INSTITUTE OF THEORETICAL
AND EXPERIMENTAL PHYSICS

ITEP -- 151-88

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SOME EXAMPLES OF INSTANTONS IN SIGMA MODELS

Moscow – ATOMINFORM – 1988

УДК 530.I45 M-I6

SOME EXAMPLES OF INSTANTONS IN SIGMA MODELS: Preprint ITEP 88-151 /
Ya.Kogan, D.Markushevich, A.Morozov et al. - M.: ATOMINFORM, 1988 -
24p.

Fig. -, ref. - 5

Fields in bosonic sigma-model describe maps of two-dimensional spaces S_p of various topologies into Riemannian manifold M. We shall consider the case of Kähleriam manifolds M. Instantons in the sigma-model are topologically non-trivial maps of Riemann surface S_p of genus p into M, satisfying classical equations of motion. If there is a holomorphic map of S_p into M in the topological class under consideration, it is known to realize the minimum of action [1]. In what follows we consider only this (holomorphic) type of instantons.

An important characteristic of instanton solution is the quantity of its zero modes. If holomorphic map $S_p \to M$ is an embedding, its image V_p is a complex submanifold of complex dimension one (i.e. it is a complex curve of genus p in M). In this case in order to find the quantity of (bosonic) zero modes of the instanton, it is necessary to evaluate the number of deformations of the curve V_p within M. One should mention, that for $p > 1$ a holomorphic map S_p M exists only for special complex structures on S_p. Zero modes of holomorphic instanton are related to deformations, preserving the complex structure, which is induced on V_p by instanton map. In string theory one should sum over all complex structures on S_p. Restriction to complex structures compatible with holomorphic instanton is allowed in the zeroth approximation in inverse size of the manifold M.

We shall consider supersymmetric sigma-models, which contain fermionic fields along with the bosonic ones. These fields are two-dimensional spinors with coefficients in the tangent bundle over M. In supersymmetric case bosonic and fermionic zero modes are of interest.

Recently instanton calculus was applied to string theory [2]. We are not going to touch string aspects of the theory. Note only, that instantons on two-dimensional complex manifold K3 to be discussed are related to those on Calabi-Yau manifolds, sinse in some cases it appears a bundle over projective plane with the fibre K3.

In s.1 we give the definition of holomorphic instantons of arbitrary genus on Kählerian manifolds and of their bosonic and fermionic zero modes. General formulae for evaluation of their quantity are presented in s.2. In ss.3 and 4 zero modes are discussed in more details for spheric and toric instantons.

Section 1. General definitions.

Ordinary N=2 supersymmetric σ-model on a Kähler manifold M corresponds to a choice of two-dimensional sphere (to be denoted by S_0) for euclidean compact space-time.[1] In this paper we do not fix space-time topology, and consider surfaces S_p of arbitrary genus p instead of S_0. However, the main conclusions will be done for two cases: spere S_0 and torus S_1.

[1] In string theory another terminology is used: S_0 is reffered to as string world sheet; while space-time is identified with the target space M of the sigma-model.

Let us assume, that the complex structure on M is in the standard form, $J_\mu^\nu = i\delta_\mu^\nu$ $(\nu,\mu = 1,\dots n = \dim_{\mathbb{C}} M)$ and fix Kahler metric $G_{\mu\bar\nu}(x)$. Bosonic fields $X^\mu(\xi_1,\xi_2)$ take values in the manifold M and they are scalars with respect to general coordinate transformations on S_p. The metric on S_p will be denoted by $g_{\alpha\beta}$. Bosonic part of the action looks like

$$S_{bos} = \int_{S_p} d^2\xi \sqrt{g}\, g^{\alpha\beta} G_{\mu\bar\nu}\, \partial_\alpha X^\mu \partial_\beta \bar X^\nu. \tag{1.1}$$

Kahler metric may be expanded in generators $\omega^{(j)}$ of the integer cohomology group $H^{(1,1)}(M,\mathbb{Z})$:

$$G_{\mu\bar\nu} = \sum_{j=1}^{\rho} G^{(j)}_{\mu\bar\nu} = i\sum_{j=1}^{\rho} n^{(j)} \omega^{(j)}_{\mu\bar\nu}\ ,\quad \rho = \dim_{\mathbb{C}} H^{(1,1)}(M,\mathbb{C}).$$

Note, that metric $G_{\mu\bar\nu}$ being non-degenerate and positively definite, restricts in a certain way the possible values of real parameters $n^{(j)}$. Let ε_γ^β be complex structure on S_p, compatible with the metric, i.e. $\varepsilon_\gamma^\beta = \varepsilon_{\gamma\alpha} g^{\alpha\beta}$ with

$$\varepsilon_{\alpha\gamma} = -\varepsilon_{\gamma\alpha}\ ,\quad \varepsilon_{1,2} = \sqrt{\det g_{\alpha\beta}} = \sqrt{g}$$

Then a standard reasoning suggests, that inequality

$$(\partial_\alpha X^\mu \pm i\varepsilon_\alpha^\beta \partial_\beta X^\mu)\, G^{(j)}_{\mu\bar\nu}\, g^{\alpha\gamma} (\partial_\gamma \bar X^\nu \pm i\varepsilon_\gamma^\beta \partial_\beta \bar X^\nu) \geq 0$$

implies restrictions on components $S^{(j)}$ of the action S (1.1), which correspond to constituents $G^{(j)}$ of the metric:

$$S \geq \sum |Q^{(j)} n^{(j)}|.$$

Here $Q^{(j)}$ are topological charges,

$$Q^{(j)} = i\cdot \int_{S_p} d^2\xi \sqrt{g}\, \varepsilon^{\alpha\beta} \omega^{(j)}_{\mu\bar\nu}\, \partial_\alpha X^\mu \partial_\beta \bar X^\nu \tag{1.2}$$

Minimum of the action with given values of $Q^{(j)}$ is achieved on solutions of (anti)autoduality equations,

$$\partial_\alpha X^\mu \pm i\,\varepsilon_\alpha^\beta\,\partial_\beta X^\mu = 0 \qquad (1.3)$$

to be reffered as (anti)holomorphic instantons of genus p. Let us stress that the definition of (anti)holomorphic instanton depends on both complex structures on S_p and on M. Sinse there is only one complex structure on the sphere, the holomorphic property of genus 0 instanton depends only on complex structure on M.

