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THE q–DEFORMED $SU(2)$ HEISENBERG MODEL IN 3–DIMENSIONS

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Abstract

A q-deformed $SU(2)$ Heisenberg (3-dimensional) spin model is set up, and the q-deformed spin-wave solution is obtained through the q-analogous Holstein-Primakoff transformation. The result is given for small $\gamma = \ln q$, which is the quantity characterizing the nonlinearity of the Hamiltonian. A mean-field treatment is arranged to preserve (at least some of) the nonlinearity, and the ordinary ferromagnet ground state is shown as the exact ground state of the new system. Interesting results are obtained for this nonlinear model: (i) There is an energy gap between the ground state and the first excited one, thus the ground state is stable under small perturbation of the background; (ii) the specific heat per volume is modified by a small term proportional to the 1/2-th power of temperature and the square of γ, which is qualitatively different from the conventional model, and (iii) the magnetization $M(T)$ is modified by a factor that depends on γ.

1. Introduction

In this paper, we give the modified spin-wave theory in a nonlinear ferromagnet model of q-analogous Heisenberg type. To get this model, we replace the spin-operators S_α (with $\alpha = +, -, 3$) by their quantum-group [1][2][3][4][5] counterparts $\tilde{S}_\alpha$. The new model will be solved by mean-field treatment, under the condition of low temperature and small nonlinearity. From the consideration of symmetry and algebra, we also gain some global knowledge of the system, such as the good quantum number and ground state, before the mean-field treatment.

As the quantum group $SU_q(2)$ coincides with the Lie group when spin at each site is $1/2$, we will assume in the following that the spin is higher than $1/2$, (i.e. $S \geq 1$) and then the new model is highly nonlinear. Throughout this paper, we will assume that the deformation parameter q is not a root of unity, and therefore the representations of the quantum group $SU_q(2)$ are in one-one correspondence with those of the Lie algebra $SU(2)$. Thus the new model of q-spins has the same vacuum as the conventional model. This point will be explained more explicitly. As the vacuum for the new model is an exact one, the mean-field treatment carried out in the forthcoming sections is safe. This is a result that does not depend on the deformation parameter or temperature.

Unlike the conventional model which is isotropic in three space dimensions, the new model is anisotropic, but is isotropic in the x-y plane. In other words, any rotation about z-axis does not change the Hamiltonian. Thus there is a good quantum number corresponds to this symmetry, and in the following we will see that the conserved quantity is the z-component of the total q-spin. This is another result that has no dependence on mean-field treatment.

It is well known that the ordinary spin-wave theory for the isotropic ferromagnet (which will be referred to as conventional one) are obtained by bilinear-approximation and through Holstein-Primakoff transformation [6][7]. The dependence of both specific heat per volume and magnetization on temperature is $T^{3/2}$, which is verified qualitatively by experiments. The ground state of the system is well-known, and there is no energy gap between the ground state and the excited ones, i.e., any perturbation of infinitesimal amount of energy can be carried away by the propagating spin waves. As can be seen in the following, the conventional model is recovered as the limit of present theory at $q \to 1$.

It should be stressed that there is no such physical entity as deformed spin, and the physical operators in our model are S_α, not $\tilde{S}_\alpha$. The latter appear as effective quantities. The nonlinear interaction between neighbouring sites is represented effectively by q-spin operators. When excited, spin deviations propagate like ordinary spin waves, but with the amplitudes modulated by the nonlinear interaction. The propagating spin waves modulated by nonlinear effect will be properly named in this paper as q-spin waves.

When q deviates, though slightly, from 1, the Hamiltonian for the nonlinear model can be approximated as the conventional Hamiltonian plus a perturbation. As the readers may notice in the following, the nonlinearity in the perturbation term is carefully preserved by a mean-field treatment and through the so-called q-analogous HPT, and we obtain the q-analogous spin-wave theory. The magnetization and specific heat are both evaluated at the mean-field level. It is novel to find that there is a qualitative modification in the specific heat of zero-th order (conventional spin-wave theory) result, i.e., a term proportional to $T^{1/2}$. A modifying factor appears in the formula of magnetization, which depends on γ^2. When $q \to 1$, this factor goes to unity.

An interesting phenomenon worth noting is that there is an energy gap between the ground state and the first excited one, and therefore any excitation will cost *finite* amount of energy. If the perturbation is small enough, the magnet at the ultimately-ordered state (conventional ferromagnetic ground state) is stable. In other words, when the background thermal temperature is smaller than a certain T_0, the magnet sitting in the ground state is difficult to be excited. This unusual property is obviously due to the nonlinear perturbation terms in the Hamiltonian, which turns out to tend to keep the ground state stable against foreign disturbances.

This paper is organized in the following way: In section II, we give a brief review of the ordinary spin-wave theory, with HPT stressed. In section III, we recall some essential preliminary of the quantum algebra $SU_q(2)$, and then construct the q-analogous HPT which is in fact a new realization of the quantum algebra $SU_q(2)$ via q-oscillators. Section IV is devoted to the construction of the deformed Hamiltonian with q-deformed operators replacing the $SU(2)$ operators. To show explicitly the nonlinearity of the new system, the Hamiltonian is expanded into Taylor series. The conventional Hamiltonian H_0 appears in the expansion as the zero-th order term in γ. The first-order terms ($\propto \gamma^2$), denoted H_I, contains the forth-powers of spin generators, e.g., $S_\ell^+ S_{\ell+1}^- (S_\ell^z)^2$, representing some self-interaction that modulates the amplitudes of waves propagating from site to site. The conventional ferromagnetic ground state is shown as the exact ground state of our new model. In section V we evaluate the thermal-dynamical quantities, i.e., the specific heat and magnetization. The modifications to the conventional results are found, and the physical implication of the modifications is discussed. In the last section some short discussions and

concluding remarks are presented.

