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ON THE EXISTENCE OF n-DIMENSIONAL INDECOMPOSABLE VECTOR BUNDLES

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ABSTRACT

Let X be an arbitrary smooth irreducible complex projective curve of genus g with $g \geq 4$. In this paper we extend the existence theorem of special divisors to high dimensional indecomposable vector bundles. We give a necessary and sufficient condition on the existence of n–dimensional indecomposable vector bundles E with $deg(E) = d$, $dim H^0(X,E) \geq h$. We also determine under what condition the set of all such vector bundles will be finite and how many elements it contains.

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1 Introduction

Let X be an arbitrary smooth irreducible complex projective curve of genus g. Based on Riemann-Roch Theorem and Clifford Theorem, the Brill-Norther theory studies the dimension of the cohomology group $H^0(X,L)$ of line bundles L of given degree(see[1]). A basic problem in it is to determine under what condition there exists a line bundle L with deg(L)=d, $dim H^0(X,L) \geq r+1$. This problem was formulated by Brill-Noether and proved by Kempf[4], Kleimann-Laksov[5] and Griffiths-Harris[3]. It can be formulated as follows.

The existence theorem of special line bundles: Let X be an arbitrary smooth irreducible complex projective curve of genus g, then for existent a line bundle L on X with deg(L)=d, $dim H^0(X,L) \geq r+1$, it is necessary and sufficient that g, d and r satisfy $g-(r+1)(g-d+r) \geq 0$. Furthermore, if $g-(r+1)(g-d+r) = 0$, then for general curve X, there exists exactly

$$g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$$

line bundles L on X with deg(L)=d, $dim H^0(X,L) \geq r+1$.

Here X is a general curve means X do not contained in a low dimensional subvariety A of the moduli space M_g of all smooth projective curves of genus g.

Let $J_d(X)$ be the Jacobi variety of X consisting of all line bundles of degree d, and set

$$W_d^h(X) = \{L \in J_d(X) \mid dim H^0(X,L) \geq r+1 =: h\}$$

$W_d^h(X)$ is a subvariety of $J_d(X)$. In 1980, Griffiths-Harris[3] proved that for general curve X, $dim W_d^h(X) = g - h(g-1-d+h)$. So in the following when we say X is of general type, we means X is a general curve in the sense of Griffiths-Harris, in particular, if $g > 2$, X is then a non-hyperelliptic curve.

Analogous to the case of line bundles, it is natural to study the Brill-Noether theory for high dimensional vector bundles. But if a vector bundle E is decomposable then to bound $dim H^0(X,E)$ by deg(E) will be meaningless. So we have to restrict at least to the case of indecomposable vector bundles. Furthermore, for indecomposable vector bundles, we do have the Vanishing theorem and Clifford theorem [8], so based on Riemann-Roch, we may ask when deg(E) is given (which means the topological type of E is given), how $dimH^0(X,E)$ varies and how to classify. In particular, as in the case of line bundles, we should determine what condition we need to put on g,n,d and h to guarantee that there does exist n-dimensional indecomposable vector bundle E with deg(E)=d and $dimH^0(X,E) \geq h$. This paper will try to answer this question, here we generalize the existence theorem of special line bundles to high dimensional indecomposable vector bundles, we have

Theorem 1: Let X be an arbitrary smooth irreducible projective curve of genus g with $g > 3$, h and d are given integers with $d(modn) = t$, and $-(n-l+1)n(g-1) \leq d-t \leq -(n-l)n(g-1)$ for some integer l. Then for existent an n-dimension indecomposable vector bundles E on X with deg(E)=d, $dimH^0(X,E) \geq h$, it is necessary and sufficient that g,n,d and h satisfy

$$n^2g - [-\frac{l}{2}(d-t) - \frac{nl(2n-l-2)}{4}(g-1) - nmax\{0, \frac{(2t+l-2n)}{2}\}$$
$$+nh] \times [-\frac{(l-2)}{2}(d-t) - \frac{n(l-2)(2n-l)}{4}(g-1)$$
$$-nmax\{0, \frac{(2t+l-2n-2)}{2}\} + nh] \geq 0$$

if l is even;

$$n^2g - [-\frac{(l-3)}{2}(d-t) - \frac{n(l-3)(2n-l+1)}{4}(g-1) - nmax\{0, \frac{(2t+l-2n-3)}{2}\}$$
$$+nh] \times [-\frac{(l-1)}{2}(d-t) - \frac{n(l-1)(2n-l-1)}{4}(g-1)$$
$$-nmax\{0, \frac{(2t+l-2n-1)}{2}\} + nh] \geq 0$$

if l is odd.

Same as in the case of line bundles, set

$$W^h_{d,n}(X) = \{E \mid E \text{ an n-dimensional indecomposable vector bundle}$$
$$\text{with}, deg(E) = d, dimH^0(X,E) \geq h\}$$

with the condition on $W^h_{d,n}(X) \neq \emptyset$, the next problem in Brill-Noether theory is to determine how large $W^h_{d,n}(X)$ may be. In particular, under what condition it will be finite, and how many elements it contains. For this we have

Theorem 2: Assume $g > 3$, $-(n-l+1)n(g-1) \leq d \leq -(n-l)n(g-1)$ for some integer l, then for general curve X, $W^h_{d,n}(X)$ is finite if and only if $d(modn) = 0$ and

$$n^2g - [-\frac{l}{2}d - \frac{n(l-3)(2n-l-2)}{4}(g-1) + nh] \times$$
$$[-\frac{(l-2)}{2}d - \frac{n(l-2)(2n-l)}{4}(g-1) + nh] = 0.$$

if l is even;

$$n^2g - [-\frac{(l-3)}{2}d - \frac{n(l-3)(2n-l+1)}{4}(g-1) + nh] \times$$
$$[-\frac{(l-1)}{2}d - \frac{n(l-1)(2n-l-1)}{4}(g-1) + ng] = 0.$$

if l is odd.

