Table 2. One can see that the lowest approximation for  $E_{0\ell}$  gives practically exact result.

#### 4.4 The logarithmic potential

Here we give the results for the logarithmic potential

$$V(r) = g \ln \cdot \frac{r}{r_0}.$$

The application of all formulas formulated above lead to

$$\begin{aligned} E_{0\ell}^{(0)} &= -\frac{g}{2} \ln g \tau_0^2 + \frac{g}{2} \left[ 1 + \psi \left( \ell + \frac{3}{2} \right) + \ell n \left( \ell + \frac{3}{2} \right) \right] \\ E_{0\ell}^{(2)} &= +\frac{g}{2} \left[ -1 + \left( \ell + \frac{3}{2} \right) \psi' \left( \ell + \frac{3}{2} \right) \right] \end{aligned}$$

and

$$E_{0\ell} = E_{0\ell}^{(0)} + E_{0\ell}^{(2)} = -\frac{g}{2} \ell n g r_0^2 + \frac{g}{2} \left\{ \ell n \left( \ell + \frac{3}{2} \right) + \psi \left( \ell + \frac{3}{2} \right) + \left( \ell + \frac{3}{2} \right) \psi' \left( \ell + \frac{3}{2} \right) \right\}.$$

The value of

$$\delta_\ell = \left| \frac{E_{0\ell}^{(2)}}{E_{0\ell}^{(0)} + \frac{g}{2} \ell n g r_0^2} \right|$$

is given in Table 2.

## 5. THE ASYMPTOTIC BEHAVIOUR OF WAVE FUNCTIONS

It is known that all numerical calculations in quantum mechanics require a test wave function to have the correct asymptotical behaviour for large  $r$ . Usually the explicit form of this asymptotical behaviour can be found. The oscillator representation means that unknown true wave function is developed over the oscillator Gaussian functions. This expansion exists mathematically but it cannot be good enough numerically, i.e. the lowest terms of this expansion cannot give the result which is close to the true one.

Our experience shows that the oscillator representation works good if the energy level  $E$  is deep enough, i.e.  $E/V_0 \gtrsim 1$  where  $V_0$  is the depth of our potential. In this case the asymptotic behaviour of the true wave function is not so important and the oscillator approximation gives good results in the nearest perturbation orders.

But the situation is completely different if the energy level is shallow, i.e.  $E/V_0 \ll 1$ . For example, this situation takes place for deuteron. Then the asymptotic behaviour of wave function for large  $r$  plays the important role and the oscillator representation is not good in the lowest perturbation orders.

What can we do in this case? Let us consider a radial equation for the ground state in  $\mathbb{R}^3$

$$\left( -\frac{1}{2} \frac{d^2}{dr^2} + W(r) \right) R(r) = ER(r). \quad (5.1)$$

Usually we can find the asymptotic behaviour of the function  $R(r)$  for  $r \rightarrow \infty$ . Let this asymptotic be

$$R(r) \sim \exp(-a(r)). \quad (5.2)$$

Our oscillator functions behave as

$$|0\rangle \sim \exp(-r^2). \quad (5.3)$$

Therefore we should introduce a new variable  $r = r(s)$  in such a way that

$$a(r(s)) \sim s^2 \quad \text{for } s \rightarrow \infty$$

and

$$R(r(s)) \sim \exp\{-s^2\}.$$

As a result we can apply the oscillator representation to the equation written in this new variable.

Let us do these transformations. Substituting

$$r = r(s)$$

into (5.1) one can get

$$\left[ -\frac{1}{2} \frac{d^2}{ds^2} + \frac{1}{8} \left( \frac{r''(s)}{r'(s)} \right)^2 - \frac{1}{4} \left( \frac{r''(s)}{r'(s)} \right)' + (r'(s))^2 (W(r(s)) - E) \right] \phi(s) = 0. \quad (5.4)$$

Now the problem is to apply the oscillator representation to this equation.

We consider two examples. First, let the potential  $W(r)$  in (5.1) be the Coulomb potential with an orbital term

$$W(r) = \frac{\ell(\ell+1)}{2r^2} - \frac{\alpha}{r}. \quad (5.5)$$

We know that  $a(r) \sim r$  in this case, therefore we introduce

$$r = \frac{s^2}{2}. \quad (5.6)$$

Eq.(5.4) looks as

$$\left( -\frac{1}{2} \frac{d^2}{ds^2} + \frac{1}{2s^2} \left[ \frac{3}{4} + 4\ell(\ell+1) \right] - 2\alpha - Es^2 \right) \phi(s) = 0.$$

Observing that

$$\frac{3}{4} + 4\ell(\ell+1) = L_d(L_d+1), \quad d = 4 + 4\ell \quad (5.7)$$

we can see that this equation coincides with the equation

$$\left( -\frac{1}{2} \Delta_d + \frac{(-2E)}{2} s^2 \right) \phi(s) = 2\alpha\phi(s) \quad (5.8)$$

in the space  $\mathbb{R}^d$  for the ground state.

Thus the Coulomb problem is equivalent to the pure oscillator potential in  $\mathbb{R}^d$ . This result is well known.

The energy spectra and the radial wave functions can be found in this case. The wave functions are

$$|n_r\rangle = N (a_i^+ a_i^-)^{n_r} |0\rangle, \quad N = \sqrt{\frac{\Gamma(\frac{d}{2})}{2^{2n} \Gamma(\frac{d}{2} + n)}}$$

where  $n_r$  is a number of zeroes of this radial function.

$$H = \omega a_i^\dagger a_i + \frac{d}{2} \omega, \quad \omega = \sqrt{-2E}$$

$$H|n_r\rangle = \omega \left( 2n_r + \frac{d}{2} \right) |n_r\rangle = 2\alpha |n_r\rangle$$

and

$$E_{n\ell} = -\frac{\alpha^2}{2n^2}, \quad n = n_r + \ell + 1 \quad (5.9)$$

where  $n$  is the principal quantum number.