If complex coordinates are introduced on the surface S_p, eq. (1.3) acquires the well known form:

$$\bar{\partial} X^\mu = 0 \qquad (\text{or } \partial X^\mu = 0).$$

In what follows we discuss solutions of the former equations and therefore deal with holomotphic instantons.

Note also, that minimum of the action can be achieved also on topologically non-trivial configurations, which do not satisfy eqs.(1.3). In other words a strict inequality $S^{(j)} > |Q^{(j)} n^{(j)}|$ may hold. These solutions are referred to as non-holomorphic instantons. The reason for them to occur is the following. Let us consider the mappings $\varphi : S_0 \to M$ and let M be simply connected. Then mappings φ are classified by the homotopic group $\pi_2(M)$ isomorphic to $H^2(M,Z)$. The group $H^2(M,Z)$ contains $H^{(1,1)}(M,Z)$ and therefore holomorphic genus 0 instantons, related to $H^{(1,1)}$ are accounted for by this classification. However, H^2 contains also cycles with non-trivial projections on $H^{(2,0)}$ and $H^{(0,2)}$ wich correspond to non-holomorphic instantons. In what follows we discuss only holomorphic instantons.

2. Proceed now to the fermionic piece of the action. Let us fix complex structure on the Riemann surface S_p. Fermionic fields live in the tangent bundle TM of M, and they are half-integer differentials with respect to S_p. In order to explain the meaning of these words we introduce a more general object - the (j,k)-differential on S_p. Let us cover S_p by coordinate cards. Transition functions on overlaps are holomorphic with respect to the given complex structure. Therefore holomorphic and antiholomorphic functions are defined on S_p: $f(\xi)=f'(\xi'=\xi'(\xi))$. More general objects are (1,0) and (0,1)-forms $f_{(1)}(\xi,\bar\xi)d\xi$ and $f_{(1)}(\xi,\bar\xi)d\bar\xi$. After the transformation from coordinate ξ to ξ' (1,0)-form $f_{(1)}(\xi,\bar\xi)d\xi$ turns into $f'_{(1)}(\xi,\bar\xi)d\xi'$ with $f'_{(1)}(\xi,\bar\xi)=f_{(1)}(\xi,\bar\xi)\frac{d\xi}{d\xi'}$, i.e. its form remains the same. The same is trough about the (0,1)-form. For example, the sphere $S_0=\mathbb{C}P^1$ may be covered by two maps with coordinates related by $\xi'=-1/\xi$. (1,0)-forms on the sphere transform as

$$f'_{(1)}\left(-\frac{1}{\xi},-\frac{1}{\bar\xi}\right)=\xi^2 f_{(1)}(\xi,\bar\xi).$$

More general objects, transforming in a more general way on the overlaps of cards, are also of interest. For example, (j,0)-differentials on $\mathbb{C}P^1$ may be considered. They are given by a pair of expressions $f_{(j)}(\xi,\bar\xi)d\xi^j$ and $f'_{(j)}(\xi',\bar\xi')d\xi'^j$. Since $d\xi'^j=\xi^{-2j}d\xi^j$ then $f'_{(j)}\left(-\frac{1}{\xi},-\frac{1}{\bar\xi}\right)=\xi^{2j}f_{(j)}(\xi,\bar\xi)$. Let us stress that this "gluing" is well defined for any half-integer j. The pair of functions $f_{(j)}$ and $f'_{(j)}$ defines a section of linear bundle of (j,0)-differentials over $\mathbb{C}P^1$. This bundle is denoted by Ω_j. The integer number $-2j$ is the degree of the linear bundle. This construction is easily generalized for arbitrary (j,k)-differentials on Riemann surfaces of arbitrary genus S_p. In this case (j,0)-differentials form linear bundle of degree $2j(p-1)$.

Since product of j-differential and (-j)-differential transforms as a scalar, $\Omega_j^* = \Omega_{-j}$. The sections of $\mathfrak{a}_{p,q}$, - the bundles of (p,q)-differentials,- in local coordinates $\xi, \bar{\xi}$ look like $f(\xi,\bar{\xi})\, d\xi^p d\bar{\xi}^q$. In these terms Ω_j coincides with $\mathfrak{a}_{j,0}$, and also an operator $\bar{\partial}: \Gamma(\mathfrak{a}_{j,0}) \to \Gamma(\mathfrak{a}_{j,1})$ is defined: $\bar{\partial}(f_{(j)} d\xi^j) = -\frac{\partial}{\partial \bar{\xi}} f_{(j)}\, d\xi^j d\bar{\xi}$.
($\Gamma(\mathfrak{a})$ is the space of sections of $\mathfrak{a}$.)

