SCALING AND CRITICAL BEHAVIOUR IN NUCLEAR FRAGMENTATION

X. Campi
Division de Physique Théorique;
Institut de Physique Nucléaire, 91406, Orsay Cedex, France

Abstract

These notes review recent results on nuclear fragmentation. An analysis of experimental data from exclusive experiments is made in the framework of modern theories of fragmentation of finite size objects. We discuss the existence of a critical regime of fragmentation and the relevance of scaling and finite size scaling.
1 Introduction

One of the present goals of nuclear science is the determination of the equation of state of infinite nuclear matter. The knowledge of this object in a wide range of physical conditions is fundamental for future developments of nuclear theory as well as a basic input for astrophysics. Particularly interesting is the behaviour of the equation of state in the vicinity of phase transitions. For example, one expects that at a temperature of about 6 to 10 MeV a sort of liquid-gas phase transitions takes place in infinite nuclear matter. We ignore if it is possible to study this phase transition in the laboratory. By energetic heavy ions collisions we are able to produce during short times very small pieces of nuclear matter at the required conditions, but we don't know what are the signals of a "phase transition" in a transient state of so small a system. In addition, energetic collisions are always associated with nuclear fragmentation. We will see that this fragmentation has many similarities with a phase transition.

The aim of these lectures is to discuss nuclear fragmentation in the general framework of new theories of fragmentation. The study of particle fragmentation is a new and fast developing field of statistical mechanics which concerns many physical systems of the microscopic world, including polymers, gels, atomic clusters, nuclei and elementary particles. The main goal is to investigate what features are common to various fragmentation processes regardless the nature of the elementary constituents or the binding interactions of the fragmenting object. A particularly significant outcome of this research is the realization that in many circumstances the fragment size distribution and related quantities are scale invariant. This realization clears the way to a model independent classification of the various fragmentation mechanisms in terms of the scaling properties. The concept of scaling with respect to the size.
(mass or charge) of the fragmenting system may be very fruitful in nuclear physics. In fact this concept is commonly used but loosely defined. For example, how do we define an Intermediate Mass Fragment (IMF)? Obviously this mass is not the same in Uranium fragmentation and Carbon fragmentation. What is the dependence (linear, logarithmic...) on the mass of the fragmenting system? We will see that various fragmentation regimes coexist and that this mass dependence changes from one regime to another.

These notes are organised as follows. Section 2 is devoted to review some basic ideas on critical behaviour, scaling and finite size scaling. For pedagogical purposes these concepts are developed in the framework of very simple geometrical models. The fragmentation of atomic nuclei and the relevance of the above concepts are discussed in section 3. Section 4 contains some final remarks.

2 Scaling and critical behaviour

First we recall some basic definitions concerning critical phenomena. For a detailed discussion, the book of Stanley [1] remains the standard reference in the field.

Let's start with a mathematical introduction of scaling. Many simple functions $F(x,y)$ of two variables approach to the leading order the simpler form

$$F(x, y) = x^A f(y/x^B) \quad (2.1)$$

if both $x$ and $y$ approach zero. (If a variable goes to infinity, use the reciprocal value of this variable). Equation (2.1) is not a mathematical theorem, but most simpler functions have this property. For example, $F(x,y) = (x + \sin(x,y))(y + xy)$ looks very complicated, but if $x$ and $y$ go to zero with fixed ratio then we get asymptotically...
\[ F(x,y) = \frac{x}{y} \sin(\frac{x}{y})/x. \] Thus in equation (2.1), \( A = -1 \), \( B = 1 \) and \( f(z) = \sin(1/z)/z \).

This mathematical property applies in many domains of physics, particularly in phase transitions. For example, in the Fisher droplet model [2] the number of clusters of size \( s \) at temperature \( T \) is given by

\[ \eta(s, T) \sim s^{-\tau} \exp(-\text{const} (T - T_c) s^\sigma) \] (2.2)

i.e. by a scaling invariant part \( s^{-\tau} \) and a scaling function \( f(s) \). Here \( \tau \) and \( \sigma \) are two critical exponents and \( T_c \) is the critical temperature. We recall that critical phenomena are the phenomena that manifest in the vicinity of the transition point of a second order phase transition. A critical exponent is a number that describes the behaviour of a physical quantity near that critical point. Consider a real and non-negative function \( f(x) \) defined in the interval \( ]0, x_0[ \). If the limit

\[ \lambda = \lim_{x \to 0^+} \frac{\log f(x)}{\log(x)} \] (2.3)

exists when \( x \) goes to zero on the positive side, than \( \lambda \) is called the critical exponent to be associated with the function \( f(x) \). (The definition is extended straightforwardly to the interval \( [x_1, 0[ \) when \( x \to 0^- \). It is important to stress that the shorthanded notation that will be frequently used

\[ f(x) \sim x^\lambda \] (2.4)

does not imply that

\[ f(x) = Ax^\lambda \] (2.5)

although the converse is true.