II. Holstein-Primakoff Transformation

To set up notations and facilitate forth-coming descriptions, we give a brief review of the well-known Holstein-Primakoff transformation and spin-wave theory. Consider the ferromagnet model in three space dimensions with the following Hamiltonian

$$\begin{aligned} H_0 &= -J \sum_{\ell,\delta}{}' \vec{S}_\ell \cdot \vec{S}_{\ell+\delta} , \\ &= -J \sum_{\ell,\delta}{}' \left[S_\ell^z S_{\ell+\delta}^z + \frac{1}{2} \left(S_\ell^+ S_{\ell+\delta}^- + S_\ell^- S_{\ell+\delta}^+ \right) \right] \qquad (J > 0) \end{aligned} \tag{1}$$

where $\vec{S}_\ell = (S_\ell^x, S_\ell^y, S_\ell^z)$ is the angular momentum operator, and $S_\ell^\pm = S_\ell^x \pm i S_\ell^y$ are raising and lowering operators. $\sum'$ means $\delta \neq 0$ and the summation is over nearest neighbours. The operators satisfy

$$[S_\ell^x, S_{\ell'}^y] = i S_\ell^z \delta_{\ell\ell'} \quad (x, y, z \quad \text{cycl. permut.}) , \tag{2}$$

or

$$\begin{aligned} \left[S_\ell^\pm \ , \ S_{\ell'}^z \right] &= \mp S_\ell^\pm \delta_{\ell\ell'} , \\ \left[S_\ell^+ \ , \ S_{\ell'}^- \right] &= 2 S_\ell^z \delta_{\ell\ell'} , \end{aligned} \tag{3}$$

which constitute the $SU(2)$ algebra, with the following representations (Dicke spaces)

$$D_S = \{ |S, m\rangle \ , \quad m = S, S-1, \cdots, -S \} , \tag{4}$$

where S is (positive) half integer, i.e., the quantum number of angular moment. The actions of the generators in the Dicke space D_S yield

$$\begin{aligned} S_\ell^\pm |S, m\rangle &= \sqrt{(S \mp m)(S \pm m + 1)} |S, m \pm 1\rangle , \\ S_\ell^z |S, m\rangle &= m |S, m\rangle . \end{aligned} \tag{5}$$

The actions of $S_\ell^\pm$ on the highest weight vector $|S, S\rangle_\ell$ result in an increase or decrease of the z-component of spin vector at site ℓ, or the deviation of the spin vector from z-direction.

It is easy to see that the ground state for the system (1) is

$$|0\rangle = |S,S\rangle_1|S,S\rangle_2\cdots|S,S\rangle_N\ , \tag{6}$$

i.e., the spins at all positions are at exactly the same direction (z-axis). The ground state is an ultimately-ordered state that breaks $SO(3)$ symmetry respected by the Hamiltonian. According to Goldstone's theorem [8], the system has gapless excitations (Goldstone bosons) as the elementary excitations (magnons).

As

$$S_\ell^+|0\rangle = 0 \tag{7}$$

the ground state energy is

$$E_0 = -JNZS^2|0\rangle\ , \tag{8}$$

where Z is the coordination number.

Now we consider the low-excitations in the spin system. Suppose that there is a deviation of the spin at site ℓ. This state is

$$|(S-1)_\ell\rangle = |S,S\rangle_1|S,S\rangle_2\cdots|S,S\rangle_{\ell-1}|S,S-1\rangle_\ell|S,S\rangle_{\ell+1}\cdots|S,S\rangle_N\ , \tag{9}$$

and the action of the Hamiltonian transfers the excitation at the ℓ-th position to the excitation at the $(\ell+1)$-th position:

$$|(S-1)_{\ell+1}\rangle = |S,S\rangle_1|S,S\rangle_2\cdots|S,S\rangle_\ell|S,S-1\rangle_{\ell+1}|S,S\rangle_{\ell+2}\cdots|S,S\rangle_N\ . \tag{10}$$

That is, the deviation of spins propagates in the lattice.

To uncover the mathematical structure, we introduce the HPT, which expresses the spin operators *via* the creation, annihilation and number operators ($a^\dagger$, a and

$N=a^\dagger a$) of certain boson fields :

$$\begin{aligned} S^+ &= \sqrt{2S-N}\,a \\ S^- &= a^\dagger\sqrt{2S-N} \\ S^z &= S-N\ . \end{aligned} \tag{11}$$

The operators $a^\dagger$, a and N span the Heisenberg algebra $H(4)$ with the following relations

$$[a\ ,\ a^\dagger]=1\ ,\ \ [N\ ,\ a]=-a\ ,\ \ [N\ ,\ a^\dagger]=a^\dagger\ . \tag{12}$$

The representation of the Heisenberg algebra is infinite dimensional, namely the Fock space

$$F=\left\{|n\rangle=\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle\ ,\quad n=0,1,2,\cdots\right\}, \tag{13}$$

and the actions of the generators in F yield

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle\ , \\ a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle\ , \\ N|n\rangle &= n|n\rangle\ , \end{aligned} \tag{14}$$

And the HP transformation induces the transformation of the representation of the simple harmonic oscillator into those of the spin algebra:

$$n = S-m\ , \tag{15}$$

and

$$\begin{aligned} S^+|n\rangle &= \sqrt{2S-n+1}\sqrt{n}|n-1\rangle\ ,\quad (0<n\le 2S)\ , \\ S^-|n\rangle &= \sqrt{2S-n}\sqrt{n+1}|n+1\rangle\ ,\quad (0\le n<2S)\ , \\ S^+|0 &>= 0\ , \\ S^-|2S &>= 0\ . \end{aligned} \tag{16}$$

It is obvious that the raising and lowering operators play the roles similar to those of the annihilation and creation operators. In the above transformation, S is an (positive) half-integer, and when S runs over all possible values, we obtain from the transformation all the spin-S Dicke spaces.