If metric $ds^2 = g_{\xi,\bar{\xi}}\, d\xi\, d\bar{\xi}$ is given on Riemann surface S_p, then hermitean scalar products in fibres of bundles $\mathfrak{a}_{j,k}$ and in the spaces of their sections may be introduced. Let $f_{(j)} d\xi^j$ and $h_{(j)} d\xi^j$ be two j-differentials on S_p. Their scalar product is defined through the following integral:

$$\langle f_{(j)} | h_{(j)} \rangle = \int_{S_p} d\xi\, d\bar{\xi}\, (g_{\xi,\bar{\xi}})^{1-j} f_{(j)}\, \bar{h}_{(j)} . \tag{1.4}$$

The integrand is obvious (1,1)-form,as required. With the help of scalar product (1.4) an operator $(\bar{\partial})^+$ conjugate to $\bar{\partial}$ may be introduced: $(\bar{\partial})^+ = -(g_{\xi,\bar{\xi}})^{j-1}\, \partial\, (g_{\xi,\bar{\xi}})^{-j}$

$$\bar{\partial}: \Gamma(\mathfrak{a}_{j,0}) \to \Gamma(\mathfrak{a}_{j,1})$$

$$(\bar{\partial})^+: \Gamma(\mathfrak{a}_{j,1}) \to \Gamma(\mathfrak{a}_{j,0}).$$

Now one can define Laplace operator,acting in the space of (j,0)-differentials:

$$\Delta = \bar{\partial}^+ \partial = -(g_{\xi,\bar{\xi}})^{j-1}\, \partial\, (g_{\xi,\bar{\xi}})^{-j}\, \bar{\partial} \tag{1.5}$$

3. Let us identify now left-hand fermions $\psi_L(\xi,\bar\xi)$ and (1/2,0)--differentials. The fermionic part of the action is compatible with chiral graduation. It has the form of

$$S_{ferm} = \int d\xi\, d\bar\xi\; G_{\mu,\bar\nu}(x)\left(\psi_L{}^{\mu}\bar{D}\psi_L{}^{\nu} + \psi_R{}^{\mu} D\psi_R{}^{\nu}\right) - \frac{1}{12} R_{\bar\mu\bar\nu\gamma\delta}\left(\psi_L{}^{\bar\mu}\psi_R{}^{\bar\nu}\right)\left(\psi_L{}^{\gamma}\psi_R{}^{\delta}\right).^{*)} \tag{1.6}$$

Here $R_{\bar\mu\bar\nu\gamma\delta}$ is curvature tensor of Kahlerian manifold M, and the action of covariant derivative $D\psi^{\mu} = \partial\psi^{\mu} + \Gamma^{\mu}_{\nu\delta}\partial x^{\nu}\psi^{\delta}$ reflects the fact, that fermionic fields take values in the tangent bundle TM. The action is invariant with respect to supersymmetric transformation, which in the case of tori S_1 looks like

$$\delta x^{\mu} = \varepsilon_L\psi_L + \bar\varepsilon_R\bar\psi_R$$

$$\delta\psi_L{}^{\mu} = -\varepsilon_L\left(\partial x^{\mu} - \Gamma^{\mu}_{\nu\gamma}(x)\, x^{\nu}x^{\gamma}\right)$$

$$\delta\psi_R{}^{\mu} = -\varepsilon_R\left(\bar\partial x^{\mu} - \Gamma^{\mu}_{\nu\gamma}(x)\, x^{\nu}x^{\gamma}\right)$$

with ε_L (ε_R) being covarantly constant (-1/2,0) and (0,-1/2)--differentials.

*) It is possible in principle to add to the action the gravitino field (3/2-differentials). However it is not needded for our considerations.

§2 ZERO MODES IN INSTANTONIC FIELD.

1. Holomorphic instantons are locally solutions of eq.(1.3), while globally they describe holomorphic mappings $\varphi: S_p \to M$. The image of such mapping is a curve in the manifold M. Holomorphic bosonic zero modes in the instanton field are arbitrary holomorphic deformations of the mapping φ and its image. Each deformation is defined by a vector field on $V_p = \mathrm{Im}\,\varphi$: at each point of V_p a vector is defined, which describes the shift of the point under the deformation. Therefore holomorphic deformations are related to holomorphic sections of the tangent bundle TM over M, lifted to $S_p, \varphi^*(TM)$. The quantity of holomorphic zero-modes n_B is equal to the quantity of independent holomorphic sections, which is dimension of the cohomology group:

$$\tilde{n}_B = \dim H^0(TM|_{S_p}) \qquad (2.1)$$
$$H^0(TM|_{S_p}) := H^0(S_p, \varphi^*(TM)).$$

Generally speaking, some holomorphic fields may not lift to deformations of the mapping φ of S_p into M. Therefore in general situation $\tilde{n}_B \leq \dim H^0(TM|_{V_p})$. In most applications however, there is an equality.

Let φ be an embedding of S_p in M, and NV_p be normal bundle over $V_p = \mathrm{Im}\,\varphi$, defined by exact sequence

$$0 \to TV_p \to TM|_{V_p} \to NV_p \to 0.$$

This implies exact sequense of cohomologies [3]:

$$0 \to H^0(TV_p) \to H^0(TM|_{V_p}) \to H^0(NV_p) \to H^1(TV_p) \to \ldots$$

If the group $H^1(TV_p)$ is trivial, then

$$\tilde{n}_B = \dim H^0(TV_p) + \dim H^0(NV_p). \tag{2.2}$$

The group $H^1(TV_p)$ is related to deformations of complex structure of Riemann surface S_p. Its dimension is known to be

$$\dim_{\mathbb{C}} H^1(TV_p) = \begin{cases} 0 & \text{for } p=0 \\ 1 & \text{for } p=1 \\ 3p-3 & \text{for } p>1. \end{cases} \tag{2.3}$$

Generically (for $p \geqslant 1$) the quantity of zero modes will be less than (2.2): in fact it contains $\dim \mathrm{Ker}(H^0(NV_p) \to H^1(TV_p))$ rather than to $\dim H^0(NV_p)$. Dimension of the group $H^0(TV_p)$ coincides with the number of Killing vectors:

$$\dim H^0(TV_p) = \begin{cases} 3 & \text{for } p=0 \\ 1 & \text{for } p=1 \\ 0 & \text{for } p>1. \end{cases} \tag{2.4}$$