(Take for example the function \( f(x) = Ax^\lambda (1 + ax^\mu) \), \( \mu \to 0 \) that has \( \lambda \) as critical exponent).
2.1 - Percolation: A simple example of critical phenomena

Many geometrical models of clusterization exhibit particularly simple critical behaviour. These models consist in ensembles of points in a space, which are linked by some mechanisms. The points represent the positions of the constituents in the space (coordinate, phase space...). The positions of the points are either fixed (static models) or change with time (kinetic models). The linkage mechanisms are in general very simple random mechanisms based on proximity rules. Remark that "geometrical" is used here in a very loose sense. We will describe percolation models in some details with the purpose to introduce critical phenomena and finite size scaling manifestations in cluster production.

A percolation model is a collection of static points or sites distributed in space, certain pairs of which are said to be adjacent or linked \[5\]. Whether or not two sites are linked is governed by a random mechanism the details of which depend on the context in which the model is used. There are two classes of percolation known as bond and site percolations. In the site problem, the sites are occupied at random with a probability \( p \). All bonds between occupied sites located at less than a certain distance are active. In the bond problem, all sites are occupied and bonds are active with probability \( p \). The sites may be partitioned into clusters such that pairs of occupied sites in the same cluster are connected by active bonds but there is no path between sites in different clusters.

When \( p \) is close to zero most sites will be isolated or form very small clusters. In the opposite, if \( p \) is close to one then nearly all sites are connected forming one large cluster, which occupies most of the available sites. This is called a "percolation" cluster. It is observed that in sufficiently large systems (see below), there is either one or none, but never two or more such percolation clusters. For infinite systems a sharply defined percolation threshold \( p_c \) exists such that for
\( p < p_c \) no percolating cluster exists and for \( p > p_c \) one percolating (infinite) cluster exists. The transition from a non-percolating state to a percolating state is a kind of phase transition. The main difference between percolation and other phase transitions is the absence of a Hamiltonian. The percolation transition is a purely geometrical phenomenon in which the clusters are clearly defined static objects.

For theoretical purposes it is simpler to consider percolation models on a regular lattice. This reduction is done without any loss of generality because the behaviour of the percolation model near the critical point is independent of the details of the lattice. This behaviour is characterized by the critical exponents, which are "universal" to all short range linkage models of the same euclidean dimension \( d \). This means, for example, that the exponents for triangular and square lattices \((d = 2)\), site or bond percolation, are strictly the same although the percolation thresholds \( p_c \) may vary by a factor of two.

One of the main problems in percolation theory is that of determining the distribution of cluster sizes. The average number of clusters per lattice site has been demonstrated [3] to follow approximately a scaling relation near \( p_c \) for large cluster sizes \( s \):

\[
    n_s(e) \sim s^{-\tau} f(s/s_\xi(e)) + ...
\]

where \( e \propto p - p_c \) and \( s_\xi(e) \) is the characteristic cluster size that diverges at \( p \rightarrow p_c \) as

\[
    s_\xi \sim e^{-1/\sigma}
\]

Here \( \tau \) and \( \sigma \) are two critical exponents and \( f \) a scaling function satisfying \( f(0) = 1 \), i.e. at the critical point \( n_s \sim s^{-\tau} \). Remark that \( \tau \) and \( \sigma \) are "universal" once \( d \) is fixed, but \( f \) is model dependent.
For illustration, we will derive (2.6) in one dimension. Consider a linear chain of \( L \) sites. Each of these sites is randomly occupied with probability \( p \) (empty with probability \( 1 - p \)) and that all the bonds between nearest neighbours are active (site percolation model). The probability that \( s \) arbitrary sites are occupied is \( p^s \). The probability of one end having an empty neighbour is \( (1 - p) \) and therefore the total probability of finding an \( s \) - cluster is \( p^s (1 - p)^2 \). When \( L \) is large, boundary effects are negligible. Then \( L p^s (1 - p)^2 \) is the total number of \( s \) - clusters, and the number of \( s \) - cluster per lattice site is

\[ n_s^{\text{site}}(p) = p^s (1 - p)^2 \]  

(2.8)

Analogous reasoning gives for the bond percolation

\[ n_s^{\text{bond}}(p) = p^{s-1} (1 - p)^2 \]  

(2.9)

We now discuss the result (2.8).

