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EXACTLY SOLUBLE MATRIX MODELS

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ABSTRACT

We study examples of one dimensional matrix models whose potentials possess an energy spectrum that can be explicitly determined. This allows for an exact solution in the continuum limit. Specifically, step–like potentials and the Morse potential are considered. The step–like potentials show no scaling behaviour and the Morse potential (which corresponds to a $\gamma = -1$ model) has the interesting feature that there are no quantum corrections to the scaling behaviour in the continuum limit.

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1 Introduction

Recent progress in the discretized approach to two dimensional gravity has shown that interesting information can be extracted from studying the continuum limit of appropriate matrix models by using a suitable double scaling in this limit[1]. For matrices which depend on a single 'time' coordinate($c = 1$ models), the problem of evaluating the free energy is equivalent to finding the ground state energy of a system of N non–interacting fermions in an external potential[2,3]. One can then expand the free energy in powers of the cosmological constant in an appropriate double scaling limit[4]. The external potential in question is determined by the nature of the discretization used to put together the two dimensional surface. For a triangulated surface, the potential is cubic in the matrix variable. This quantum mechanical problem of course is, strictly speaking, not well defined, owing to the unbounded nature of the potential. The semi–classical expansion in powers of $1/N$ that is generally used to determine the behaviour in the continuum limit is correspondingly of limited validity.

One can ask therefore if there are methods other than the semi–classical expansion that can be used to study the continuum limit. Thus one is led naturally to consider potentials whose eigenvalue spectrum can be determined directly. One might also wonder if a 'regularized' version of two dimensional gravity could be formulated by using not only triangles, but polygons of all possible shapes, to discretize the surface. Although it is not clear at present how to construct such a model, in this context it is useful to study potentials which are not polynomial and which are bounded from below; in this case one would be dealing with a well defined quantum mechanical problem.

With these motivations in mind, we shall study in this letter some matrix models whose potentials have a spectrum that can be explicitly determined. In the next section, we shall briefly describe the evaluation of the free energy of the matrix model and its reduction to an eigenvalue problem in quantum mechanics. In the following section, we consider two potentials–the double square well 'in a box' potential and the double square well potential. Both of these do not show, in the limit $N \to \infty$ and $g \to g_c$, any scaling behaviour, where g_c is the critical value of the coupling constant.

In section 4, we investigate the scaling of the matrix model with the Morse potential. Since the spectrum of this potential is known exactly, one can directly evaluate, in the double scaling limit, the dependence of the free energy on the renormalized coupling constant without resorting to the semi-classical expansion. Remarkably, we find that the result is the same as the classical one, and that the free energy scales with a susceptibility exponent of -1. Thus this matrix model does not receive any higher genus contributions.

Finally, we end with our conclusions.

2 Review of c=1 models

The partition function for a one-dimensional hermitian one-matrix model can be written as

$$Z = \int \mathcal{D}\Phi(t) \exp\left(-\beta \int dt \mathrm{Tr}[\frac{1}{2}\dot{\Phi}^2(t) + U(\Phi)]\right), \quad (1)$$

where β is related to the cosmological constant. Diagonalizing Φ and carrying out the integrations over the 'angular' variables of the matrix, the evaluation of the free energy $\ln Z(\beta)$ of this model reduces to the computation of the ground state energy[2,3] of N fermions at zero temperature, with the hamiltonian

$$H = \beta \sum_{i=1}^{N} \left(-\frac{1}{2\beta^2}\frac{d^2}{d\lambda_i^2} + U(\lambda_i)\right),$$

where λ_i, $i = 1,..,N$ are the N eigenvalues of the matrix Φ. Here the eigenvalues λ_i form the coordinates of the configuration space for a fermionic wave function. Therefore the ground state energy of this non-interacting system of N fermions is simply the sum of the first N eigenvalues of the one-body hamiltonian

$$\tilde{H} = \beta\left(-\frac{1}{2\beta^2}\frac{d^2}{d\lambda^2} + U(\lambda)\right).$$