If we add to the Coulomb potential a linear part then instead of (5.9) one obtains

$$\left( -\frac{1}{2} \Delta_d + \frac{(-2E)}{2} s^2 + \frac{h}{4} s^4 \right) \Phi = 2\alpha \Phi. \quad (5.10)$$

Thus we have an anharmonic potential in  $\mathbb{R}^d$ .

The second example is the orbital excitations of an anharmonic oscillator in  $\mathbb{R}^3$

$$\left( -\frac{d}{2} \frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{2r^2} + hr^4 \right) R(r) = ER(r). \quad (5.11)$$

The true asymptotic of the function  $R(r)$  in (5.11) is

$$R(r) \sim \exp(-r^3)$$

therefore we introduce the new variable

$$r(s) = \frac{3}{2} s^{2/3}. \quad (5.12)$$

Then Eq.(5.4) looks as

$$\left( -\frac{d^2}{2ds^2} + \frac{1}{2s^2} \left( -\frac{5}{16} - \frac{4\ell(\ell+1)}{g} \right) + \frac{gh}{4} s^2 - \frac{E}{s^{2/3}} \right) \psi = 0.$$

We have

$$-\frac{5}{16} + \frac{4\ell(\ell+1)}{g} = L_d(L_d+1), \quad d = \frac{8+4\ell}{3}. \quad (5.13)$$

Eq.(5.14) obtains the form

$$\left( -\frac{1}{2} \Delta_d + \frac{gh}{4} s^2 - \frac{E}{s^{2/3}} \right) \psi = 0. \quad (5.14)$$

The nonentire dimension  $d$  in this equation should not confuse us because all formulas depend continuously on the parameter  $d$ .

Eq.(5.13) describes the harmonic oscillator with an attractive weakly singular potential in the space  $\mathbb{R}^d$ . The coupling constant of this singular potential is our unknown energy. Thus the spectrum of the anharmonic oscillator (5.11) coincides with the value of the "coupling constant  $E$ " in the potential (5.14) for which the ground state energy of this Eq.(5.14) in the space  $\mathbb{R}^d$  ( $d = \frac{1}{3}(8+4\ell)$ ) equals zero.

Generally speaking a new variable can be introduced in the form

$$r(s) = s^\sigma$$

and the parameter  $\sigma$  can be considered as an additional variational parameter.

## 6. RELATIVISTIC SCHRÖDINGER EQUATION

Here we consider the Schrödinger equation in  $\mathbb{R}^3$  of the following type

$$\left[ \sqrt{p^2 + m^2} + V(r) \right] \psi = E\psi. \quad (6.1)$$

This equation is an attempt to take into account the relativistic character of interaction of quarks or other elementary particles. All motivations can be found, for example, in [13–17].

Here we apply our oscillator representation for calculation orbital excitations energy of the ground state, i.e.

$$\psi(r, \theta, \varphi) = Y_{\ell m}(\theta, \varphi) \psi_\ell(r) \quad (6.2)$$

where  $\psi_\ell(r)$  has no zeroes for  $r > 0$ . Substituting (6.2) into (5.1) one obtains

$$\left[ \sqrt{-\frac{1}{r} \left( \frac{d}{dr} \right)^2 r + \frac{\ell(\ell+1)}{r^2} + m^2} + V(r) \right] \psi_\ell^{(2)} = E\ell \psi_\ell^{(2)}. \quad (6.3)$$

Let us introduce the function

$$\psi_\ell(r) = r^\ell R_\ell(r),$$

$$L_d = \frac{d-3}{2} = \ell, \quad d = 2\ell + 3.$$

Eq.(6.3) becomes the form

$$\left[ \sqrt{-\frac{1}{r^{\frac{d-1}{2}}} \left( \frac{d}{dr} \right)^2 r^{\frac{d-1}{2}} + \frac{L_d(L_d-1)}{r^2} + m^2} + V(r) \right] R(r) = ER(r).$$

This equation coincides with the radial Schrödinger equation in the space  $\mathbb{R}^d$ :

$$\left[ \sqrt{-\Delta_d + m^2} + V(r) \right] \psi_d(r) = E\psi_d(r). \quad (6.4)$$

Now we can use the oscillator representation for the Hamiltonian

$$H = \sqrt{p^2 + m^2} + V(r) \quad (6.5)$$

in the space  $\mathbb{R}^d$ . Let us rewrite this Hamiltonian in the form

$$H = \frac{1}{2\mu} (p^2 + \kappa^2 r^2) + \left[ \sqrt{p^2 + m^2} - \frac{p^2}{2\mu} \right] + \left[ V(r) - \frac{\kappa^2}{2\mu} r^2 \right] \quad (6.6)$$

and introduce the oscillator canonical variables:

$$r_i = \frac{Q_i}{\sqrt{\kappa}}, \quad Q_i = \frac{a_i + a_i^\dagger}{\sqrt{2}} \quad (i = 1, \dots, d), \quad (6.7)$$

$$p_i = \sqrt{\kappa} P_i, \quad P_i = \frac{a_i - a_i^\dagger}{i\sqrt{2}}$$

Then one can get

$$\langle 0 | r_i r_j | 0 \rangle = \frac{\delta_{ij}}{2\kappa}, \quad \langle 0 | p_i p_j | 0 \rangle = \delta_{ij} \frac{\kappa}{2}.$$