When \( p = 1 \) all sites are occupied and the chain contains a single infinite cluster called percolation cluster. For every value \( p < 1 \) the chain will have on average \( (1 - p)L \) empty sites and there is no percolation cluster. The percolation threshold is at \( p = 1 \). Because \( p < 1 \), there is no phase transition in this one dimensional model, as usual. Nevertheless we will see that the system approaches a critical behaviour like in (2.6) when \( p \to p_c = 1 \).

We can write [7],

\[ p = \exp (\ell np) \approx \exp (1 - p) = \exp (p - p_c) \]

and

\[ n_s^{\text{site}}(p) \approx s^{-2} |(p - p_c)s|^2 \exp [(p - p_c)s] \]

\[ = s^{-r} f(z) \]  

(2.10)

7
with \( r = 2, z = (p - p_c) s^\sigma, \sigma = 1 \) and

\[
f(z) = z^2 e^z.
\]

Remark that \( n_a \) decreases exponentially with \( s \) when \( p \ll p_c \) and as a power law when \( p \to p_c \).

The average size of the clusters

\[
S = \frac{\sum s^2 n_s(p)}{\sum s n_s(p)}
\]

\[= (1 + p)(1 - p)^{-\gamma}
\]
diverges at threshold with an exponent \( \gamma = 1 \).

The correlation function \( g(r) \) is defined as the probability that a site at a distance \( r \) away from an occupied site belongs to the same cluster. Obviously \( g(0) = 1, g(1) = p \) and in general in one dimension,

\[
g(r) = p^r = \exp \left[ -\frac{r}{\xi} \right]
\]

where

\[
\xi = \frac{1}{\ln(p)} \sim (p_c - p)^{-\nu} (p \to p_c)
\]
is the correlation length and \( \nu = 1 \) a new critical exponent.

Cluster size distributions like (2.8) or (2.9) may be seen as signatures of fragmentation of 1-dimensional objects. In most practical application it is better to work with the moments of \( n_a \),

\[
m_k = \sum s^k n_s
\]

For example for 1-dimensional bond percolation we have

\[
\begin{align*}
m_0 &= 1 - p \\
m_1 &= 1 \\
m_2 &= \frac{(1 + p)}{(1 - p)} \\
\vdots
\end{align*}
\]
Generally is better to eliminate the parameter $p$ which is difficult to measure, for example by taking the reduced multiplicity of clusters $m_0$ as driving parameter

$$m_2 = 2/m_0 - 1$$  \hspace{1cm} (2.16)

In higher dimensions it has been verified numerically that formula (2.6) works very well. In addition, when $p > p_c$ then exists an infinite percolation cluster. The intensity $P(p)$ of this percolation cluster, i.e. the number of sites that belong to this cluster per number of lattice sites goes to zero as

$$P(p) \sim (p - p_c)^\beta, \quad p \to p_c, \quad p > p_c$$  \hspace{1cm} (2.17)

where $\beta$ is a new critical exponent. This quantity plays the role of the order parameter in percolation theory (it is null in the most symmetric phase and takes a finite value in the other).

At the critical value $p_c$ the cluster size distribution $n_s$ is singular because all moments $m_k$ with $k > \tau - 1$ diverge. This is shown by [3]

$$m_k(\varepsilon) = \sum s^k n_s(\varepsilon)$$
$$\approx \int_0^{s_{\text{max}}} s^{k-\tau} f(s^{s^\varepsilon}) ds$$
$$= |\varepsilon|^{(\tau - k - 1)/\sigma} \int_0^{s^{s_{\text{max}}}} |z|^{(1+k-\tau)/\sigma} z^{-1} f(z) dz$$
$$= C_k^{\pm}(\varepsilon). |\varepsilon|^{(\tau - k - 1)/\sigma}$$  \hspace{1cm} (2.18)

where we have introduced the variable $z = \varepsilon s^{s^\varepsilon}$ and replaced the sum by an integral. For an infinite system the largest cluster $s_{\text{max}}$ is infinite and the integral has constant values $C^{\pm}$, depending on the sign of $\varepsilon$. The exponents of the moments $k = 0, 1$ and 2 are called $2 - \alpha, \beta$ and $-\gamma$ respectively, in analogy to thermal critical exponents. Then exponent relations like

$$\gamma + 2\beta = 2 - \alpha = (\tau - 1)/\sigma$$
are automatically fulfilled.