To get spin-wave theory, only the bilinear part of the Hamiltonian is taken into consideration, i.e.,

$$H_0 = E_0 + \sum_k \hbar\omega_k b_k^\dagger b_k \ , \quad \hbar\omega_k = 2JSa_0^2k^2 \tag{17}$$

for cubic lattice with a_0 being the spacing length. It can be seen from ω_k that there is no energy gap and therefore any small perturbation of infinitesimal amount of energy will result in the deviation of the spins and will be soon carried away by the spin-waves propagating in the magnet. Thus

$$\langle n_k\rangle_T = \left\langle b_k^\dagger b_k\right\rangle_T = \left[\exp\left(\frac{\hbar\omega_k}{k_BT}\right) - 1\right]^{-1} . \tag{18}$$

At low temperature, the magnetization is

$$M(T) = M(0)\left[1 - \frac{\beta}{S}\left(\frac{k_BT}{2SJ}\right)^{3/2}\right] \tag{19}$$

where β is some constant, and

$$\Delta M(T) = M(0) - M(T) = M(0)\frac{\beta}{S}\left(\frac{k_BT}{2SJ}\right)^{3/2} . \tag{20}$$

The specific heat per volume obeys a power-law, too

$$C_V = \nu\mathcal{N}k_B\left(\frac{k_BT}{2JS}\right)^{3/2} , \tag{21}$$

where ν is a constant.

Generally, at the spin-wave level, the factors like $S_\ell^+S_{\ell+1}^-$ in the Hamiltonian represent the propagation of the spin-deviations.

III. Quantum $SU(2)$ Algebra and q-Analogous HPT

We briefly review the quantum Heisenberg group $H_q(4)$ and quantum special unitary group $SU_q(2)$. Through out this paper, we assume that q and $\gamma = \ln q$ are real.

The quantum Heisenberg group $H_q(4)$ is spanned by the q-analogous creation and annihilation operators $\tilde{a}^\dagger$, $\tilde{a}$ and number operator $N = \lim_{q\to 1}\tilde{a}^\dagger\tilde{a}$, with the following relations

$$\begin{aligned}\left[\tilde{a}\ ,\ \tilde{a}^\dagger\right] &= [N+1]_q - [N]_q = \frac{\cosh\left[\gamma\left(N+\frac{1}{2}\right)\right]}{\cosh\frac{\gamma}{2}} , \\ [N\ ,\ \tilde{a}] &= -\tilde{a}\ , \quad \left[N\ ,\ \tilde{a}^\dagger\right] = \tilde{a}^\dagger ,\end{aligned} \tag{22}$$

with the Hopf operations and Yang-Baxter equation constructed in [9]. When $q \longrightarrow 1$ or $\gamma \to 0$, the simple harmonic oscillator algebra is recovered. A simple relation between the q-operators and the operators of the simple harmonic oscillator can be revealed in the following

$$\tilde{a} = a\sqrt{\frac{[N]_q}{N}}\ , \quad \tilde{a}^\dagger = \sqrt{\frac{[N]_q}{N}}a^\dagger . \tag{23}$$

The q-oscillator algebra gives the realizations of the quantum enveloping algebras of simple or semi-simple Lie algebras [4][10][11]. The simplest example is the realization of the $SU_q(2)$ algebra, which are intensively discussed in [4][5][10][11]. But the discussions are concentrated on the q-analogous Jordan-Schwinger representation. In the following we give a new realization, namely, the q-analogous HP transformation:

$$\begin{aligned}\tilde{S}^+ &= \sqrt{[2S-N]_q}\ \tilde{a}\ , \\ \tilde{S}^- &= \tilde{a}^\dagger\sqrt{[2S-N]_q}\ , \\ \tilde{S}^z &= S - N\ .\end{aligned} \tag{24}$$

The relations for $SU_q(2)$ algebra can be easily checked

$$\begin{aligned}\left[\tilde{S}^+\ ,\ \tilde{S}^-\right] &= \left[2\tilde{S}^z\right]_q\ , \\ \left[\tilde{S}^z\ ,\ \tilde{S}^\pm\right] &= \pm\tilde{S}^\pm\ .\end{aligned} \tag{25}$$

By this realization we can give the transformation from (infinite dimensional) representation of the $H_q(4)$ algebra to those (finite dimensional ones) of the $SU_q(2)$. We choose to start our investigation from the infinite dimensional representation of the q-Heisenberg algebra, i.e., the q-Fock space

$$F_q = \left\{ |n\rangle_q = \frac{(\tilde{a}^\dagger)^n}{\sqrt{[n]_q!}}|0\rangle \ , \quad n = 0, 1, 2, \cdots \right\} \ . \tag{26}$$

From (23), we get to know that $|n\rangle_q = |n\rangle$, so we will not distinguish them in the following, without causing any confusion. The actions of the q-annihilation and creation operators yield

$$\begin{aligned} &\tilde{a}^\dagger|n\rangle = \sqrt{[n+1]_q}|n+1\rangle \ , \\ &\tilde{a}|n\rangle = \sqrt{[n]_q}|n-1\rangle \ , \\ &N|n\rangle = n|n\rangle \ . \end{aligned} \tag{27}$$

Therefore the q-Dicke states are easily obtained by noting that

$$D_S = \{ |S, S-m\rangle = |n\rangle \ , \quad n = S - m \ , \quad m = S, S-1, \cdots, -S \ \} \ . \tag{28}$$

The actions of the generators of the $SU_q(2)$ algebra yield

$$\begin{aligned} &\tilde{S}^+|n\rangle = \sqrt{[2S-n+1]_q [n]_q}|n-1\rangle \ , \quad (0 < n \leq 2S) \ , \\ &\tilde{S}^-|n\rangle = \sqrt{[2S-n]_q [n+1]_q}|n+1\rangle \ , \quad (0 \leq n < 2S) \ , \\ &\tilde{S}^+|0\rangle = 0 \ , \\ &\tilde{S}^-|2S\rangle = 0 \ . \end{aligned} \tag{29}$$

Therefore the q-Dicke spaces

$$D_S = \{ |n\rangle \ , \quad n = 0, 1, 2, \cdots, 2S \ \} \tag{30}$$

are irreducible and indecomposable representations.