Now let us substitute the representation (6.7) in (6.6), go over to the normal product of the operators  $a_i$  and  $a_i^+$  in the Hamiltonian and require that the interaction Hamiltonian should not contain terms with  $: p^2 :$  and  $: r^2 :$ . After some transformations one can obtain

$$\begin{aligned} H &= H_0 + H_I + E_0, \\ H_0 &= \omega a_i^+ a_i, \quad \omega = \frac{\kappa}{\mu}, \\ H_I &= \int \left( \frac{d\rho}{\sqrt{\pi}} \right)^d e^{-\rho^2} \left[ \sqrt{\rho^2 \kappa + m^2} : e^{-2\rho P - P^2} - 1 + P^2 \left( 1 - \frac{2\rho^2}{d} \right) : \right. \\ &\quad \left. + V \left( \frac{\rho}{\sqrt{\kappa}} \right) : e^{-2\rho Q - Q^2} - 1 + Q^2 \left( 1 - \frac{2\rho^2}{d} \right) : \right] = \\ &= \int \left( \frac{du}{2\pi} \right)^d \left[ \tilde{K}_d(u^2) e^{-\frac{u^2}{\kappa}} : e_2^{iup} : + \tilde{V}_d(u^2) e^{-\frac{u^2}{\kappa}} : e_2^{iur} : \right] \end{aligned} \quad (6.8)$$

where

$$\begin{aligned} \tilde{K}_d(u^2) &= \int (d\rho)^d e^{iup} \sqrt{\rho^2 + m^2}, \\ e_2^z &= e^z - 1 - z - \frac{z^2}{2}, \\ E_0 &= \min_{\kappa} \int \left( \frac{d\rho}{\sqrt{\pi}} \right)^d e^{-\rho^2} \left[ \sqrt{\rho^2 \kappa + m^2} + V \left( \frac{\rho}{\sqrt{\kappa}} \right) \right] = \\ &= \min_{\kappa} \int_0^{\infty} \frac{du u^{\frac{d}{2}-1} e^{-u}}{\Gamma(\frac{d}{2})} \left[ \sqrt{u\kappa + m^2} + V \left( \sqrt{\frac{u}{\kappa}} \right) \right]. \end{aligned}$$

The parameters  $\kappa$  and  $\omega$  are defined by the equations

$$\begin{aligned} \int_0^{\infty} du u^{\frac{d}{2}-1} e^{-u} \frac{d}{du} \left[ \sqrt{u\kappa + m^2} - V \left( \sqrt{\frac{u}{\kappa}} \right) \right] &= 0, \\ \omega &= \frac{2}{\Gamma(\frac{d}{2} + 1)} \int_0^{\infty} du u^{\frac{d}{2}-1} e^{-u} \cdot \frac{d}{du} V \left( \sqrt{\frac{u}{\kappa}} \right). \end{aligned} \quad (6.9)$$

As an example let us consider the Hamiltonian

$$H = \sqrt{p^2} - \frac{\alpha}{r} + hr. \quad (6.10)$$

Eq.(5.9) can be solved easily:

$$\begin{aligned} \kappa &= \frac{h}{1 - \alpha_{\ell}}, \quad \alpha_{\ell} = \frac{\alpha}{\ell + 1} \\ \omega &= \sqrt{\frac{h}{1 - \alpha_{\ell}}} \cdot \frac{\Gamma(\ell + 2)}{\Gamma(\ell + \frac{3}{2})}. \end{aligned} \quad (6.11)$$

One can see if  $\alpha > 1$  then there exists "the downfall on the centre", i.e. there is no stable state in this system (see [16, 17]).

The Hamiltonian in the oscillator representation looks as

$$\begin{aligned} H &= \omega a_i^+ a_i + H_I + E_0, \\ H_I &= -\sqrt{\frac{h}{1 - \alpha_{\ell}}} \cdot \frac{\Gamma(\ell + 2)}{\pi^{\ell+2}} \int \frac{(du)^d}{(u^2)^{\ell+2}} e^{-\frac{u^2}{\kappa}} \left[ : e_2^{iup} : + \left( 1 - \alpha_{\ell} + \frac{\alpha_{\ell}}{2} u^2 \right) : e_2^{iuQ} : \right], \\ E_0 &= 2\sqrt{h(1 - \alpha_{\ell})} \cdot \frac{\Gamma(\ell + 2)}{\Gamma(\ell + \frac{3}{2})}. \end{aligned} \quad (6.12)$$

The second correction can be calculated. We give this formula for the case  $\alpha = 0$ . The result is

$$\begin{aligned} E_2 &= -E_0 \delta_{\ell} \\ \delta_{\ell} &= \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\Gamma(\ell + 2) \Gamma(2n + \frac{1}{2})}{\Gamma(2n + \ell + \frac{1}{2})} \cdot \frac{2^{4n-1} \Gamma^2(2n - \frac{1}{2})}{\Gamma(4n + 2)} \end{aligned} \quad (6.13)$$

and

$$E = E_0 + E_2 = 2\sqrt{h} \frac{\Gamma(\ell + 2)}{\Gamma(\ell + \frac{3}{2})} (1 - \delta_{\ell}).$$

The numerical values of  $\delta_{\ell}$  are given in Table 2. One can see that the second correction in the case of the linear potential are very small. Thus we can say that the formula (6.12) gives the orbital excitations of the ground state with very good accuracy.