For a finite size system the value of the integral (2.18) depends on $\varepsilon$, and in particular goes to zero when $\varepsilon \to 0$. Then $m_\varepsilon(\varepsilon)$ takes the largest but finite values in the critical regions. We will study finite size effects in more details in the next Section.

Table 1 gives the values of the critical exponents of percolation theory for various space dimensions. We insist on the fact that these values are "universal", i.e. independent on the details of the model (bond or site percolation, the type of lattice). For a comprehensive study of percolation theory, see the excellent book of Stauffer [3].
2.2 - Finite Size Scaling

The above considerations strictly apply to infinite systems. Monte-Carlo simulations show that in finite systems the transition at $p_c$ is broader but equation (2.6) remains quantitatively valid in a restricted domain of cluster size $s$. For example, the behaviour of equation (2.6) is illustrated in figure 1 for the bond percolation model in a cubic lattice with $4 \times 4 \times 4$ sites. Figures 1a and 1d show the cluster size distribution far below and above the critical point respectively. We remark that in both situations the distribution of light fragments is a very fast (exponential-like) decreasing function. In addition, above the critical point exists a distribution of large clusters (centered in this example around $s = 155$) which is not accounted by eq.(2.6). Right at the critical point (fig.1b) the distribution is a power law (remark the change to a log-log plot). Slightly above the critical point (1c) one sees the rise of the large cluster at $s = 35$. This behaviour is quite general. Analogous distributions appear in thermal phase transitions and in kinetic processes of cluster formation.

When the scaling ansatz (2.6) is valid, how do quantities like $n_e, mk$ or $P$ behave in the critical region as a function of the (finite) size $S_T$ of the system? This behaviour is rather unexpected. One observes that the largest cluster that is present in the system (or percolation cluster) has a very ramified structure, containing holes of all sizes in the interior. Figure 2 shows a percolation cluster in a square lattice of size $50 \times 50$. It is useful to characterize this structure by a single number that we will call a "fractal dimension" $D_f$. In our context, this dimension has the following geometrical interpretation. Let's take boxes of linear size $L$ containing $N$ occupied sites of the cluster. If we can determine the relation

$$N \propto L^{D_f} \quad (2.19),$$

11
when $L \to \infty$ with $D_f \neq d$, the object is called a "fractal" of dimension $D_f$. Equation (2.19) means for the density

$$\rho \propto L^{D_f - d} \quad (2.20)$$

In a finite size cluster, the relation (2.19) may hold of course only for a restricted range of values of $L$ and as Monte Carlo average over many percolation clusters of the same system size.

Table 1 gives the values of the fractal dimension $D_f$ for the percolation cluster right at the critical point for various space dimensions $d$.

From equation (2.19) we deduce that the size $s_L$ of the percolation cluster grows with the total size $S_T = L^d$ of the system as

$$s_L \propto S_T^{D_f/d} \quad (2.21)$$

This result is plotted in a log-log scale in figure 2 for Monte Carlo data of the triangular ($d = 2$) lattice. The slope of this curve gives the fractal dimension. One obtains $D \simeq 1.9$, which is in very good agreement with the value given in table 1. What is really remarkable in this result is that the relation (2.21) is fulfilled even for extremely small lattices, thus expecting that small systems of the real world may follow the same relations.

Now we will discuss the behaviour of the other characteristic quantities. Here we follow closely the reasoning of reference [4]. In a finite box of size $L$ at $p = p_c$ the characteristic size is limited by $s_L \propto L^{D_f}$. From equation (2.7) one gets

$$m_k(L) \propto L^{-(r-k-1)D_f} \quad (2.22)$$

The correlation length $\xi$ is limited by 1. It diverges as indicated by equation (2.13) with

$$\nu = (\sigma D_f)^{-1} \quad (2.23)$$
Because the first moment $m_1$ is the density of occupied sites of the percolation cluster, one gets the relation

$$D_f = d/(r - 1) \quad (2.24)$$

This corresponds to the hyperscaling relation

$$d\nu = 2 - \alpha$$

from thermal critical phenomena. Using this relation and equation (2.6) gives the finite size scaling of the cluster size distribution function

$$n_s(L) = L^{-D_f - d} \tilde{f}(s/L^{D_f}) \quad (2.25)$$

with a new scaling function $\tilde{f}$.