The properties of normal bundle can be studied not only in terms of bundles over $V_p \subset M$ but over entire M. Consider the case, when $\dim_{\mathbb{C}} M = 2$ (complex surfaces). In this situation V_p is effective divisor on M, and exact sequence of the form

$$0 \to O_M \to O_M(V) \to \overset{\substack{NV \\ \parallel}}{O_M(V)|_V} \to 0$$

holds, which implies the long exact sequence of cohomologies,

$$0 \to H^0(\overset{\substack{\mathbb{C} \\ \parallel}}{O_M}) \to H^0(O_M(V)) \to H^0(NV) \to H^1(O_M)$$

Thus whenever $H^1(O_M) = 0$ (this is the case, for example, for $\mathbb{C}P^2$, Hirzebruch surfaces F_n, K3-surfaces and Enriques surfaces), we have

$$h^0(NV) = h^0(O_M(V)) - 1 .$$

In order to determine $h^0(O_M(V)) = h^0(V)$ one may use the Noether formula [3],

$$h^0(V) - h^1(V) + h^2(V) = \frac{V \cdot V - K \cdot V}{2} + p_a(M) \qquad (2.5)$$

K standing for canonical divisor, and $p_a(M) = \frac{1}{12}(K \cdot K + c_2(M))$ - for arithmetic genus of the surface. $V \cdot V$, $K \cdot V$, $K \cdot K$ are intersection indices of the divisors.

Equations for bosonic zero modes in the field of instanton follow form equations of motion, implied by the action S_{bos} (1.1):

$$(\partial G_{\bar{\mu}\nu})\bar{\partial} X^\nu + (\bar{\partial} G_{\bar{\mu}\nu})\partial X^\nu - \frac{\partial G_{\bar{\lambda}\nu}}{\partial X^\mu}(\bar{\partial} \bar{X}^\lambda \partial X^\nu + \partial \bar{X}^\lambda \bar{\partial} X^\nu) = 0$$

after the substitution $X^\nu = X^\nu_{inst} + \Phi^\nu$. They look as follows:

$$\partial G_{\bar{\mu}\nu}(X_{inst})\bar{\partial}\Phi^\nu = 0. \qquad (2.6)$$

2. Let us proceed to fermionic zero modes. Equation for left-hand zero modes follows from the action S_{ferm} (1.6):

$$\bar{\partial} \psi_L^\nu = 0 \qquad (2.7)$$

since holomorphic instanton satisfies $\bar{\partial} X^\nu_{inst} = 0$ (comp. with (1.3)). Therefore left-hand fermionic zero modes are holomorphic 1/2-differentials, i.e. holomorphic sections of the bundle $\Omega_{1/2}$. Left hand zero-modes cary also a vector index, since these are sections of the bundle TM. Thus we need the number of linearly independent holomorphic sections of the bundle $\Omega_{1/2} \otimes TM$.

Expression for this quantity, analogous to (2.1), is

$$n_L = \dim H^0(\Omega_{1/2} \otimes TM|_{V_p}). \qquad (2.8)$$

Similar formula exists also for the number of right-hand zero modes. First of all, note that equation for right-hand modes is

$$\partial G_{\bar{\mu}\nu}(x_{inst})\,\Psi_R^{\nu} = 0. \qquad (2.9)$$

This implies, that the right-hand modes are antiholomorphic sections of the bundle of (-1/2,0)-differentials $\bar{\Omega}_{-1/2}$ after they are multiplied by the metric $G_{\bar{\mu}\nu}(x_{inst})$ induced on the curve $V_p = \operatorname{Im} S_p$ and (as it was discussed in 1.2) by $g^{1/2}_{\xi\bar{\xi}}$. With respect to their transformation properties the objects of this kind can be described in terms of a bundle, dual to $\Omega_{-1/2} \otimes TM|_{V_p}$. Right-hand zero modes are related to the sections of $(\Omega_{-1/2} \otimes TM|_{V_p})^* =$ $= \Omega_{1/2} \otimes TM^*|_{V_p}$ and

$$n_R = \dim H^0(\Omega_{1/2} \otimes TM^*|_{V_p}). \qquad (2.10)$$

§3. SPHERICAL INSTANTONS

1. We restrict ourselves here with discussion of only non-singular instantons S_0. This means, that the image of sphere $S_0 = \mathbb{C}P^1$ is smooth rational curve in M. Let us assume, that M is defined by a system of equations in $\mathbb{C}P^n$ and has an induced metric. Then the degree q of our curve is related to the charges $Q^{(j)}$ (1.2) as follows:

$q = \sum Q^{(j)} n^{(j)}$ and in this case $n^{(j)}$ are integers.

In order to calculate the dimension of the space of zero-modes one should know the structure of the bundle $TM|_{V_0}$. The fibers of this bundle are d-dimensional complex tangent spaces to M at the points of the base V_0 . Each bundle over $\mathbb{C}P^1$ can be expanded in the sum of linear bundles (i.e.with one-dimensional fibers). All linear bundles over $\mathbb{C}P^1$ are exhausted by already familiar bundles of j-differentials, $\Omega_j = O(m)$, $m=-2j$. Let us remind, that any bundle over sphere is determined by its transition function. Transition function is determined on a spere with two punctures and takes values in the group, which acts in the fibers. From the point of view of their homotopic properties such functions are classified by elements of the group π_1 (fibre group). In the case of linear bundles the corresponding invariant is reffered to as the first Chern class. The Chern class of the bundle $O(m)$ is $c_1(O(m))=m$. Any linear bundle over $\mathbb{C}P^1$ coincides with some $O(m)$, m being equal to the first Chern class of the bundle. In particular, $O(m_1)\otimes O(m_2) = O(m_1+m_2)$. Therefore