In the conclusion we want to note that for large  $\ell$  the energy (6.12) behaves as

$$E_0 = E_{0\ell} \simeq 2\sqrt{h\ell}.$$

For the linear potential in the case of the standard quantum mechanics we have

$$E_0 = E_{0\ell} \simeq \frac{3}{2} (h\ell)^{2/3}.$$

## 7. THREE BODY PROBLEM

In this section we shall consider the application of OR in the Coulomb three-body problem, namely we shall compute the ground state energy. It is known that the harmonic oscillator model is one of the most important models in molecular physics. There are plenty of papers and books which are concerned with the applications of the harmonic oscillator to the study of the spectra of molecules (see, for example, [9,10, 18-22] and so on). Our contribution into development of this oscillator approach to the Coulomb three-body problem contains two new points.

First, we choose the coordinate system where there are no any angle variables and all variables are changed in the infinite limits, so that we can apply OR for each variable.

Second, we dump the Coulomb repulsion in this three-body system by going to a space of the highest dimension as it was done in the previous sections.

As a result we obtain the oscillator representation of the three-body Hamiltonian which is equivalent to the initial one. In this paper we shall consider the lowest approximation only. The next corrections can be calculated but we shall not compute them here.

Now let us consider the Hamiltonian:

$$H = \frac{1}{2} \left( \frac{\vec{p}_1^2}{m_1} + \frac{\vec{p}_2^2}{m_2} + \frac{\vec{p}_3^2}{m_3} \right) + \frac{Z_1 Z_2 \alpha}{|\vec{r}_1 - \vec{r}_2|} - \frac{Z_1 Z_3 \alpha}{|\vec{r}_2 - \vec{r}_3|} - \frac{Z_2 Z_3 \alpha}{|\vec{r}_2 - \vec{r}_3|}. \quad (7.1)$$

There are three charged particles which have the following charges

$$Q_1 = Z_1 e, \quad Q_2 = Z_2 e, \quad Q_3 = -Z_3 e$$

where  $Z_1, Z_2, Z_3$  are numbers of the same sign.

Let us introduce the standard transformations

$$\begin{aligned} \vec{r}_1 &= a \vec{s}_1 + b \vec{u}_1 + \vec{R}, \\ \vec{r}_2 &= (a-1) \vec{s}_1 + b \vec{u}_1 + \vec{R}, \\ \vec{r}_3 &= +(b-1) \vec{u}_1 + \vec{R}, \\ a &= \frac{m_1}{m_1 + m_2}, \quad b = \frac{m_3}{m_1 + m_2 + m_3}. \end{aligned}$$

The next transformation is

$$\vec{s}_1 = \frac{\vec{s}}{Z_1 Z_2 e^2 M}, \quad \vec{u}_1 = \frac{\vec{u}}{Z_1 Z_2 e^2 \sqrt{M \mu}}$$

where

$$M = \frac{m_1 m_2}{m_1 + m_2}, \quad \mu = \frac{(m_1 + m_2) m_3}{m_1 + m_2 + m_3}.$$

The Hamiltonian (7.1) can be written in the form

$$H = \theta_0 \left\{ \frac{1}{2} (\vec{p}_s^2 + \vec{p}_u^2) + \frac{1}{|\vec{s}|} - \frac{A_1}{|\vec{u} - \gamma_1 \vec{s}|} - \frac{A_2}{|\vec{u} + \gamma_2 \vec{s}|} \right\}. \quad (7.2)$$

Here the following notations are introduced:

$$\begin{aligned} A_j &= \frac{Z_j}{Z_j} \delta, \quad \gamma_j = \frac{m_j}{m_1 + m_2} \delta, \quad (j = 1, 2) \\ \delta^2 &= \frac{\mu}{M}, \quad \theta_0 = (Z_1 Z_2 \alpha)^2 M. \end{aligned} \quad (7.3)$$

In the Hamiltonian (7.2) we have omitted the kinetic energy of the total system.

One can see that in (7.2) there is the repulsive term and two attractive ones in the interaction potential. Our idea is the following.

First, we divide the vector  $\vec{u} \in \mathbb{R}^3$  into two parts namely one is perpendicular to  $\vec{s}$ :

$$\vec{v} = \vec{u}_\perp \in \mathbb{R}^2, \quad (\vec{v} \vec{s}) = 0,$$

and another one is parallel to  $\vec{s}$ :

$$q = \vec{u}_\parallel \in \mathbb{R}^1, \quad (\vec{u}_\parallel \vec{s}) = |\vec{s}| q.$$

Then we have

$$(\vec{u} - \gamma \vec{s})^2 = \gamma^2 s^2 + v^2 + q^2 - 2\gamma s q$$

where  $s = |\vec{s}|, v = |\vec{v}|$ . Thus our potential in (7.2) depends on  $s = |\vec{s}|$  only.

In this representation our Hamiltonian (7.2) and the Schrödinger equations look as

$$\begin{aligned} H = \theta_0 \left\{ -\frac{1}{2} \left[ \left( \frac{1}{s} \cdot \frac{d^2}{ds^2} s \right) + \left( \frac{1}{\sqrt{v}} \frac{d^2}{dv^2} \sqrt{v} - \frac{1}{4v^2} \right) + \frac{d^2}{dq^2} \right] + \right. \\ \left. + \frac{1}{s} - \frac{A_1}{\sqrt{\gamma_1^2 s^2 + v^2 + q^2 - 2\gamma_1 s q}} - \frac{A_2}{\sqrt{\gamma_2^2 s^2 + v^2 + q^2 + 2\gamma_2 s q}} \right\}, \\ H \psi(s, v, q) = E \psi(s, v, q). \end{aligned} \quad (7.4)$$

Let us discuss a qualitative character of the potential in this Hamiltonian. Over the variable  $v$  the potential has minimum for small values of  $v$  and, therefore, we can hope that OR will be good for this variable. Very likely the same situation takes place for the variable  $q$ . Here we have the one-dimensional Schrödinger equation and generally speaking we should introduce a displacement into the potential  $q \rightarrow q + q_0$  as it was done in Section 2. The parameter  $q_0$  should be considered as an additional variational parameter. However, we will not introduce this parameter  $q_0$  in this paper.