IV. q-Deformed Ferromagnetic Spin Waves

Now we consider a multi-spin system with the following Hamiltonian

$$H_q = -J \sum_{\ell\delta}{}' \left\{ \tilde{S}^z_\ell \tilde{S}^z_{\ell+\delta} + \frac{1}{2}\left(\tilde{S}^+_\ell \tilde{S}^-_{\ell+\delta} + \tilde{S}^-_\ell \tilde{S}^+_{\ell+\delta} \right) \right\} \ , \tag{31}$$

which is hermitian. It is apparent that H_q is obtained by replacing the the spin operators in (1) by their q-analogues. The above compact expression can be expanded into Taylor series to gain physical intuition,

$$\begin{aligned} \tilde{S}^+_\ell \tilde{S}^-_{\ell+\delta} &= \left(\tfrac{\gamma}{\sinh\gamma}\right)^2 S^+_\ell S^-_{\ell+\delta} \left(1 + \tfrac{\gamma^2}{6}\left[\tfrac{(2S+1)^2}{2} + \left(S^z_\ell + \tfrac{1}{2}\right)^2 + \left(S^z_{\ell+\delta} - \tfrac{1}{2}\right)^2\right]\right) \ , \\ \tilde{S}^-_\ell \tilde{S}^+_{\ell+\delta} &= \left(\tfrac{\gamma}{\sinh\gamma}\right)^2 \left(1 + \tfrac{\gamma^2}{6}\left[\tfrac{(2S+1)^2}{2} + \left(S^z_\ell + \tfrac{1}{2}\right)^2 + \left(S^z_{\ell+\delta} - \tfrac{1}{2}\right)^2\right]\right) S^-_\ell S^+_{\ell+\delta} \ , \end{aligned} \tag{32}$$

if one notices the relations between the q-spin operators and the conventional spin-operators[10],

$$\begin{aligned} \tilde{S}^+_\ell &= S^+ \Theta(S, S^z_\ell) \ , \\ \tilde{S}^-_\ell &= \Theta(S, S^z_\ell) S^- \ , \\ \tilde{S}^z_\ell &= S^z_\ell \ , \end{aligned} \tag{33}$$

where

$$\Theta(S, S^z) = \sqrt{\frac{[S - S^z]_q [S + S^z + 1]_q}{(S - S^z)(S + S^z + 1)}} \ . \tag{34}$$

Therefore the Hamiltonian is written into two parts,

$$H_q = H_0 + H_I \ , \tag{35}$$

where H_0 is defined in (1) and

$$\begin{aligned} H_I = & -\frac{\gamma^2 J}{6} \sum_{\ell\delta}{}' \left(S^+_\ell S^-_{\ell+\delta} \left[\frac{(2S+1)^2}{2} - 2 + \left(S^z_\ell + \frac{1}{2}\right)^2 + \left(S^z_{\ell+\delta} - \frac{1}{2}\right)^2 \right] \right. \\ & \left. + \left[\frac{(2S+1)^2}{2} - 2 + \left(S^z_\ell + \frac{1}{2}\right)^2 + \left(S^z_{\ell+\delta} - \frac{1}{2}\right)^2 \right] S^-_\ell S^+_{\ell+\delta} \right) \ . \end{aligned} \tag{36}$$

There is no such physical quantities as the q-spins. The q-deformed operators appear just as effective and compact symbols, representing certain (now small) nonlinear interaction, as indicated in the above. This interaction may result from anisotropic crystal fields or other influences from the background.

Before we go to the detailed mean-field treatment, we will first observe some global properties of the new Hamiltonian.

1. For q not a root of unity, the relations in (33,34) make it clear that the representations of the quantum $SU(2)$ are in one-one correspondence with those of their Lie counterparts [10][5]. A direct result is that the highest weight vectors of the two algebras can be identified (up to some normalization constants). This point is significant for the mean-field treatment carried out in the following, as it guarantees that the new model has an exact vacuum state, as the conventional model does, regardless of the nonlinearity of the former.

2. As can be checked, the two components of the q-spin operators in x-y plane constitute a vector. To see this point, we can choose fixed eigenvalues of S_ℓ^z's, so from (34), $\Theta(S, S_\ell^z)$ are constants, and therefore $\tilde{S}_\ell^x$ and $\tilde{S}_\ell^y$ are proportional to S_ℓ^x and S_ℓ^y. It is a direct calculation to see that H_q is invariant under a rotation of the q-spin in the plane perpendicular to z-axis.

3. Another way to see the invariance of the Hamiltonian H_q under the $SO(2)$ rotation is the direct construction of the conserved dynamical operator $\tilde{S}^z_{\rm total} = \sum_\ell{}' \tilde{S}_\ell^z$. A simple computation shows $[\tilde{S}^z_{\rm total}\ ,\ H_q] = 0$, in other words, the eigenvalue of $S^z_{\rm total}$ is a good quantum number. The existence of conserved quantity verifies the deduction in item 2. (Note the $\tilde{S}^z_{\rm total}$ and tensored representations should be constructed according to [1][2].)

4. As H_q is not $SO(3)$ symmetric but $U(1)$ symmetric, the Goldstone theorem does not apply. No Goldstone boson is expected. And therefore we do not feel strange if the excitations are gapless.

These conclusions are from the knowledge of algebra and symmetry property of the system, and are important to our forth-coming mean-field treatment.