$$TM|_{V_0} = \sum_{K=1}^{d} O(m_K). \tag{3.1}$$

This means, that provided V_0 is rational curve, the group of sections of the bundle $TM|_{V_0}$ possesses a basis, consisting of sections, which transform as $(j_k,0)$-differentials $(j_k = -m_k/2)$ under transition from one card to another. Now (3.1) implies, that

$$\tilde{h}_B = \sum_{K=1}^{d} \dim H^0(O(m_K)). \tag{3.2}$$

Let us calculate the dimensions of these cohomology groups. What we need are holomorphic sections of the bundle $\Omega_j = O(m)$ over CP^1 ,i.e. pairs of holomorphic functions $f_{(j)}(\xi)$ and $f'_{(j)}(\xi')$ on the sphere,related by $f'_{(j)}(-1/\xi) = \xi^{2j} f_{(j)}(\xi)$. Both functions should not have poles,ramification points and other singularities. Thus they are polynomials. Let $f_{(j)}(\xi) = \xi^K$. Then $f'_{(j)}(-\frac{1}{\xi}) = \xi^{2j+K}$ and $f'_{(j)}(\xi') = (-1)^j (\xi')^{-2j-K}$. Non-singularity conditions $K \geqslant 0$ and $-2j-K \geqslant 0$ imply,that Ω_j possesses holomorphic sections only for $j \leqslant 0$, their number being exactly $2|j|+1$: $d\xi^j, \xi d\xi^j, \ldots |\xi|^{|2j|} d\xi^j$

$$\dim H^0(CP^1 | \Omega_j) = \begin{cases} 0 & j > 0 \\ 2|j|+1 & j \leqslant 0. \end{cases} \qquad (3.3)$$

In order to find the quantity of right-hand zero-modes, we need to know the number of sections of Ω_j ,which satisfy the equation

$$\partial \left(g^{-j}_{\xi,\bar\xi}(\xi,\bar\xi) f_{(j)}(\xi,\bar\xi) \right) = 0 \quad {}^{*)}$$

We shall reffer to these sections as antiholomorphic. This equation implies,that $f_{(j)}(\xi,\bar\xi) = \bar\xi^K g^j_{\xi,\bar\xi}$. Then $f'_{(j)}(-1/\xi, -1/\bar\xi) =$ $= \xi^{2j} f_{(j)}(\xi,\bar\xi) = \xi^{2j} \bar\xi^K g^j_{\xi,\bar\xi}(\xi,\bar\xi)$ and $f'_{(j)}(\xi',\bar\xi') = \bar\xi'^{2j-K} g_{\xi,\bar\xi}(\xi',\bar\xi')$. It follows from non-singularity conditions $K \geqslant 0$, $2j-K \geqslant 0$ that "antiholomorphic" sections of Ω_j over CP^1 exist,only if $j \geqslant 0$ their number being $2j+1$:

$$[g_{\xi,\bar\xi} d\xi d\bar\xi]^{-j} d\xi^j = g^{-j}_{\xi,\bar\xi} d\bar\xi^{-j}, g^{-j}_{\xi,\bar\xi} \bar\xi d\bar\xi^{-j}, \ldots g^{-j}_{\xi,\bar\xi} \bar\xi^{2j} d\bar\xi^{-j}$$

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*) Let us remind,that the metric on CP^1 is $g_{\xi,\bar\xi} = (1 + \frac{|\xi|^2}{R^2})^{-2}$ and $g_{\xi,\bar\xi}(-\frac{1}{\xi}, -\frac{1}{\bar\xi}) = \xi^2 \bar\xi^2 g_{\xi,\bar\xi}(\xi,\bar\xi)$.

It is clear now, that antiholomorphic sections of Ω_j are in one-to-one correspondence with the holomorphic sections of Ω_{-j}. We already mentioned in ss.1,2, that these bundles are dual,

$$\Omega_j^* = \Omega_{-j} \quad \text{and}$$

$$\dim H^0(\mathbb{C}P^1, \Omega_j^*) = \dim H^0(\mathbb{C}P^1, \Omega_{-j}) = \begin{cases} 2j+1 & j \geq 0 \\ 0 & j < 0 \end{cases}. \qquad (3.4)$$

Now we can proceed to calculation of the quantity of zero-modes in the field of instanton with the charge unity. From eqs. (3.2) and (3.3) it follows, that

$$\tilde{n}_B = \sum_{i=1}^{d} (m_i + 1). \qquad (3.5)$$

Introduce now another number, n_B, which is the quantity of instanton deformations, which do not touch the image of infinite point of $S_0 = \mathbb{C}P^1$. Among holomorphic sections of each bundle $O(m_i)$: $1, \xi, \ldots \xi^{m_i}$ only the last one, ξ^{m_i}, does not vanish at infinite point (in the map, to which this point belongs, the same sections look like $\xi'^{m_i}, \ldots \xi', 1$ and only the last one is non-vanishing at $\xi' = 0$). Therefore, in order to obtain n_B one should subtract unity from each of non-vanishing items in the formula (3.2) for $\dim H^0(\mathbb{C}P^1, O(m_i))$. Thus

$$n_B = \sum_{\substack{i=1 \\ m_i \geq 0}}^{d} m_i \qquad (3.6)$$

(comp. with eq.(3.5)).