The most interesting situation is over the variable  $s$ . For small  $s$  we have a strong repulsion and an attractive force exists only for large  $s$ . Thus the wave function of the ground state has its maximum for a large  $s$ .

Hence we can assume that the wave function  $\psi(s, v, q)$  in (7.4) has its maximum for

$$v = q = 0 \quad \text{and} \quad s \approx s_0 \neq 0.$$

Now we can proceed in two ways. One of them is to go over  $s$  to a space  $\mathbb{R}^d$  for  $d \geq 3$  as it was done in Section 3. The other way is to introduce a new variable instead of  $s$  in order to take into account the true asymptotic behaviour of our wave function as it was done in Section 5. We will consider both these approaches.

## 7.1 The space $\mathbb{R}^d \otimes \mathbb{R}^2 \otimes \mathbb{R}^1$

The Hamiltonian in this case looks as

$$H = \theta_0 \left\{ \frac{1}{2}(p_s^2 + p_v^2 + p_q^2) + \frac{1}{s} - \frac{(d-1)(d-3)}{8s^2} - V \right\}, \quad (7.5)$$

$$V = \frac{A_1}{\sqrt{\gamma_1^2 s^2 + v^2 + q^2 - 2\gamma_1 sq}} + \frac{A_2}{\sqrt{\gamma_2^2 s^2 + v^2 + q^2 + 2\gamma_2 sq}}.$$

Here  $s \in \mathbb{R}^d, v \in \mathbb{R}^2, q \in \mathbb{R}^1$ .

Now we can apply OR to (7.5), we have

$$H = \theta_0 \left\{ \frac{1}{2}(p_s^2 + \Omega^2 s^2) + \frac{1}{2}(p_v^2 + \omega^2 v^2) + \frac{1}{2}(p_q^2 + \omega_q^2 q^2) + \left[ \frac{1}{s} - \frac{(d-1)(d-3)}{8s^2} - V - \frac{1}{2}(\Omega^2 s^2 + \omega^2 v^2 + \omega_q^2 q^2) \right] \right\} = \theta_0 \{ \Omega a_i^+ a_i + \omega \bar{a}_v^+ \bar{a}_v + \omega_q a_q^+ a_q + H_I + \varepsilon_0 \}. \quad (7.6)$$

The interaction Hamiltonian  $H_I$  in (7.6) can be written in the way described above but we will not write out it here because in this paper we restrict ourselves by calculations of the lowest approximation only.

The main point of our interest here is the energy

$$E_0 = \theta_0 \varepsilon_0,$$

$$\varepsilon_0 = \min_{(\Omega, \omega, \omega_q, d)} \left\{ \frac{d}{4} \Omega + \frac{1}{2} \omega + \frac{1}{4} \omega_q + \int d\Sigma \left[ \frac{\sqrt{\Omega}}{s} - \frac{(d-1)(d-3)}{8} \cdot \frac{\Omega}{s^2} \right] - \bar{V}_d(\Omega, \omega, \omega_q) \right\},$$

$$\bar{V}_d(\Omega, \omega, \omega_q) = \int d\Sigma \cdot V \left( \frac{s}{\sqrt{\Omega}}, \frac{v}{\sqrt{\omega}}, \frac{q}{\sqrt{\omega_q}} \right),$$

$$d\Sigma = \left( \frac{ds}{\sqrt{\pi}} \right)^d e^{-s^2} \cdot \left( \frac{dv}{\sqrt{\pi}} \right)^2 e^{-v^2} \cdot \left( \frac{dq}{\sqrt{\pi}} \right) e^{-q^2}.$$

Let us introduce the new variables

$$\frac{1}{\omega} = \frac{\delta^2}{4\Omega} a, \quad \frac{1}{\omega_q} = \frac{\delta^2}{4\Omega} b, \quad D = \frac{d}{2}.$$

Then for  $\bar{V}_d(\Omega, \omega, \omega_q)$  one can get

$$\bar{V}_d(\Omega, \omega, \omega_q) = \sqrt{\Omega} N_D(a, b),$$

$$N_D(a, b) = B_1 K_D(a, b; c_1) + B_2 K_D(a, b; c_2),$$

$$K_D(a, b; c) = \int \frac{d\Sigma}{\sqrt{c^2 s^2 + av^2 + bq^2 \pm 2c\sqrt{b} sq}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{dt(t+b)^{D-\frac{1}{2}}}{(t+a)(t+b+c^2)^D} \quad (7.7)$$

where

$$B_j = \frac{2Z_j}{Z_j}, \quad c_j = \frac{2m_j}{m_1 + m_2} \quad (j = 1, 2).$$

We can compute the minimum over  $\Omega$ . After simple transformations we obtain

$$E_0 = -\frac{(Z_1 Z_2 \alpha)^2}{4} \mu U$$

$$R = \max_{a, b, D} \frac{F_D(a, b)}{\sqrt{\frac{2}{a} + \frac{1}{b} + \delta^2} \cdot \frac{4(D-3)}{8(D-1)}},$$

$$F_D(a, b) = N_D(a, b) - \frac{\Gamma(D - \frac{1}{2})}{\Gamma(D)}, \quad U = R^2. \quad (7.8)$$

This formula (7.8) defines the ground state energy in the lowest approximation.