Denoting the eigenvalues of this hamiltonian by e_i, the ground state energy is then

$$E_0 = e_1 + e_2 + ... + e_N.$$

In terms of the density of states

$$\rho(e) = \sum_n \delta(\epsilon - e_n),$$

we have the normalization condition

$$N = \int_0^{e_F} \rho(e) de$$

where $e_F \equiv e_N$ is the Fermi energy. It is more convenient to work with the eigenvalues ϵ of the rescaled hamiltonian $h = \tilde{H}/\beta$. Then we have

$$g \equiv \frac{N}{\beta} = \int_0^{\epsilon_F} \rho(\epsilon) d\epsilon$$

and

$$E_0 = \beta^2 \int_0^{\epsilon_F} \rho(\epsilon)\epsilon d\epsilon.$$

The continuum limit is defined in terms of the coupling constant g approaching a critical value g_c as $N \to \infty$; in this limit E_0(the free energy of the matrix model) scales non-trivially as a function of $(g - g_c)$, which corresponds to the average area of the triangulated surface being divergent. Correspondingly, the Fermi energy reaches the top of the potential well of U; let the critical value of the potential at the top be ϵ_c. It is convenient to define the variable $\eta \equiv \epsilon_c - \epsilon_F$. Then we can write

$$\frac{dg}{d\eta} = -\rho(\epsilon_F)$$

and

$$\frac{dE_0}{d\eta} = -\beta^2 \epsilon_F \rho(\epsilon_F).$$

These equations effectively determine E_0 as a function of $(g - g_c)$; the string susceptibility exponent γ is defined in the classical limit from

$$E_0 \sim (g - g_c)^{2-\gamma}.$$

In the continuum limit $N \to \infty$, E_0 actually depends on the renormalized coupling constant, which is generically of the form $g_R = N^a(g - g_c)$, where a is some positive number. Quantum corrections determine E_0 in terms of a perturbative expansion in powers of g_R. In the following sections we shall directly evaluate E_0 as a function of g_R for some potentials.

3 Double square well potentials

Consider the case when the potential for the eigenvalues of Φ takes the 'double square well in a box' form:

$$\begin{aligned} U(x) &= \infty, \text{ for } x > b/2 \text{ and } x < -b/2 \\ U(x) &= 0, \text{ for } a/2 < x < b/2 \text{ and } -b/2 < x < -a/2 \\ U(x) &= V, \text{ for } -a/2 < x < a/2. \end{aligned} \quad (2)$$

The wave function $\psi(x)$ for states with eigenvalues $\epsilon < V$, for the hamiltonian

$$h = -\frac{1}{2\beta^2}\frac{d^2}{dx^2} + U(x),$$

is oscillatory for $-b/2 < x < -a/2$ and for $a/2 < x < b/2$; it takes the form of a hyperbolic cosine for $-a/2 < x < a/2$. Matching ψ and its derivative at $x = \pm a/2$, one finds the equation for the eigenvalues:

$$\tan y = -\frac{2y}{\alpha(b-a)}\coth\frac{a\alpha}{2} \quad (3)$$

where

$$k = \beta\sqrt{2\epsilon} \text{ and } \alpha = \beta\sqrt{2(V-\epsilon)},$$

and

$$y = \frac{(b-a)}{2}k = \frac{(b-a)}{2}\beta\sqrt{2\epsilon}.$$

The intersections of the function on the right hand side above(which can be written as a single valued function of y) with the $\tan y$ curves determine the eigenvalues y_n. These intersections, for large n, always occur at

$$y_n = \frac{(2n-1)}{2}\pi + \delta, \quad (4)$$

where δ is a number of order unity. The critical value of the coupling constant, namely g_c, is determined in the continuum limit $N \to \infty$ by the requirement that exactly N eigenvalues fit into the potential well. This means that

$$y_N = \frac{(b-a)}{2}\frac{N}{g}\sqrt{2\epsilon_N} = N\pi + O(1) \quad (5)$$

or, as $N \to \infty$ and $\epsilon_N \to \epsilon_c = V$,

$$g_c^2 = (b-a)^2\frac{V}{2\pi^2}. \quad (6)$$

The eigenvalues for states close to the top of the well are given by

$$\epsilon_n = \frac{n^2}{N^2}\pi^2\frac{2g^2}{(b-a)^2};$$

the density of states $\rho(\epsilon_F)$ at the top is given by

$$\rho(\epsilon_F) = \frac{1}{\beta}\frac{dn}{d\beta}\bigg|_{n=N} = \frac{1}{\beta}\frac{N(b-a)^2}{4\pi^2 g^2}$$

or

$$\rho(\epsilon_F) = \frac{(b-a)^2}{4\pi^2 g^2}. \quad (7)$$

So $\rho(\epsilon_F)$ does not scale with $(g-g_c)$ or with $(\epsilon_c - \epsilon_F)$, which means that this potential does not possess a good continuum limit.