Now we are in the position to investigate the low excitations in this Hamiltonian system by the q-analogous HPT. The solutions are approximate ones under the condition of low temperature and $q \sim 1$, but the nonlinear effects are preserved. From (24), we rewrite the above Hamiltonian,

$$\begin{aligned} H_q \;=\; & -J \sum_{\ell\delta}{}' \{(S - N_\ell)(S - N_{\ell+\delta}) + \\ & \tfrac{1}{2}\sqrt{[2S - N_\ell]_q}\,\tilde{a}_\ell \tilde{a}^\dagger_{\ell+\delta}\sqrt{[2S - N_{\ell+\delta}]_q} \\ & + \tfrac{1}{2}\tilde{a}^\dagger_\ell \sqrt{[2S - N_\ell]_q}\sqrt{[2S - N_{\ell+\delta}]_q}\,\tilde{a}_{\ell+\delta}\}\,, \end{aligned} \tag{37}$$

where $N_\ell = \tilde{a}^\dagger_\ell \tilde{a}_\ell$. At low temperature, the excited states are generally near the ground state, and few spins in the crystal deviate from the z-axis, thus $\langle N_\ell \rangle$ is small compared with $2S$, and can be neglected. When we cancel out the terms proportional to the forth power of the q-analogous annihilation or creation operators, we have the approximate Hamiltonian

$$H_q = -NZJS^2 + 2ZJS \sum_\ell a^\dagger_\ell a_\ell - \frac{1}{2} J [2S]_q \sum_{\ell,\delta}{}' \left(\tilde{a}_\ell \tilde{a}^\dagger_{\ell+\delta} + \tilde{a}^\dagger_\ell \tilde{a}_{\ell+\delta} \right) \ . \tag{38}$$

The first term gives the ground state energy E_0, and the second term gives the deviation energy of the spin at the ℓ-th position, and the third term gives the coupling between different positions. It is not strange that this Hamiltonian is similar in form with the conventional one, in that when $q \to 1$ the q-operators are replaced by the ordinary operators.

The Fourier transformations the operators $\tilde{a}_\ell$ and $\tilde{a}_\ell^\dagger$ at each position yield

$$\tilde{a}_\ell = \mathcal{N}^{-1/2}\sum_k e^{i\vec{k}\cdot\vec{\ell}}\tilde{b}_k \ , \quad \tilde{a}_\ell^\dagger = \mathcal{N}^{-1/2}\sum_k e^{-i\vec{k}\cdot\vec{\ell}}\tilde{b}_k^\dagger \ , \tag{39}$$

and have the following commutation relations for their counterparts in momentum space:

$$\begin{aligned}\alpha_{kk'} \equiv \left[\tilde{b}_k \ , \ \tilde{b}_{k'}^\dagger\right] &= \tfrac{1}{\mathcal{N}}\sum_{\ell,\ell'} e^{-i\vec{k}\cdot\vec{\ell}}e^{i\vec{k}'\cdot\vec{\ell}'}\left(\tilde{a}_\ell\tilde{a}_{\ell'}^\dagger - \tilde{a}_{\ell'}^\dagger\tilde{a}_\ell\right) \ , \\ &= \tfrac{1}{\mathcal{N}}\sum_\ell e^{-i\vec{\ell}\cdot(\vec{k}-\vec{k}')}\left([N_\ell+1]_q - [N_\ell]_q\right) \ .\end{aligned} \tag{40}$$

As α_{kk} is independent of k, and will be denoted as α in the following text.

To express the third term in (38) into the momentum space, we calculate

$$\begin{aligned}\sum_{\ell,\delta}{}' \left(\tilde{a}_\ell\tilde{a}_{\ell+\delta}^\dagger + \tilde{a}_\ell^\dagger\tilde{a}_{\ell+\delta}\right) &= \sum_\delta{}' \left(\sum_k e^{-i\vec{k}\cdot\vec{\delta}}\tilde{b}_k\tilde{b}_k^\dagger + \sum_k e^{i\vec{k}\cdot\vec{\delta}}\tilde{b}_k^\dagger\tilde{b}_k\right) \ , \\ &= \sum_k \left(\overline{\nu}_k\tilde{b}_k^\dagger\tilde{b}_k + \nu_k'\alpha\right) \ ,\end{aligned} \tag{41}$$

where

$$\overline{\nu}_k = 2\sum_\delta{}' \cos(k\delta) = 2\nu_k \ , \quad \nu_k' = \sum_\delta{}' e^{-i\vec{k}\cdot\vec{\delta}} \ . \tag{42}$$

Because of the symmetry of the lattice, $\sum_k \nu_k' = 0$. Similarly, the second term in the Hamiltonian (38) is expressed in momentum space as follows

$$\sum_\ell a_\ell^\dagger a_\ell = \sum_k b_k^\dagger b_k \ . \tag{43}$$

Therefore the total Hamiltonian (38) is expressed in the momentum space

$$\begin{aligned}H_q &= -\mathcal{N}ZJS^2 + 2ZJS\sum_k b_k^\dagger b_k - \frac{1}{2}J[2S]_q\sum_k\left(\overline{\nu}_k\tilde{b}_k^\dagger\tilde{b}_k + \nu_k'\alpha\right) \ , \\ &= -\mathcal{N}ZJS^2 + \sum_k\left(2ZJSb_k^\dagger b_k - J[2S]_q\nu_k\tilde{b}_k^\dagger\tilde{b}_k\right)\end{aligned} \tag{44}$$

Now let us observe the important second term in the above Hamiltonian,

$$\tilde{b}_k^\dagger\tilde{b}_k = \frac{1}{\mathcal{N}}\sum_{\ell\ell'} e^{i\vec{k}\cdot(\vec{\ell}-\vec{\ell}')}\sqrt{\frac{[N_\ell]_q}{N_\ell}}a_\ell^\dagger a_{\ell'}\sqrt{\frac{[N_{\ell'}]_q}{N_{\ell'}}} \ . \tag{45}$$

By assuming that $\gamma N_\ell \ll 1$ for all ℓ, we have

$$\frac{[N_\ell]_q}{N_\ell} \doteq \frac{\gamma}{\sinh\gamma}\left(1 + \frac{\gamma^2 N_\ell^2}{3!} + \cdots\right) \ . \tag{46}$$

Thus we can take the following approximation under the condition of low temperature

$$\frac{[N_\ell]_q}{N_\ell} \doteq \frac{\gamma}{\sinh\gamma}\left(1 + \frac{\gamma^2\langle N_\ell^2\rangle}{3!} + \cdots\right) \ . \tag{47}$$