In the symmetrical case

$$Z_1 = Z_2 = Z, \quad Z_3 = Z_0,$$

$$m_1 = m_2 = m, \quad m_3 = m_0,$$

$$c_1 = c_2 = 1, \quad M = \frac{m}{2}, \quad \mu = \frac{2m m_0}{2m + m_0},$$

$$\delta^2 = \frac{\mu}{M} = \frac{4m_0}{2m + m_0} \quad (7.9)$$

the expression (7.8) gets simplified a little bit:

$$E_0 \leq -\frac{(Z^2 \alpha)^2}{2} \cdot \frac{m m_0}{2m + m_0} \cdot R^2,$$

$$R = \max_{a, b, D} \frac{\left[ \frac{4Z_0}{2} K_D(a, b; 1) - \frac{\Gamma(D - \frac{1}{2})}{\Gamma(D)} \right]}{\sqrt{\frac{2}{a} + \frac{1}{b} + \frac{m_0}{2m + m_0}} \cdot \frac{4(D-3)}{2(D-1)}} \quad (7.10)$$

## 7.2 The "asymptotical" approach

Let us in the Hamiltonian and the Schrödinger equation (7.4) carry out the substitutions

$$s \rightarrow \frac{s^2}{2}, \quad \psi \rightarrow \frac{1}{s^2} \psi.$$

According to (5.4) and (5.8) one obtains the following Schrödinger equation:

$$\begin{aligned} & \left[ -\frac{1}{2} \Delta_4 + \frac{Q}{2} s^2 - s^2 \left[ \frac{1}{2} \Delta_2 + \frac{1}{2} \Delta_1 \right] - W \right] \psi = -2\psi, \\ W &= \frac{A_1 s^2}{\sqrt{\frac{\gamma_1}{4} s^4 + v^2 + q^2 - \gamma_1 s^2 q}} + \frac{A_2 s^2}{\sqrt{\frac{\gamma_2}{4} s^4 + v^2 + q^2 + \gamma_2 s^2 q}}, \\ Q &= -\frac{2E}{\theta_0}. \end{aligned} \quad (7.11)$$

Over the variable  $s$  we should go to a space  $\mathbf{R}^d$  because it is not known where the wave function  $\psi(s, u, q)$  has its maximum. The Hamiltonian looks as

$$H = \frac{1}{2} p_s^2 + s^2 \left[ \frac{1}{2} p_v^2 + \frac{1}{2} p_q^2 \right] - \frac{d(d-4)}{8s^2} + \frac{Q}{2} s^2 - W \quad (7.12)$$

where  $s \in \mathbf{R}^d, v \in \mathbf{R}^2, q \in \mathbf{R}^1$ .

Now let us write down this Hamiltonian in OR:

$$\begin{aligned} H &= \frac{1}{2} (p_s^2 + \Omega^2 s^2) + \frac{1}{2\mu} (p_v^2 + \kappa^2 v^2) + \frac{1}{2\nu} (p_q^2 + \sigma^2 q^2) + \\ &+ \left[ \frac{1}{2} p_v^2 \left( s^2 - \frac{1}{\mu} \right) + \frac{1}{2} p_q^2 \left( s^2 - \frac{1}{\nu} \right) \right] - W + \frac{Q}{2} s^2 - \\ &- \frac{d(d-4)}{8s^2} - \frac{\Omega^2}{2} s^2 - \frac{\kappa^2}{2\mu} v^2 - \frac{\sigma^2}{2\nu} q^2 = \\ &= \Omega a_i^\dagger a_i + \frac{\kappa}{\mu} \bar{a}_v^\dagger \bar{a}_v + \frac{\sigma}{\nu} a_q^\dagger a_q + H_I + \varepsilon_0. \end{aligned} \quad (7.13)$$

Using the results of the previous sections one can get

$$\frac{1}{\mu} = \frac{1}{\nu} = \frac{d}{2\Omega},$$

and

$$\begin{aligned} \varepsilon_0 &= \min \left\{ \left( \frac{d}{4} - \frac{d(d-4)}{4(d-2)} \right) \Omega + \frac{d}{4} \left[ \kappa + \frac{1}{2} \sigma + Q \right] \frac{1}{\Omega} - \overline{W}(\Omega, x, \sigma) \right\}, \\ \overline{W}(\Omega, x, \sigma) &= \int d\mathcal{L} W \left( \frac{s}{\sqrt{\Omega}}, \frac{v}{\sqrt{\kappa}}, \frac{q}{\sqrt{\sigma}} \right). \end{aligned}$$

As examples, let us apply the formula (7.10) to three systems: helium, ion  $H_2^+$  and ion ( $ZZe$ ) where  $Z \gg 1$ .

In the case of helium we have

$$Z = 1, \quad Z_0 = 2, \quad m = m_e, \quad m_0 = m_{He}$$

and

$$\begin{aligned} E_0 &\leq -\frac{\alpha^2}{2} m_e U \\ R &= \max_{a,b,D} \frac{\left[ 8 K_d(a, b; 1) - \frac{\Gamma(D-\frac{1}{2})}{\Gamma(D)} \right]}{\sqrt{\frac{2}{a} + \frac{1}{b} + \frac{4D-3}{2(D-1)}}}. \end{aligned}$$

The numerical calculation gives

$$U = R^2 = 5.24$$

for

$$D = 3.1, \quad a = 0.51, \quad b = 1.65.$$

The experimental value of  $U$  is

$$U_{\text{exp}} = 5.80.$$

The standard oscillator variational estimation (see [9]) gives

$$U_{\text{var}} = 4.6.$$

One can see that our OR result in the lowest approximation gives accuracy  $\sim 90\%$  and practically coincides with the standard perturbation calculations (see, for example, [18]).