Equations (2.22)-(2.25) give the expected scaling at the critical point. In the vicinity of that point one expects [3] a scaling of the type

$$L^{-z/\nu} F((p - p_c)L^{1/\nu}) \quad (2.26)$$

for a quantity varying as $|p - p_c|^{z/\nu}$.

3 - ANALYSIS OF NUCLEAR DATA

3.1 - Signals of critical behaviour

First we concentrate on the possible manifestation of a critical behaviour in nuclear multifragmentation. We turn now to the following question: Is there or is there not some critical phenomenon which manifest itself in nuclear fragmentation and if so, what form does it take? We believe the answer to this...
question is positive. The method we use here [9] consists in a comparison of observable quantities that behave in a qualitatively different way when a phase transition is present or not. We consider the partitions of the following systems:

a) A one-dimensional bond percolation system containing $S_T$ lattice sites. Bonds between sites are activated with probability $p$. Varying $0 \leq p \leq 1$ we change the shape of the fragment size distribution but we know that in this system we don't have a phase transition (percolation in 1 dimension).

b) A three-dimensional bond percolation model in a cubic lattice containing $S_T$ sites. In this case we know that when $S_T \to \infty$ and $p \to p_c \approx 0.25$ we have a sharp second order phase transition.

c) A gold nucleus with $0 < E_k < 200$ GeV kinetic energies is fragmented in collisions with emulsion plates [10]. For each of the events the charges $z_i$ of the fragments have been measured. We believe that all fragmentation regimes have been covered by this experiment because fragment multiplicities range from 1 to 79 (the total charge number of gold). Low energy fission events are not included in our analysis.

For system c) we have at our disposal 367 individual events. For systems a) and b) we generate the same number of events varying randomly the parameter $p$. Each event $i$ generates a partition

$$S_T \to \{s_1 \leq s_2 \leq \ldots \leq s_k\}$$  \hspace{1cm} (3.1)

with $k^i$ fragments. For the $k - 1$ lighter fragments containing $s$ particles, we define $n_s^i$ as the number of such $s$-fragments per total number $S_T$ of particles in the system. The largest fragment $s_k^i$ is studied separately. We will use the notations

$$s_{\max}^i = s_k^i$$

$$P^i = s_{\max}^i / S_T$$  \hspace{1cm} (3.2)
We also define a reduced multiplicity of the event $i$ by

$$m^i = (k^i - 1)/S_T$$  \hspace{1cm} (3.3)

The separation of the largest fragment $s_{max}$ is reminiscent of infinite percolation systems, for which above the threshold exist one infinite cluster with specific properties.

Now we characterize the fragment size distribution in each event $i$ by the moments

$$M_k^i = \sum_s s^kn_s^i$$  \hspace{1cm} (3.4)

where the sum runs over all fragment sizes present in the event, the largest one excluded. Because of particle number conservation we have

$$M_1^i + P^i = 1$$  \hspace{1cm} (3.5)

and by definition

$$M_0^i = m^i$$  \hspace{1cm} (3.6)

In order to improve the statistical significance of the results we also consider the conditional moments

$$M_k(m) = \sum_i \delta(m^i - m)M_k^i / \sum_i \delta(m^i - m)$$  \hspace{1cm} (3.7)

i.e. the average value of the $k$ - moment for events with reduced multiplicity $m$. Similarly,

$$P(m) = \sum_i \delta(m^i - m)P^i / \sum_i \delta(m^i - m)$$  \hspace{1cm} (3.8)
is the average reduced size of the largest fragment for events with reduced multiplicity $m$. We may also define an average fragment size distribution per events with reduced multiplicity $m$ by

$$n_s(m) = \frac{\sum \delta(m_i - m)n_s^i}{\sum \delta(m_i - m)} \quad (3.9)$$

and we have of course

$$M_k(m) = \sum_s s^k n_s(m) \quad (3.10)$$

In what follows, $m$ will play the role of the driving parameter $p$ of eqs. (2.7).

From (2.18) we expect an enhancement in the critical region of the moments $M_k$ for $k > r - 1$, with $r > 2$ as in most critical phenomena. We expect $M_0$ and $M_1$ to be smooth and $M_2$ to show a maximum in the critical region. This is what is observed for models b) and c) while for model a) $M_2$ decreases monotonously when $m$ increases [9].