Or equivalently, we have the following mean-field result

$$\tilde{b}_k^\dagger\tilde{b}_k \doteq \frac{1}{\mathcal{N}}\sum_{\ell,\ell'} e^{i\vec{k}\cdot(\vec{\ell}-\vec{\ell}')}\eta a_\ell^\dagger a_{\ell'} \ , = \eta b_k^\dagger b_k \tag{48}$$

where

$$\eta = \left\langle\frac{[N_\ell]_q}{N_\ell}\right\rangle \ , \tag{49}$$

which is a quantity (pure number) independent of ℓ. At low temperature, $\langle N_\ell^2\rangle \ll 1$, therefore for small $|\gamma|$

$$\eta \doteq \frac{\gamma}{\sinh\gamma}\left[1 + \frac{\gamma^2}{6}\langle N_\ell^2\rangle\right] \sim 1 \ . \tag{50}$$

Therefore the Hamiltonian reads

$$H_q = -\mathcal{N}ZJS^2 + \sum_k \hbar\omega(\vec{k})b_k^\dagger b_k \ , \tag{51}$$

if we denote that

$$\hbar\omega(\vec{k}) = 2ZJS - J[2S]_q\nu_k \ . \tag{52}$$

Again, when the temperature is low enough, only the spin-waves with long wavelengths can be excited, i.e., $|\vec{k}\cdot\vec{\delta}| \ll 1$. Thus for the cubic lattice, we may expand ν_k into

$$\nu_k = \sum_\delta{}' \cos\left(\vec{k}\cdot\vec{\delta}\right) = \sum_\delta{}'\left[1 - \tfrac{1}{2}\left(\vec{k}\cdot\vec{\delta}\right)^2\right] \ , \tag{53}$$

and

$$\sum_{\delta}{}'\left(\vec{k}\cdot\vec{\delta}\right)^2=\left[\frac{1}{Z}\sum_{\delta}{}'Z\delta^2\cos^2\theta\right]k^2=\frac{1}{3}6a_0^2k^2=2a_0^2k^2\ , \tag{54}$$

where $\cos\theta=\dfrac{\vec{k}\cdot\vec{\delta}}{|\vec{k}|\cdot|\vec{\delta}|}$ and a_0 is the spacing length of the lattice. Denote that $y=J\,[2S]_q\,\eta$ and $\Delta=2JZS-yZ$, then we have

$$\hbar\omega(\vec{k})=\Delta+ya_0^2k^2\ . \tag{55}$$

From (50), some simple algebra shows that for small $|\gamma|$, $\Delta=2ZJS\left(1-\eta\frac{[2S]_q}{2S}\right)>0$. Thus at low temperature and for small deformation parameter $|\gamma|$, ω is always positive. Because the ground state is the ultimately-ordered state $|0)$, and the excitations are simply the deviations of the spins from the z-axis, it is reasonable to expect that the above dispersion relation is the only possible one (without other branches).

And in that the term Δ appears, there should be a gap between the first excited state and the ground one. Thus when the magnet is in the ground state (i.e., with all spins ordered in the z-axis) any excitation of the system costs a finite amount of energy.

V. Thermal-dynamical Quantities

Because b_k and $b_k^\dagger$ are bosonic operators,

$$\langle n_k\rangle_T=\left\langle b_k^\dagger b_k\right\rangle_T=\left[\exp\left(\frac{\hbar\omega(\vec{k})}{k_BT}\right)-1\right]^{-1}\ . \tag{56}$$

The total excitation for the k-modes is

$$\sum_{k\in BZ}\langle n_k\rangle_T=\frac{V}{8\pi^3}\int\frac{d^3\vec{k}}{\exp\left(\frac{\hbar\omega(\vec{k})}{k_BT}\right)-1}\doteq\frac{V}{8\pi^3}\int\frac{d^3\vec{k}}{\exp\left(\frac{ya_0^2k^2}{k_BT}\right)-1}\ , \tag{57}$$

where the second equal holds under the condition of $b\gamma\to 0$, or equivalently, $\Delta\doteq 0$ (and BZ means the reduced Brillouin zone). An experienced reader would expect an exponential factor of $\exp(-\Delta/k_BT)$ because the system is gapful. In fact such a factor really occurs before the exponential in the denominator during calculation, but physical considerations show that it is safe to replace it by 1. Actually, $\Delta/k_BT\sim J\gamma^2/K_BT$, and when T is in $Kelvin$, $\Delta/k_BT\sim\gamma^2/T\ll 1$ for the temperature of most of the experiments concerning ferromagnets and for our assumption of $\gamma\ll 1$.

As the temperature is low, $ya_0^2k_{\max}^2\gg k_BT$, thus we may have approximate result for the above integral

$$\frac{1}{N}\sum_{\vec{k}}\langle n_k\rangle_T=\left(\frac{V}{2Na_0^3}\right)\left(\frac{k_BT}{y}\right)^{3/2}\frac{1}{2\pi^2}\Gamma(\frac{3}{2})\ \xi\left(\frac{3}{2}\ ;\ 1\right)\ , \tag{58}$$

where $\Gamma\left(\frac{3}{2}\right)=\sqrt{\pi}/2$, and the Riemann function $\xi\left(\frac{3}{2}\ ;\ 1\right)\doteq 2.612$. We denote that $\alpha=\left(\frac{V}{2Na^3}\right)^3\left(\frac{1}{2\pi^2}\right)\Gamma\left(\frac{3}{2}\right)\ \xi\left(\frac{3}{2}\ ;\ 1\right)$, therefore

$$\frac{1}{N}\sum_{\vec{k}}\langle n_k\rangle_T=\alpha\left(\frac{k_BT}{y}\right)^{\frac{3}{2}}\ . \tag{59}$$

The magnetization of the system can also be calculated (at low temperature)

$$M(T)=g\mu_B\sum_{\ell}\left(S-\left\langle a_\ell^\dagger a_\ell\right\rangle_T\right)=Ng\mu_BS-g\mu_B\sum_{k}\left\langle b_k^\dagger b_k\right\rangle_T\ . \tag{60}$$

Using the mean-field approach indicated in (48), we have

$$M(T)=M(0)\left[1-\frac{\alpha}{S}\left(\frac{k_BT}{y}\right)^{\frac{3}{2}}\right] \tag{61}$$

where $M(0)=Ng\mu_BS$ is the saturated magnetization. As the spin-wave is excited, the magnetization decreases

$$\begin{aligned}\Delta M(T)&=M(0)-M(T)=M(0)\left[1-\frac{\alpha}{S}\left(\frac{k_BT}{y}\right)^{\frac{3}{2}}\right]\\&=M(0)\frac{\alpha}{S}\left(\frac{k_BT}{\eta J[2S]_q}\right)^{3/2}\ ,\end{aligned} \tag{62}$$

As can be seen, though the qualitative feature of magnetization in this above approach is similar to the conventional one, the present result gives quantitative modification. q (or γ) is a free parameter to be determined phenomenologically.