The quantity of left-hand zero-modes may be found from eqs. (2.8), (3.1) and (3.3):

$$n_L = \dim H^0(\mathbb{C}P^1, O(-1) \otimes \sum_{i=1}^{d} O(m_i)) = \sum_{i=1}^{d} \dim(\mathbb{C}P^1, O(m_i - 1)) = \sum_{i=1}^{d} m_i \quad m_i \geq 0. \qquad (3.7)$$

We see, that this quantity is the same as n_B : $n_B = n_L$

For calculation of right-hand zero-modes we need the dual bundle $TM^*|_{CP^1} = \sum_{i=1}^{d} O(m_i)^* = \sum_{i=1}^{d} O(-m_i)$. In this case (see eq. (2.1)):

$$n_R = \dim H^0(CP^1, O(-1)\otimes TM^*|_{CP^1}) =$$
$$= \sum_{i=1}^{d} \dim H^0(CP^1, O(-m_i-1)) = \sum_{\substack{i=1\\ m_i<0}}^{d} |m_i| . \qquad (3.8)$$

The difference between quantities of left- and right-hand zero--modes coincides with the first Chern class of the bundle :

$$n_L - n_R = \sum_{i=1}^{d} m_i = c_1(TM|_{CP^1}). \qquad (3.9)$$

This is the consequense of index theorem: axial anomaly in σ-model reads $\partial_a j_a^5 = R_{\mu\bar\nu}\partial X^\mu \bar\partial \bar X^\nu$ while the first Chern class is given by the integral $\int_{S_0} R_{\mu\bar\nu}(X_{inst})\, dX^\mu_{inst}\, d\bar X^\nu_{inst}$. Eq.(3.9) is valid for instantons of arbitrary genus.

From eqs.(3.7)-(3.9) the values of n_B, n_L, n_R may be easily determined for holomorphic spherical instantons in two interesting classes of sigma-models: those on Ricci-flat Kahlerian*) and on homogeneous Kahlerian manifolds. (Some time ago these numbers were determined by explicit enumeration of all zero--modes in these models [4].)

For Ricci-flat Kahlerian manifold M the first Charn class of the tangent bundle TM is zero. Therefore for spherical instanton $V_0 \subset M$ we have: $\sum_{i=1}^{d} m_i = 0 \quad d = \dim_{\mathbb{C}} M$. (3.10)

From the relations $0 = c_1(TM|_{V_0}) = c_1(TV_0) + c_1(NV_0)$, $c_1(TV_0)=2$ we conclude, that for the normal bundle $c_1(NV_0) = -2$. In the

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*) Ricci-flat Kahlerian manifolds are also reffered to as Calabi--Yau manifolds. Particular set of this class consists of hyperkahlerian manifolds.

case of d=2 [*] this is enough to find the quantity of zero-modes Indeed, for d=2 NV_0 is a one-dimensional bundle over the sphere V_0. Therefore from $c_1(NV_0)=-2$ it follows, that $NV_0 = O(-2)$. Thus $H^0(NV_0)=0$ and $H^0(TM|_{V_0}) = H^0(TV_0)$, $\dim H^0(TM|_{V_0}) = 3$. After all $n_B = \dim H^0(TM|_{V_0}) - 1 = 2$. Analogous reasoning allows one to find also, that $n_L = n_R = 2$ and thus the following relations hold:

$$n_B = n_L = n_R\ , \quad n_F = n_L + n_R = 2n_B. \tag{3.11}$$

For arbitrary Ricci-flat manifold M denote non-negative m_j through m_j^+, and negative m_j -- through m_j^-. Then $\sum m_j^+ = \sum |m_k^-|$, $n_B = \sum m_j^+$, $n_L = \sum m_j^+$, $n_R = \sum |m_j^-|$. Thus relations (3.11) hold for all Ricci-flat manifolds, including three-dimensional Calabi-Yau ones, which are of special interest for string compactifications.

Note also, that for hyperkahlerian manifolds (for which explicit hyperkahlerain metrics are known, see, e.g. ref.[5]),

$$T(T^*CP^n)|_{V_0} = O(2) + (n-1)O(1) + O(-2) + (n-1)O(-1) \tag{3.12}$$

if $V_0 \simeq CP^1$ is embedded in $CP^n \subset T^*CP^n$ by linear equation, and

$$T(T^*CP^n)|_{V_0} = 2O(3) + (n-2)O(2) + 2O(-3) + (n-2)O(-2) \tag{3.13}$$

for quadratic embedding of $V_0 \simeq CP^1$.

Proceed now to homogeneous kahlerian manifolds. For them all the $m_i \geqslant 0$ (Holomorphic sections of $TM|_{V_0}$ arise, in particular, from Killing vectors on M, and because of transitive action of the group, Killing vectors at any point give rise to the whole TM.) Therefore in the corresponding sigma-models $n_B = n_L$

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[*] Compact two-dimensional manifolds with $c_1 = 0$ are tori and so called K3-surfaces. The latter ones will be considered separately.

and n_R . Let us list now expansions of tangent bundles $TM|_{V_0}$ for all Kahlerian symmetric manifolds M and for a class of non--symmetric homogeneous M (for flag spaces with Einstein metric) in the case of linear embedding of $V_0 \simeq \mathbb{C}P^1$ into M.

$$T\mathbb{C}P^n = O(2) + (n-1)O(1)\,, \quad n_B = n_L = 2 + (n-1) = n+1.$$

Generically, for Grassmanians

$$T(SU(m+n)/S(U(m)\oplus U(n)) = O(2) + (m-1)O(1) + (nm-n-m)O(0)$$
$$d = nm\,, \quad n_B = n_L = m+n\,;$$
$$T(Sp(n)/SU(n)\oplus U(1)) = O(2) + (n-1)O(1) + \frac{n(n-1)}{2}O(0)$$
$$d = \frac{n(n+1)}{2}\,, \quad n_B = n_L = n+1\,;$$
$$T(SO(2n)/SO(2n-2)\oplus SO(2)) = O(2) + (n-1)O(1) + O(0)$$
$$d = 4n\,, \quad n_B = n_L = 2 + (n-1) = n+1\,;$$
$$T(U(n)/U(1)^n) = O(2) + (n^2-n-1)O(1)$$
$$d = n^2-n\,, \quad n_B = 2 + (n^2-n-1) = n^2-n+1.$$