In the case of ion  $H_2^+$  we have

$$Z = Z_0 = 1, \quad m = m_p, \quad m_0 = m_e$$

and  $\frac{m_x}{m_p} \ll 1$

$$\begin{aligned} E_0 &= -\frac{\alpha^2}{2} m_e U, \\ R &= \max_{a,b,D} \frac{\left[ 4 K_d(a, b; 1) - \frac{\Gamma(D-\frac{1}{2})}{\Gamma(D)} \right]}{\sqrt{\frac{2}{a} + \frac{1}{b}}}. \end{aligned}$$

The maximum is for large  $D$ . If  $1 \ll D \ll \frac{m_p}{m_e}$  then introducing the new variables  $a, b, t \rightarrow Da, Db, Dt$  one can get

$$R = \max_{a,b,D} \frac{\left[ \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{dt}{\sqrt{t+b}(t+a)} \exp\left(-\frac{1}{t+b}\right) - 1 \right]}{\left[ \frac{2}{a} + \frac{1}{b} \right]}.$$

The numerical computing gives

$$U = 1.068.$$

The exact result is

$$U_{\text{exp}} = 1.194.$$

One can see that the accuracy is about  $\approx 90\%$ .

At last let us consider an ion ( $ZZe$ ) with  $Z \gg 1$  and  $m \gg m_e$ , then

$$E_0 = -\frac{(Z\alpha)^2}{4} m_e U$$

$$R = \max_{a,b,D} \frac{\left[ \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{dt(t+b)^{D-\frac{1}{2}}}{(t+a)(t+b+1)^D} - Z \frac{\Gamma(D-\frac{1}{2})}{\Gamma(D)} \right]}{\sqrt{\frac{2}{a} + \frac{1}{b}}}$$

$D$  is large and we get

$$R = \max_{a,b} \frac{\left[ \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{dt}{(t+a)\sqrt{t+b}} \exp\left(-\frac{1}{t+b}\right) - Z \right]}{\sqrt{\frac{2}{a} + \frac{1}{b}}}$$

If  $Z \gg 1$  then the parameter  $a$  should be small and

$$R = \max_{a,b} \frac{\left[ \frac{4}{\sqrt{\pi}} \frac{1}{\sqrt{b}} e^{-\frac{1}{2} \ln \frac{1}{a}} - Z \right]}{\sqrt{\frac{2}{a} + \frac{1}{b}}} = \frac{4}{\sqrt{\pi e^3}} e^{-\frac{1}{2} Z \sqrt{\pi}}$$

and

$$E_0 = -\frac{(Z\alpha)^2}{2} m_e \cdot \frac{8}{\pi e^3} e^{-Z \sqrt{\pi}}$$

One can see that there exists an extremely strong repulsion in this system but nevertheless the bound state exists.

Thus we can say that the OR gives the 90% accuracy in the lowest approximation.

#### Acknowledgments

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste. The author also wishes to thank V.B. Belyaev, P. Colangelo, S.B. Gerasinov, A.B. Govorkov, A.V. Matveenko, S. Randjbar-Daemi and J. Wrzecionko for instructive and enjoyable discussions.

#### REFERENCES

- [1] C. Quigg and J.L. Rosner, *Physics Reports* **56** (1979) 167.
- [2] J. Zinn-Justin, *Physics Reports* **70** (1981) 109.
- [3] B.L. Schumaker, *Physics Reports* **135** (1986) 317.
- [4] E. Papp, *Physics Reports* **136** (1986) 103.
- [5] A. Chatterjee, *Physics Reports* **188** (1990) 249.
- [6] W.E. Caswell, *Ann. Phys.* **123** (1979) 153.
- [7] I.D. Feranchuk and L.I. Komarov, *Phys. Lett.* **88** (1982) 211; **125** (1987) 123.
- [8] A. Messiah, *Quantum Mechanics*, Vol.II (North-Holland Publishing Co., 1965).
- [9] M. Moshinsky, *The Harmonic Oscillator in Modern Physics: From Atoms to Quarks* (Gordon and Breach, Science Publishers, New York, 1969).
- [10] *Few Body Problems in Physics*, ed. H.W. Fearing (North-Holland Publishing Co., 1990).
- [11] R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
- [12] G.V. Efimov, *Int. J. Mod. Phys. A* **4** (1989) 4977; preprint ICTP, Trieste, No.IC/90/23.
- [13] D.B. Lichtenberg, *Int. J. Mod. Phys. A* **2** (1987) 1668.
- [14] P. Colangelo, G. Nardulli and M. Pietroni, preprint, BARI No.TH/90-70.
- [15] J.L. Basdevant, P. Colangelo and G. Preparata, *Il Nuovo Cimento*, **A71** (1982) 445.
- [16] J. Finger, D. Horn and J. Mandule, *Phys. Rev.* **D20** (1979) 3253.
- [17] L. Durand, *Phys. Rev.* **D32** (1985) 1257.
- [18] C.J.H. Schutte, *The Theory of Molecular Spectroscopy* (North-Holland Publishing Co., Amsterdam, 1976).
- [19] V.V. Gusev, V.I. Puzynin, V.V. Kostykin, A.A. Kvitsinsky, S.P. Merkuriev and L.I. Ponomerev, *Few-Body Systems* **9** (1990) 137.
- [20] S. Rosanti, M. Viviani and A. Kievsky, *Few-Body Systems* **9** (1990) 1.
- [21] E. Ficocelli Varracchio, *Few-Body Systems* **8** (1990) 65.
- [22] E. Cravo and A.C. Fonseca, *Few-Body Systems* **5** (1988) 117.