Since

$$\sum_{k\in BZ}\omega(\vec{k})\left\langle b_k^\dagger b_k\right\rangle_T=\frac{V}{8\pi^2}\int\frac{\Delta+ya_0^2k^2}{\exp\left[\frac{\Delta+ya_0^2k^2}{k_BT}\right]-1}d^3\vec{k}=I+II\ , \tag{63}$$

where

$$\begin{aligned} I &\doteq \frac{V}{8\pi^3}\Delta\int\frac{d^3\vec{k}}{\exp\left(\frac{ya_0^2k^2}{k_BT}\right)-1} \\ &= \frac{V}{2\pi^2}\Delta\left(\frac{k_BT}{y}\right)^{3/2}\frac{1}{2a^3}\Gamma\left(\frac{3}{2}\right)\ \xi\left(\frac{3}{2}\ ;\ 1\right)\ , \end{aligned} \tag{64}$$

and

$$II\doteq\frac{V}{8\pi^3}\int\frac{k^2}{\exp\left(\frac{ya_0^2k^2}{k_BT}\right)-1}d^3\vec{k}=\frac{V}{4\pi^2}\frac{(k_BT)^{5/2}}{(ya_0^2)^{3/2}}Q \tag{65}$$

with

$$Q=\int_0^\infty\frac{\overline{x}^{\frac{3}{2}}}{\exp(\overline{x})-1}d\overline{x}\ . \tag{66}$$

Therefore the Hamiltonian is explicitly

$$H_q(T)=\frac{V}{2\pi^2}\Delta\left(\frac{k_BT}{y}\right)^{\frac{3}{2}}\frac{1}{2a^3}\Gamma\left(\frac{3}{2}\right)\ \xi\left(\frac{3}{2}\ ;\ 1\right)+\frac{V}{4\pi^2}\frac{(k_BT)^{\frac{5}{2}}}{(ya_0^2)^{\frac{3}{2}}}Q-\mathcal{N}ZJS^2\ , \tag{67}$$

and thus the specific heat per volume is

$$C_V=\frac{dH_q(T)}{VdT}=\frac{3\Delta}{8\pi^2a^3}\left(\frac{k_B}{y}\right)^{\frac{3}{2}}\Gamma\left(\frac{3}{2}\right)\ \xi\left(\frac{3}{2}\ ;\ 1\right)T^{\frac{1}{2}}+\frac{5a}{8\pi^2}\frac{(k_B)^{\frac{5}{2}}}{(ya_0^2)^{\frac{3}{2}}}T^{\frac{3}{2}}\ , \tag{68}$$

where Δ is proportional to $|\gamma^2|$. It is interesting to see that the specific heat (per volume) is modified by a term proportional to $T^{1/2}$, that rarely appears in the well-studied spin models. This is a qualitative modification to the zero-order result or usual spin-wave theory.

The modification to the specific heat is a characteristic of this q-modified spin-wave theory and makes it possible to put it under the tests of proper experiments.

VI. Concluding Remarks

As the new spin-wave treatment introduces a free parameter, and includes some nonlinear terms in the Hamiltonian, it makes a more realistic description of physical systems convenient and possible. This free parameter q may bear dynamical properties of the model concerned, thus deserves investigation. Same to the conventional spin-wave results, the q-analogous ones are good approximations for high spin S, they become exact for $S\to\infty$.

The Holstein-Primakoff transformation can be interpreted in the framework of para-particles. It is known that the representations for para-algebras are naturally provided by those of $SU(2)$ algebra by identifying the order of the para-particle and the spin of the representation of the unitary group [12]. For instance, the spin-$\frac{1}{2}$ representation of $SU(2)$ is the representation of the 1-st order parafermi algebra, and that of the 2-nd order parafermi algebra is identical to that of the spin-1 representation of $SU(2)$, etc..

Therefore the q-analogous HPT should be easily understood in the framework of q-analogous para-particles [13], which are the quasi-particle resulted from the spin deviations. The states of the q-para-particles are naturally provided by the representations of the quantum special unitary group $SU_q(2)$. A q-para-particle of order k violates the statistics of the usual para-particle of order k (except for the special case of $k=1$). When $k\to\infty$, the well studied q-boson is recovered that violates the Bose statistics. On the other hand, the model and q-analogous HPT developed in this paper assign the q-boson and q-para-particles an interesting physical picture.

The quantum groups as beautiful mathematical objects are now being intensively applied in studies of the exactly solvable models such as the 1-dimensional spin-1/2 Heisenberg model. As it is known, the Hamiltonian for the isotropic system is $SU(2)$ invariant, and the anisotropic (XXZ type) model has also been solved exactly by Betha ansatz and quantum group approaches[14][15].

The exact solutions are absolutely necessary to gain the global nature of the models, but as it is known, the spin models in higher dimensions and/or with higher spins are problematic in integrability. Mean-field theories are still the powerful tool in studying most of these models. The quantum groups should not be restrained away from possible applications in and generalizations of the mean-field theories of the spin models. Whether the applications and/or generalizations are physically meaningful deserves further comparing the theoretical results and constructions with the practical systems and experimental results.