§4. TORIC INSTANTONS

1. σ-model $S_1 \to M$ with a torus as a world-shhet is described by the same action as the ordinary σ-model, only the metric $d\xi\, d\bar{\xi}$ is now flat, and all the fields $X(\xi,\bar{\xi})$, $\psi_{L,R}(\xi,\bar{\xi})$ are twice periodic functions

$$X(\xi+1, \bar{\xi}+1) = X(\xi+\tau, \bar{\xi}+\bar{\tau}) = X(\xi,\bar{\xi}),$$
$$\psi_{L,R}(\xi+1, \bar{\xi}+1) = \psi_{L,R}(\xi+\tau, \bar{\xi}+\bar{\tau}) = \psi_{L,R}(\xi,\bar{\xi}).$$

Since periodic boundary conditions are imposed on both fermionic and bosonic fields and covariantly constant -1/2-differentials $\mathcal{E}_{L,R}$ exist, this σ-model posesses supersymmetry.

2. The number of zero-modes in the field of holomorphic toric instanton can be defined by the general rules of §2. It is only necessary to remember that even for non-singular instanton the image $\mathrm{Im}\, S_1$ in M should not be a torus, it may be also a sphere (or a point). The structure of bundles over torus is not so simple, as it is over a sphere. Generically a multidimensional bundle is not a direct sum of linear (1-dimensional) ones. From s.2.1 we may conclude, that

$$\tilde{n}_B \le \dim H^0(TS_1) + \dim H^0(NS_1) = 1 + \dim H^0(NS_1). \qquad (4.1)$$

Let us remind, that $\dim H^0(TS_1)$ is the quantity of (complex) Killing vectors on S_1, and the non-equality appears, because some deformations of the genus 1 curve in M may be accompanied by deformations of the complex structure of the curve itself.

Tangent bundle to $V_1 = \mathrm{Im}\,\varphi$ is a subbundle in $TM|_{V_1}$ while the normal one, $\mathcal{N}V_1$, is a factor-bundle:

$0 \to TV_1 \to TM|_{V_1} \to \mathcal{N}V_1 \to 0$. Therefore the following restrictions from below on the number of zero-modes arise (comp. with eq.(4.1)):

$$\tilde{n}_B = \dim H^0(TM|_{V_1}) \ge \dim H^0(TV_1) = \dim(\Omega_{-1}) = 1, \qquad (4.2)$$

$$n_L = \dim H^0(\Omega_{1/2} \otimes TM|_{V_1}) \ge \dim H^0(\Omega_{1/2} \otimes TV_1) =$$
$$= \dim H^0(\Omega_{-1/2}) = 1$$

$$n_R = \dim H^0(\Omega_{1/2} \otimes T^*M|_{V_1}) \ge \dim H^0(\Omega_{1/2} \otimes \mathcal{N}V_1^*) =$$
$$= \dim H^0(\mathcal{N}V_1^*)$$

The last restriction is because the dual bundle to NV_1 (i.e.conormal bundle) is a subbundle in the cotangent one, $T^*M|_{V_1}$, i.e. $0 \to NV_1^* \to T^*M|_{V_1} \to T^*V_1 \to 0$.

Linear bundles over torus are topologically classified by π_0 and π_1 of the structure group of the bundle (the elements of this group are transition functions,acting in fibers). This is because torus may be considered as a cylinder after it is cut along one non-contractable cycle (or it arises from a cylinder with the help of transition function defined on another non-contractable cycle - this is the origin of π_1 (group)). Bundles over a cylinder are equivalent to those over a circle and are charactarized by a single transition function at a single point. All linear bundles over torus are exhausted by θ -bundles, which ha θ -functions as their holomorphic sections, and dual bundles. As holomorphic bundles these are also characterized by moduli - elements of Picard group or points of Jacobian of the curve S_1 . Linear bundles Ω_j of j-differentials are trivial over torus for all j, they possess exactly one doubly periodic holomorphic section: $\mathrm{const}\cdot dz^j$. The same sections are also antiholomorphic ones. (To compare,let us remind that Ω_j exhauste all non-equivalent linear bundles over $\mathbb{C}P^1$.) The triviality of Ω_j over torus implies,that despite all the problems with computation of n_B and n_L in the case of holomorphic toric instantons, these numbers are always the same:

$$n_B = \dim H^0(TM|_{V_1}) = \dim H^0(\Omega_{1/2} \otimes TM|_{V_1}) = n_L . \qquad (4.3)$$

The difference $n_L - n_R$ is dictated by index theorem:

$$n_L - n_R = \dim H^0(\Omega_{1/2} \otimes TM|_{V_1}) - \dim H^0(\Omega_{1/2} \otimes T\bar{M}|_{V_1}) = \\ = c_1(TM|_{V_1}). \qquad (4.4)$$

3. Let us discuss briefly several simple examples.

First of all, for Ricci-flat manifolds M $c_1(TM|_{V_1})=0$ and $n_L = n_R$. Therefore $n_B = n_L = n_R$ in this case, and this implies vanishing β-function from instanton calculus.

Turning to symmetric Kahlerian manifolds, we consider only toric instanton for $M = CP^1$. Following ref.4 it is straightforward to do the same analysis for other manifolds. β-function calculated with the help of toric instantons in all cases is the same as the one from spherical instantons.