More insight in the shape of the fragment size distributions $n_s(m)$ is obtained looking at a combination of moments $M_k$. For example, the dimensionless quantity

$$\gamma_2 = M_2 M_0 / M_1^2 \quad (3.11)$$

In figure 4 we have represented $\gamma_2$ as a function of $m$ for the experimental distribution c) (top) and for model (b) (bottom). We clearly see that in both cases $\gamma_2 > 2$ around $m \approx 0.25$. In contrast there is not such maximum for distribution a) (crosses, bottom) when the phase transition is not present. The maximum of $\gamma_2$ at $m \sim 0.25$ is the manifestation of a phase transition in finite size systems b) and c). This is shown in figure 5, where $\gamma_2(m)$ is plotted for various system sizes $(50^3, 9^3, 5^3, 3^3)$ of model b). We see clearly the critical behaviour for the largest system and how it is smoothed when decreasing the size. This broadening of the
signal in small systems is clearly seen in figure 6, where we show $\gamma_2(m)$ for the fragmentation of a beam of 2.1 GeV/nucleon $^{12}\text{C}$ ions [11].

The fluctuations in the fragment size distributions are expected to be largest near the critical points. This is also true in finite systems and it is clearly shown in figure 4, although this is better seen by considering a different measure. In the left of figure 7 are plotted the distributions $P(m)$ of the largest fragment produced per event. On the right is represented the fluctuation (standard deviation) of that quantity. Again we remark that b) and c) are quite similar. The maximum of $\Delta P$ around $m \approx 0.25$ reinforces the conviction that at this point nuclei have crossed a finite size phase transition. We conclude that nuclei fragment like a finite size system that shows a phase transition in infinite size.

3.2 Finite size scaling

In the present analysis we use three different sets of experimental data points. A first set consists of data from inclusive proton-nucleus reactions at high energies. For this set, the size $s$ of the fragments will be the mass number $A$. The choice of protons as projectiles is dictated by the fact in this case the cross section for the production of isobars of mass $A$ from a target of mass $A_T$ can be factorized as

$$\sigma(A, A_T) = \sigma_R(A_T)n(A, A_T)$$

(3.12)

where $\sigma_R(A_T)$ is the total p-nucleus reaction cross section and $n(A, A_T)$ the average number of fragments of mass $A$ produced per collision. These quantities are bombarding energy dependent. However it is observed experimentally that above an energy of the order of a few GeV (for light targets) or 10 to 20 Gev (for heavy targets) $\sigma(A, A_T)$ reaches a limiting value [18]. This energy dependence and the shape of the cross sections can be understood qualitatively in terms of.
fragment size distributions of the type of figure 1. At low bombarding energies only reactions of type (d) are energetically allowed, producing U-shaped \( \sigma(A, A_T) \) distributions. Increasing the energy, reactions of type (b)-(c) become dominant for central collisions, but still a large number of peripheral ones lead to distributions of type (d), the ensemble producing the observed asymmetric V-shaped distributions (see figure 8). Still increasing the energy, processes of type (a), which produce only very light fragments become dominant for central collisions, while peripheral ones remain in the domains (b) and (d). We deduce from this picture that heavy fragments (HF) are produced in events of type (d), intermediate mass fragments (IMF) come mostly from (b)-(c) and very light fragments from (a). Almost all models of nuclear fragmentation substantiate this physical picture.

The set of data points represented in figure 8 are from references 12 to 23. Some of them require a particular comment. Total isobaric cross sections for p-Xe reactions have been deduced from data of ref. 15 using the observed isotropy of the measured differential cross sections. For the p-Ag reactions we use the compilation of data of ref. 17, but we do not include the points for \( A < 20 \) because these cross sections have not been measured but rather interpolated from krypton and xenon data assuming a \( A_T^{2/3} \) dependence (i.e. that \( n(A, A_T) \) is independent of \( A_T \)). The cross sections of p-Cu are from refs. 13 and 18. For p-Al we use the isotopic data of the compilations of refs. 19 and 20, adding the cross sections of the most abundantly produced isobars. In a few cases, when a cross section is missing, we made an estimate using information on the neighbouring isotopes and exploiting the almost total independence of the shape of isotope cross sections of light fragments on target and energy [21,24]. Data points (open circles) for p-C reactions are from ref. 23. The mass yield has been transformed in an absolute.
cross section using the sum rule

\[ \sum A \sigma(A, A_T) = \sigma_R(A_T) A_T \quad (3.13) \]

The reaction cross section \( \sigma_R \) is from ref. 25. In addition we also show a few points (crosses) deduced from the data of refs. 19, 20 and 22. The two sets of data do not agree very well.