As it is emphasized in [11], the quantum groups describe deviations from the Lie symmetries. This suggests possible applications of these new algebraic structure in studies of physical theories involving violations of perfect Lie symmetries, (induced by, e.g., interactions with the background) In [16] the deviations of the vibration and rotation from the ideal ones in the diatomic molecule and the heavy-ion resonance systems are two obvious examples and are both shown to be described by the quantum groups $H_q(4)$ and $SU_q(2)$. The nonlinear spin model considered above is another example showing that the abstract mathematical entity as Hopf algebra is useful tool in dealing with systems that are more flexible and complicated than conventional ones which are usually within the description of Lie algebras.

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References

[1] V. G. Drinfel'd, "*Quantum Groups*", in Proceedings of the International Congress of Mathematicians, (Berkeley), ed. A. M. Gleason, p. 798 (Am. Math. Soc., Providence, R.I., 1986).

L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, in "*Algebraic Analysis*", Vol. 1, eds. M. Kashiwara and T. Kawai, (Academic, Boston, 1989).

[2] M. Jimbo, *Lett. Math. Phys.* **10**, 63, (1985).

M. Jimbo, *Commun. Math. Phys.* **102**, 537, (1986).

For a review, see S. Majid, *Int. J. Mod. Phys.* **A 5**, 1, (1990).

[3] Yu. I. Manin, *Ann. Inst. Fourier*, **37**, 191, (1987). S. Woronowicz, *Publ. RIMS-Kyoto* **23**, 117, (1987).

S. Vokos, B. Zumino, and J. Wess, *Z. Phys.* **C 48**, 65, (1990).

A. Schirmacher, J. Wess and B. Zumino, *Z. Phys.* **C 49**, 317, (1991).

A. Sudbery, *J. Phys.* **A 23**, L697, (1990).

[4] L. C. Biedenharn, *J. Phys.* **A 22**, L873, (1989).

A. J. MacFarlane, *J. Phys.* **A 22**, 4581, (1989).

T. Hayashi, *Commun. Math. Phys.* **127**, 129, (1990).

Y. J. Ng, *J. Phys.* **A 23**, 1023, (1990).

P. P. Kulish and E.V. Damaskinsky, *J. Phys.* **A 23**, L415, (1990).

M. Chaichian and D. Ellinas, *J. Phys.* **A 23**, L291, (1990).

[5] M. Rosso, *Commun. Math. Phys.* **117**, 581, (1988).

P. Roche and D. Arnaudon, *Lett. Math. Phys.* **17**, 295, (1989).

G. Lusztig, *Contemp. Math.* **82**, 59, (1989).

E. Floratos, *Phys. Lett.* **B 233**, 395, (1989).

M. Couture, *J. Phys.* **A 24**, L103, (1991).

R. W. Gray and C. A. Nelson, *J. Phys.* **A 23**, L945, (1990).

[6] T. Holstein and H. Primakoff, *Phys. Rev.* **58**, 1098, (1940).

[7] See, for example, C. Kittel, "*Quantum Theory of Solid*", Chapter 4, (Willey, New York, 1963).

[8] J. Goldstone, *Nuovo Cimento* **19**, 154, (1961).

For a review, see S. Nakajima, Y. Toyozawa and K. Abe, *The Physics of Elementary Excitations*, Chapt. 4, (Springer-Verlag Berlin, Hendelberg, New York, 1980).

[9] H. Yan, *J. Phys.* **A23**, L1155, (1990).

[10] T. L. Curtright and C. K. Zachos, *Phys. Lett.* **B 243**, 237, (1990).

D. Fairlie, *J. Phys.* **A 23**, L183, (1990).

D. Fairlie, P. Fletcher and R. Sasaki, *J. Math. Phys.* **31**, 776, (1990).

A. Polychronakos, *Mod. Phys. Lett.* **A 5**, 2325, (1990).

T. L. Curtright, G. I. Ghandour and C. K. Zachos, *J. Math. Phys.* **32**(2), 676, (1991)

D. B. Fairlie and C. K. Zachos, *Phys. Lett.* **B 256**, 43, (1991).

T. L. Curtright, G. I. Ghandour and C. K. Zachos, *J. Math. Phys.* **32** (3), 171, (1991).

[11] Z. Chang, W. Chen and H.-Y. Guo, *J. Phys.* **A 23**, 4185 (1990); also Z. Chang, W. Chen, H.-Y. Guo and H. Yan, *ibid.* **A 23**, 5371 (1990); and also *ibid.* **A 24**, 1427, (1991).

[12] Y. Ohnuki, S.Kamefuchi, "*Quantum Field Theory and Parastatistics*", (Univ. of Tokyo Press, Tokyo, 1982).

[13] Hong Yan, to appear in *Phys Lett.* **B**, (1991).

[14] R. J. Baxter, *Exactly Solved models in Statistical Mechanics* (Academic, New York, 1982).

H. J. de Vega, *Int. J. Mod. Phys.* 4, 2371, (1989).

F. C. Alcaraz, U. Grimm and V. Rittenberg, *Nucl. Phys.* **B 316**, 735, (1989).

A. Klüper, *J. Phys.* **A 23**, 809, (1990).

S. Sarkar, *J. Phys.* **A 23** , L409, (1990).

[15] V. Pasquier and H. Saleur, *Nucl. Phys.* **B 330**, 523, (1990).

D. Levy, *Phys. Rev. Lett.* **64**, 499, (1990).

H. J. de Vega, *Int. J. Mod. Phys.* **4**, 2371, (1989).

M. T. Batchelor, L. Mezincescu, R. Nepomechie and V. Rittenberg, *J. Phys.* **A 23**, L141, (1990).

[16] Z. Chang and H. Yan, *Phys. Lett.* **A 154**, 254, (1991); also *ibid* **A 156**, 192, (1991), and also *Phys. Rev.* **A 43**, 6043, (1991).

For an independent work, see D. Bonatsos, P. P. Raychev, R. P. Roussev and Yu. F. Smirnov, *Chem. Phys. Lett.*, **175**, 300, (1990).