Now we turn to the case of the double square well potential. Explicitly, we choose

$$\begin{aligned} U(x) &= V, \text{ for } x > b/2 \text{ and } x < -b/2 \\ U(x) &= 0, \text{ for } a/2 < x < b/2 \text{ and } -b/2 < x < -a/2 \\ U(x) &= V, \text{ for } -a/2 < x < a/2, \end{aligned} \quad (8)$$

where V is a constant.

For states with eigenvalues $\epsilon < V$, the wave function is a decreasing exponential for $x < -b/2$ and $x > b/2$; it is oscillatory in the regions where $U = 0$ and a hyperbolic cosine for $-a/2 < x < a/2$. Matching the wave function and its derivative as before, one finds the following equation for the eigenvalues:

$$\tan\frac{k(b-a)}{2} = \frac{\alpha}{k}\frac{1-\tanh\frac{a\alpha}{2}}{1+\frac{\alpha^2}{k^2}\tanh\frac{a\alpha}{2}}. \quad (9)$$

Again, the right side is a single valued function of the variable $y = k(b-a)/2$. Its intersections with $\tan y$ determine the eigenvalues y_n(and hence ϵ_n). As $\epsilon \to V$, $\frac{\alpha}{k} \to 0$; in this limit the right hand side of (9) vanishes. So the eigenvalues are given (for large n) by

$$y_n = \frac{k_n(b-a)}{2} = n\pi$$

or

$$\epsilon_n = \frac{n^2}{N^2}\frac{2\pi^2 g^2}{(b-a)^2}, \quad (10)$$

as before. This means that this potential does not possess a well defined continuum limit either, even though one may have expected that it did since there is a continuum of states above the energy level V.

4 The Morse potential

Here we shall show that the matrix model with the Morse potential possesses a continuum limit but does not receive any quantum corrections. The Morse potential is [5]

$$U(x) = Ae^{-2ax} - 2Ae^{-ax}. \quad (11)$$

This has a minimum value of $-A$ at $x = 0$; also $U(x) \to 0$ as $x \to \infty$(this limit is the top of the potential well). The eigenvalue equation

$$\left[-\frac{1}{2\beta^2}\frac{d^2}{dx^2} + U(x)\right] = \epsilon\psi$$

becomes, with $z = \frac{2\beta\sqrt{A}}{a}e^{-ax}$ and $r = \beta\sqrt{A}/a$,

$$\left(\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} + (-\frac{1}{4} + \frac{r^2}{z^2}\frac{\epsilon}{A} + \frac{r}{z})\right)\psi = 0. \quad (12)$$

Comparing this [5] with the standard form of the hypergeometric equation

$$F'' + \frac{\alpha+1-\gamma}{z}F' + \left(-\frac{1}{4} + (\beta + \frac{1+\alpha}{2}\frac{1}{z} - \gamma(2\alpha-\gamma)\frac{1}{4z^2}\right)F = 0 \quad (13)$$

where β is a positive integer and $\alpha > 0$, we see that we have in our case $\gamma = \alpha$, $\beta + \frac{\alpha+1}{2} = r$ and $-\alpha^2/4 = r^2\epsilon/A$. Since β must be a positive integer n, we can write $\alpha = 2(r-n)-1$. Since $\alpha > 0$, it must be true that $n < r - \frac{1}{2}$. Then the eigenvlues are

$$\epsilon_n = -\frac{\alpha^2 A}{4r^2} = -\frac{A}{r^2}(r - n - \frac{1}{2})^2.$$

Using $r = \beta\sqrt{A}/a$, this can be cast in the form

$$\epsilon_n = -\frac{a^2g^2}{2}\left(\frac{\sqrt{2}}{ga} - \frac{n}{N} - \frac{1}{2N}\right)^2. \quad (14)$$