We observe that the isobaric cross sections show large fluctuations due to experimental uncertainties and to nuclear structure effects that are beyond the scope of our analysis. In order to better test the validity of the scaling hypothesis we made smooth fits of the data points (solid lines of figure 8). Doing these fits we took into account the learnings of previous works [15, 24] namely that masses \( A = 4, 11 \) and 15 are over produced, \( A = 6, 8, 9 \) are under produced, while \( A = 3, 7, 16 \) and 20-25 fall on the average curve. In addition, when the entire mass yield curve can be fitted, we impose the sum rule (3.13) as an extra constraint. The total reaction cross sections are from ref. 25.

The test of the scaling hypothesis \( A_L \propto A_T^{D_f/d} \) (equation (2.21)) is the simplest and the most unambiguous because it involves only pure mass numbers. We determine \( A_L \) by the condition

\[ \frac{d\sigma(A, A_T)}{dA} \bigg|_{A=A_L} = 0 \quad (3.14) \]

in accordance with the previous discussion on the rise of the large cluster from figures 1b. and 1c. The results are represented in figure 9. A chi-square fit determines a slope \( x = D_f/d = 0.82 \pm 0.05 \) with a chi-square deviation \( K^2 = 0.3 \). The best fit forcing \( x = 1 \) gives \( K^2 = 2.3 \).

The finite size scaling of the cluster size distribution function (2.25) has been investigated in reference 26. In this case, the use of inclusive data is more ques-
tionable. Excluding the poor data fit for carbon, scaling is obtained in the domain \( x = 0.7-0.85 \).

More insight in scaling properties of atomic nuclei is obtained using data from exclusive experiments. Unfortunately, we have at our disposal only two sets of data points, one for gold \([10]\) and another for carbon \([10]\). Data for silver fragmentation \([27]\) can't be used safely because of the arbitrary selection of the events made during the analysis of the data.

As a first step, we have verified that the above conclusion on scaling (eq. (2.21)) remains valid. Here we select unambiguously the critical events (peak of the curves in figures 4c and 6) and we calculate the average size of the largest fragment. For Au, we transform the average charge in an average mass assuming that most probable isotopes are those on the stability line. We get values that are slightly larger than with the previous method, but the slope of the line of figure 9 is unchanged.

The finite size scaling of the moments (eq. (2.22)) or combinations of moments (eq. (3.11)) can be checked straightforwardly. In figure 10, we show the scaling of \( \gamma_2 \) in the critical region. The solid line is the prediction of percolation in 3 dimensions \( (x = 0.82) \). The two points are the experimental values for carbon and gold. Again we remark an astonishing agreement between experimental data and the predictions of percolation. The dashed and dashed-dot lines are the predictions for \( x = 0.66 \) and \( x = 1 \), respectively. These curves have been obtained applying equations (2.21), (2.24) and assuming that in the critical region the size distribution is in the interval \( 1 < s < s_L - 1 \) a power law (see eq. (2.6) and fig. 1). We remark the drastic change in the slope of the curves. Hopefully, this test of scaling gives the most sensible determination of the scaling exponent \( x \).
4. FINAL REMARKS.

My first aim preparing these lectures was to introduce some ideas on a modern approach of critical phenomena. For many physicists, phase transitions suggests only Van der Waals or liquid-gas phase transitions. In my opinion, no serious student in nuclear physics is properly educated without some understanding of these modern topics and concepts such as "critical exponents", "correlation length" or "finite size scaling".

To cover this material at an introductory level, I make use of a simplified model. Using the percolation model one can keep the mathematics simple yet still describe many complex ideas in the field. Admittedly, percolation has a somewhat particular phase transition since no temperature is involved. Nevertheless, the parameter $p$ plays in many respects the role of the temperature in thermal phase transitions. Many functions diverge or vanish at one point as a function of the continuously varying parameter $p$. The scaling properties are also quite universal. I believe that the knowledge of percolation can be very helpful in understanding more complicated phase transitions.

Percolation theory has two more pedagogical advantages. It has a very intuitive interpretation in terms of simple geometry and it is very simple to work with in "numerical experiments", much simpler than with thermal phase transitions.