Topological charge of the holomorphic instanton in the σ-model on $M = CP^1$ is the number of poles of the map $S_p \to M$ on the world surface S_p. The minimal number of poles, which holomorphic function can have on S_1 is 2. According to Riemann--Roch theorem such map has 4 parameters. This map may be easily given in explicit form: $X_{inst} = \lambda \dfrac{\theta(\xi-a)\,\theta(\xi-b)}{\theta(\xi-c)\,\theta(\xi-d)}$

and periodicity requires, that $a+b=c+d$ (Abel theorem). Any other function with poles at the same points c and d and zero at a (then the second zero is automatically at b) may differ from X_{inst} only in overall constant. Thus $n_B=4$. By index theorem $n_L - n_R = 4$. In the case of torus also $n_B = n_L$. It is easy to ensure explicitly, that there are no right-hand zero--modes. Equation for left-hand fermionic zero-mode is $\bar\partial \psi_L = 0$, the norm of this mode is $\|\psi_L\|^2 = \int_{S_1} G(X_{inst})\, g_{\xi\bar\xi}^{1/2}\, |\psi_L|^2 d\xi\, d\bar\xi$ and metric in fibers, $G(X_{inst}) = (1+|X_{inst}|^2)^{-2}$, vanishes at points and . Equation for the right-hand zero-mode reads $\partial G(X_{inst})\psi_R = 0$ and implies, that $\psi_R = const/G(X_{inst})$ (any antiholomorphic differential on torus is constant). The norm of this would be zero-mode is $\int_{S_1} g_{\xi\bar\xi}^{1/2} G(X_{inst}) |\psi_R|^2 d\xi\, d\bar\xi$ Integrand has poles at

$\xi = c$ and $\xi = d$, the integral diverges, and thus right--hand zero-modes are absent.

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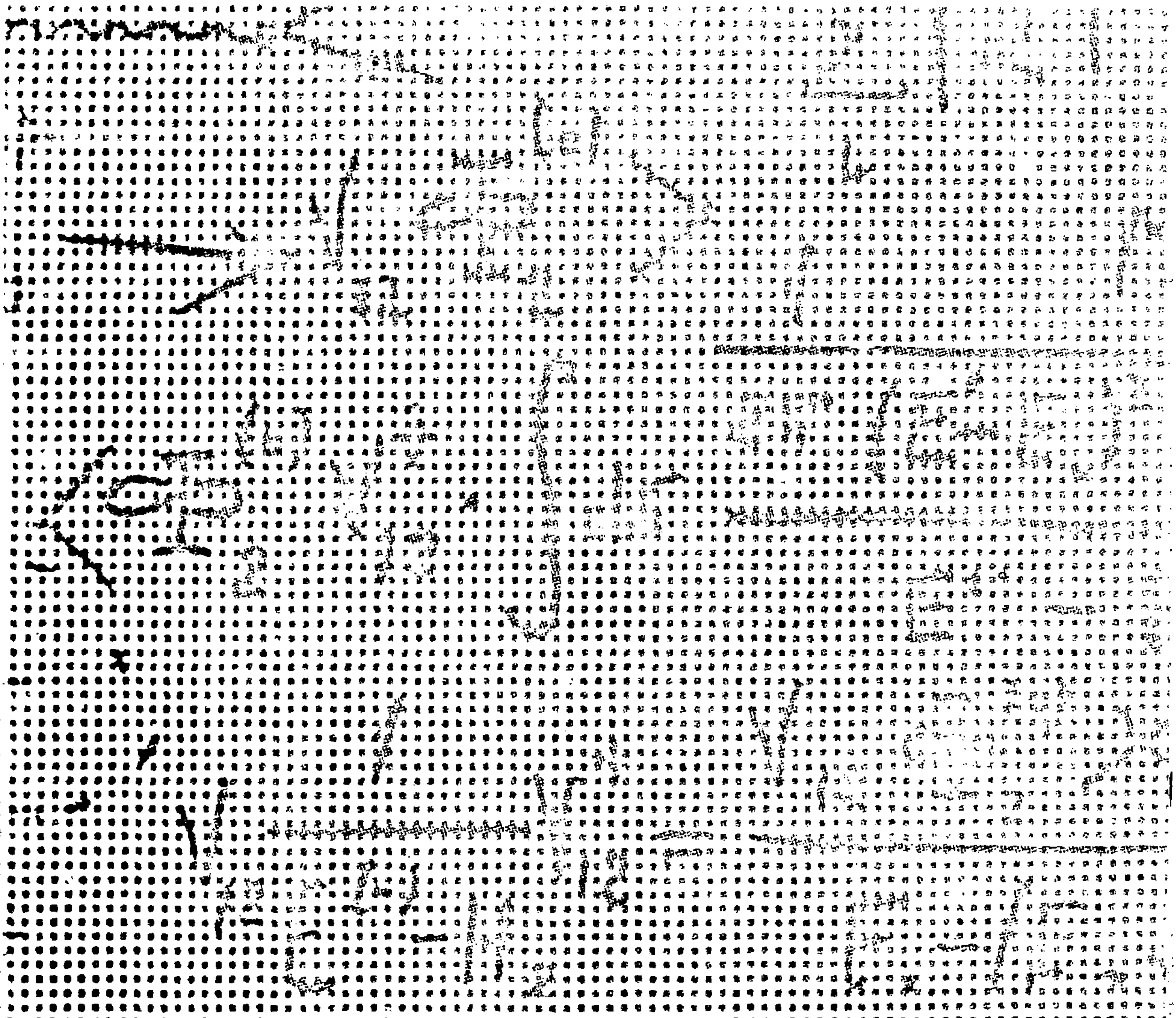
Я.Коган и др.
Некоторые примеры инстантонов в сигме-моделях.
Работа поступила в ОНТИ I5.0I.88

Подписано к печати I7.II.88 Т22340 Формат 60x90 I/I6
Офсетн.печ. Усл.-печ.л.I,5. Уч.-изд.л.I,I. Тираж 270 экз.
Заказ I5I Индекс 3624 Цена I6 коп.

Отпечатано в ИТЭФ, II7259, Москва, Б.Черемушкинская, 25

ИНДЕКС 3624



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