In the continuum limit $N \to \infty$, criticality corresponds to $\epsilon_N \to 0$, or $\sqrt{2}/ga \to 1$, which says that the critical coupling is $g_c = \sqrt{2}/a$. So we can write

$$\epsilon_n = -\frac{a^2g^2}{2}\left(\frac{g_c}{g} - \frac{n}{N} - \frac{1}{2N}\right)^2. \quad (15)$$

Then

$$\left.\frac{d\epsilon_n}{dn}\right|_{n\to N} = \frac{a^2g}{N}\left(g_c - g - \frac{g}{2N}\right).$$

Now both $(g_c - g)$ and $1/N$ go to zero in the continuum limit. In order to take the limit with care, we assume that $(g_c - g)N^\alpha \equiv g_R$ is held fixed in the double scaling limit, where α is to be determined consistently. Then we have

$$\rho(\epsilon_F) = \left.\frac{1}{\beta}\frac{dn}{d\epsilon_n}\right|_{n\to N} = \frac{N^\alpha}{a^2}\left(g_R - \frac{g_cN^{\alpha-1}}{2}\right)^{-1}. \quad (16)$$

This can be rewritten in terms of g_R as

$$-\frac{dg_R}{d\epsilon_F} = \frac{N^{2\alpha}}{a^2}\left(g_R - \frac{g_cN^{\alpha-1}}{2}\right)^{-1}. \quad (17)$$

Integration gives

$$\eta \equiv \epsilon_c - \epsilon_F = \frac{a^2}{2N^{2\alpha}}\left(g_R - \frac{g_cN^{\alpha-1}}{2}\right)^2. \quad (18)$$

Using $dE_0/dg = -\beta^2\eta$ and requiring that powers of N cancel fixes $\alpha = 2/3$ and gives

$$\frac{dE_0}{dg_R} = \frac{a^2}{2g_c^2}g_R^2 \quad (19)$$

or

$$E_0 = \frac{a^2}{6g_c^2}g_R^3. \quad (20)$$

This gives the exact dependence of the free energy on the renormalized coupling constant, and corresponds to a susceptibility exponent of $\gamma = -1$. We note that this is the same result that one expects from a zeroth order WKB(purely classical) analysis, in the case when the potential is infinitely multicritical(i.e., derivatives of the potential of all orders vanish at the critical point); a k-th order multicritical model has $\gamma_k = -\frac{k-2}{k+2}$, and $\gamma = -1$ is the corresponding $k \to \infty$ limit.

5 Conclusions

To summarise our results, we have solved some examples of one dimensional matrix models explicitly. The examples we considered all have 'flat' regions as critical points. While in the case of the Morse potential the exact result agrees with the classical result and yields a susceptibility exponent of -1, the step-like double well potentials do not show a scaling behaviour at all. This lack of scaling can be attributed to the sharp edges in these potentials which spoil the existence of a smooth continuum limit.

It is remarkable that the Morse potential does not get quantum corrections to the free energy. The non-polynomial nature of the potential(or, in the language of discretized surfaces, the inclusion of all sorts of polygons in the triangulation) results in the absence of higher genus contributions in the double scaling limit. At the same time one has a potential that is bounded from below. This 'regularization' in this case yields a sensible non-perturbative result, and it is conceivable that such more general matrix models might provide a better formulation of non-perturbative two dimensional quantum gravity.

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References

[1] E. Brezin, V.A. Kazakov, Phys. Lett. 236B (1990) 144; M.R. Douglas, S. Shenker, Nucl. Phys. B335 (1990) 635; D.J. Gross, A. Migdal, Phys. Rev. Lett. 64 (1990) 1227.

[2] E. Brezin, C. Itzykson, G. Parisi and J.-B. Zuber, Comm. Math. Phys. 59 (1978) 35.

[3] V. Kazakov and A. Migdal, Nucl. Phys. B311 (1989) 171.

[4] D. Gross and N. Milkovic, Phys. Lett. 238B (1990) 217; P. Ginsparg and J. Zinn-Justin, Phys. Lett. 240B (1990) 333; E. Brezin. V. Kazakov and Al. B. Zamolodchikov, Nucl. Phys. B338 (1990) 673; G. Parisi, Phys. Lett. 238B (1990) 209.

[5] *Quantum Mechanics*, E.U. Condon and P.M. Morse, McGraw-Hill Book Company Inc., 1929.