Last but not least, percolation theory accounts for the "gross features" of nuclear fragmentation. The reasons for that are still obscure to me. We can imagine many explanations. One is that nuclear multifragmentation is the manifestation of a kind of liquid-gas phase transition. In this case, the agreement with percolation predictions can't be more than qualitative. A deeper analysis of experimental data will show small but significant differences (in the values of the critical exponents). A better agreement with data should be obtained with a description.
in terms of lattice-gas or Ising type models. Another possibility is that nuclear fragmentation is really governed, in first approximation and in what concerns the critical regime of fragmentation, by "geometrical" properties, as in percolation. By "geometrical" properties I mean the statistics for clustering of a classical ensemble of points embedded in d-dimensional space, linked (or broken) randomly by a proximity mechanism. Many nuclear physicist dislike the idea that nuclei break up like a classical object. However we have many examples, particularly in the domain of high energy of heavy ions physics, where quantal and classical behaviours are almost identical. I will give the example of the Glauber model of scattering, and the Goldhaber and the abrasion models of peripheral collisions, where a purely geometrical picture accounts very well for the reality. The last comment concern the usefulness of the information we obtain with the present method. We believe that this information will serve as a filter for present or future theories of nuclear fragmentation. The signals we have proposed have to be present in any theory. These are in a sense, minimal requirements.
References


[19] J. Tobailem et al., Note CEA-N-1466(1) and 1466(4), CEN-Saclay, France


[22] M. Rey Campagnolle, Thesis Orsay (1972) ; unpublished


<table>
<thead>
<tr>
<th>Dimension $d$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>-</td>
<td>5/36</td>
<td>0.45</td>
<td>1</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1</td>
<td>43/18</td>
<td>1.74</td>
<td>1</td>
</tr>
<tr>
<td>$\nu$</td>
<td>1</td>
<td>4/3</td>
<td>0.88</td>
<td>1/2</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>1</td>
<td>36/91</td>
<td>0.46</td>
<td>1/2</td>
</tr>
<tr>
<td>$r$</td>
<td>2</td>
<td>187/91</td>
<td>2.20</td>
<td>5/2</td>
</tr>
<tr>
<td>$D_2(p = p_c)$</td>
<td>-</td>
<td>91/48</td>
<td>2.50</td>
<td>4</td>
</tr>
<tr>
<td>$C_2^-/C_2^+$</td>
<td>-</td>
<td>0.005</td>
<td>$\sim$ 0.1</td>
<td>1</td>
</tr>
</tbody>
</table>

- Table 1 -
Figure Captions

Fig.1 Fragment size distribution $n_a(p)$ in a percolation model with a cubic lattice containing $4^3$ sites. a) $p << p_c$; b) $p \approx p_c$; c) $p \geq p_c$; d) $p >> p_c$.

Fig.2 Percolation cluster (largest cluster at $p = p_c$) in a square lattice of size $50 \times 50$. On average these clusters have a fractal dimension $D_f \approx 1.89$.

Fig.3 Monte-Carlo data for the size $a_L$ of the largest cluster at the percolation threshold of the triangular lattice, as a function of the linear dimension $L$ of the lattice. From ref. 3.

Fig.4 The quantity $\gamma_2$ (eq.3.11) for distribution a) (crosses, bottom), b) (histogram, bottom) and c) (histogram, top) as a function of the reduced multiplicity $m$.

Fig.5 The quantity $\gamma_2$ (eq.3.11) for a percolation model in cubic lattices of linear size $L = 3, 5, 9, \text{ and } 50$.

Fig.6 The quantity $\gamma_2$ (eq.3.11) for the fragmentation of a beam of 2.1 GeV/nucleon $^{12}$C ions. Data from reference 11.

Fig.7 The average reduced size of the largest fragment produced per event of reduced multiplicity $m$ (eq.(3.8) (left) and its fluctuation $\Delta P$ (standard deviation, right). Model a) crosses; model b) open circles; experimental data c) full circles.

Fig.8 Mass yield distributions $\sigma(A, A_T)$. Data from references 12 to 25.

Fig.9 The mass $A_L$ of the largest fragment produced per event as a function of the mass of the target $A_T$. The straight line is the best chi-square fit with $x = 0.82$. 

26
Fig. 10 The quantity $\gamma_2$ (eq. 3.11) in the critical region, as a function of the total charge of the system $z_L$. 
Fig. 1

(a)

(d)

(b)

(c)
Fig. 3
Fig. 4
Fig. 7