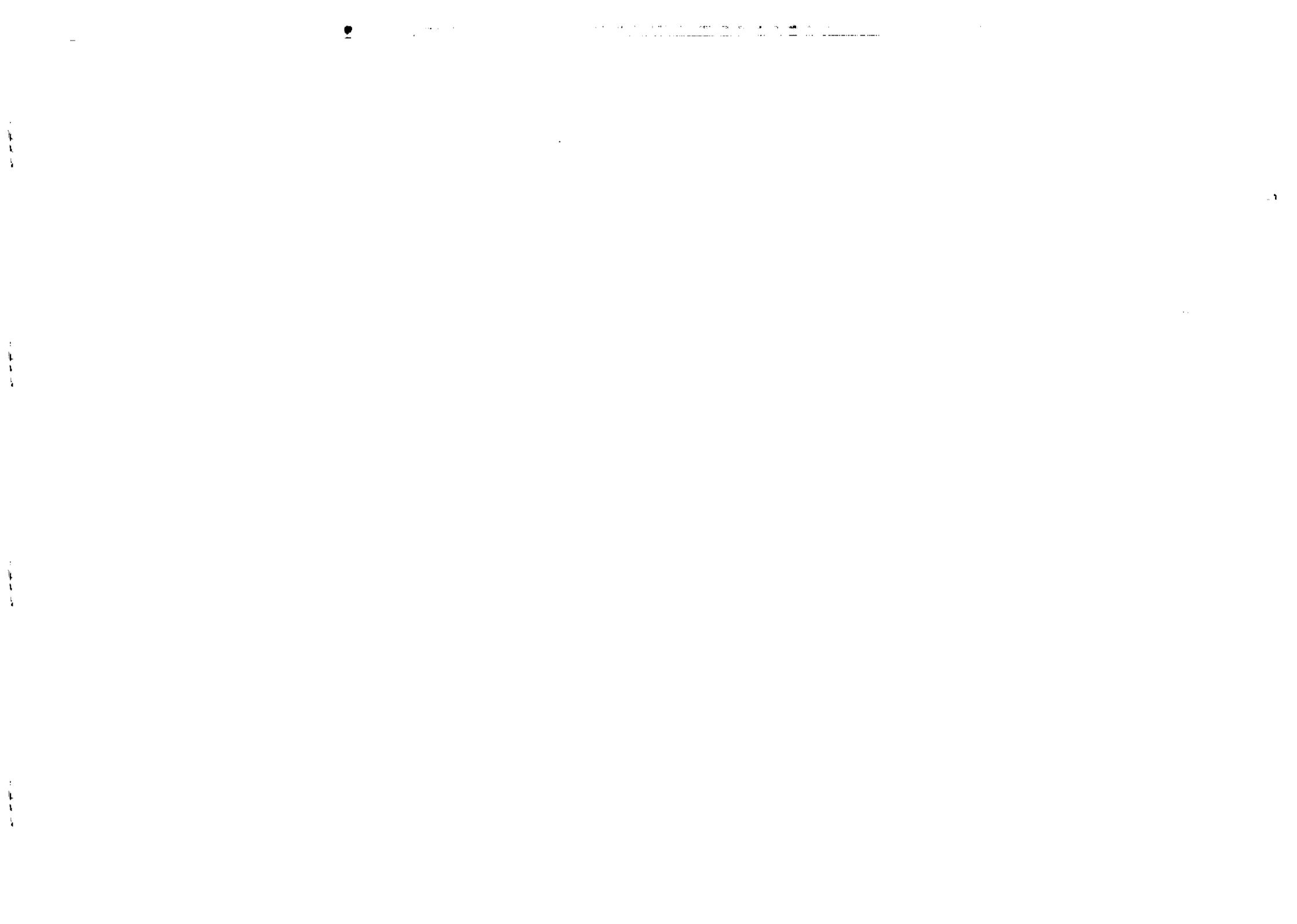
Table 1

The estimation of  $\delta(\lambda, \ell)$  in (4.8)

$\ell$	Linear potential	Coulomb potential
	$\lambda \gtrsim 1$	$\lambda \lesssim 1$
$\ell = 0.1$	$\delta \approx 1 \div 2\%$	$\delta \lesssim 15\%$
$\ell \geq 2$	$\delta < 1\%$	$\delta \lesssim 1\%$

Table 2

$\ell$	Coulomb potential			Linear potential	Logarithmic potential	Linear potential in "relativistic" equation
	$C_e^{(0)}$	$C_e^{(0)} + C_e^{(2)}$	$\delta_e$			
				$\delta_e$	$\delta_e$	$\delta_e$
0	0.849	0.948	0.116	0.003	0.279	0.006
1	0.905	0.964	0.065	0.002	0.086	0.004
2	0.931	0.973	0.045	0.001	0.047	0.003
4	0.956	0.982	0.027	0.000	0.022	0.003
6	0.967	0.986	0.020		0.014	0.002
8	0.974	0.989	0.015		0.010	0.002
10	0.979	0.991	0.013		0.008	0.002





Stampato in proprio nella tipografia  
del Centro Internazionale di Fisica Teorica