Furthermore, if $W^h_{d,n}(X)$ is finite, then it contains exactly

$$g! \prod_{i=0}^{h-1} \frac{i!}{(-d/n-(n-2)(g-1)+h+i)!}$$

elements.

Remark: If g=0, there exist no indecomposable vector bundles on X and the problem is trivial.

If g=1, Theorem 1 and Theorem 2 can be obtained from a Theorem of Atiyah[2].

The cases g=2 and g=3 are exceptional, for example, if n=2, we will still have Theorem 1 but not Theorem 2. For this, we have the following

Thorem: For g=2 or g=3, if $d \neq 2(g-1)$, Theorem 2 is still true, but

a) $W^g_{2(g-1),2}(X)$ is one-to-one corresponding to $\{X \cup E_k\}$, if g=2. Here X is the curve itself, E_k another point.

b) $W^g_{2(g-1),2}(X)$ contains two which is greater than

$$g!\prod_{i=0}^{h-1}\frac{i!}{(-d/2+h+i)!}=1$$

elements, if g=3.

For proof of this Theorem, see[9]. In this paper, we do not consider the cases g=2 and g=3.

In the next section, we will give some basic lemmas, in section 3, we will constructe an n-dimensional vector bundle E_d with the given degree d, and show that $dim H^0(X, E_d)$ is the maximal among all n-dimensional indecomposable vector bundles of degree d. By using E_d, we give the formula of Theorem 1 in section 4. Section 5 will be used to prove Theorem 2.

2 Notations and Lemmas

Let X be a smooth irreducible complex projective curve of genus g, E a vector bundle on X. If F_1 is a subbundle of E, $F_2 = E/F_1$, E is then called an extension of F_2 by F_1 and it is determined by an element $e \in H^1(X, F_1 \otimes F_2^*)$. Conversely, if $e \in H^1(X, F_1 \otimes F_2^*)$, then it determines an extension E of F_2 by F_1 with $E = F_1 \oplus F_2$ if and only if e=0.

It was proved in [2] that every vector bundle E over X can be splitted into a sequence of extensions of line bundles $L_1, ..., L_n$, we denote it by $E = (L_1, ..., L_n)$. Splitting $E = (L_1, ...L_n)$ is called maximal, if $deg(L_1)$ is the maximal among all line subbundles of E (L_1 is then called a maximal line subbundle of E), and if $(L_2, ..., L_n)$ is a maximal splitting of E/L_1.

Lemma 1: If $E = (L_1, ..., L_n)$ is a maximal splitting of E, then $deg(L_i) \geq deg(L_{i+1}) + g$.

Proof: A Theorem of Nagata ([7], see also [6]) says that if F is a rank two vector bundle, L a maximal line subbundle of F, then $deg(L) \geq (deg(F) - g)/2$.

Apply this Theorem to each pair (L_i, L_{i+1}), the Lemma follows immediately.

Lemma 2: Let E be an extension of F_2 by F_1 which is determined by $e \in H^0(X, F_1 \otimes F_2^*)$, $s \in H^0(X, F_2)$, then s can be left to be a section of E (which means $s \in Im\{H^0(X, E) \mapsto H^0(X, F_2)\}$), if and only if s e=0.

Proof: Trivial.

Vanishing Theorem :Let E be an n-dimensional indecomposable vector bundle on X, if $deg(E) > (n+1)n(g-1)$, then $H^1(X, E) = 0$; if $deg(E) < -n(n-1)(g-1)$, then $H^0(X, E) = 0$.

Proof: If n=1, this is the classical Vanishing Theorem for line bundles. Now assume this theorem for indecomposable vector bundles of dimension less or equal to n-1. Let E be an n-dimensional indecomposable vector bundle with $deg(E) < -n(n-1)(g-1)$, but $H^0(X, E) \neq 0$. Let $s \in H^0(X, E)$, $s \neq 0$, L be the line subbundle of E generated by s, then $deg(L) \geq 0$. If $E/L = G_1 \oplus ... \oplus G_l$ is the directe sum decomposition of E/L into indecomposable components, since E is indecomposable, so for each i, $i = 1, ..., l$, the extension of G_i by L in E is non-trivial, and then $H^1(X, L \otimes G_i^*) \neq 0$. By induction, we have

$$deg(L \otimes G_i^*) = r(G_i)deg(L) - deg(G_i) \leq (r(G_i)+1)r(G_i)(g-1)$$

here $r(G_i)$ is the dimension of G_i. So

$$(n-1)deg(L) - \sum_{i=1}^{l} deg(G_i) \leq$$

$$\sum_{i=1}^{l}(r(G_i)+1)r(G_i)(g-1) \leq n(n-1)(g-1).$$

but $deg(L) \geq 0$, we get

$$deg(E) = deg(L) + \sum_{i=1}^{l} deg(G_i) \geq$$

$$-n(n-1)(g-1) + ndeg(L) \geq -n(n-1)(g-1),$$

a contradiction. So if $deg(E) < -n(n-1)(g-1)$, then $H^0(X, E) = 0$, and by the Kodaira-Serre duality, we get if $deg(E) > (n+1)n(g-1)$, then $H^1(X, E) = 0$.

3 Basic construction

In this section, we are going to constructe an n-dimensional indecomposable vector bundle E_d of the given degree d and show that $dimH^0(X, E_d)$ is the maximal among all n-dimensional indecomposable vector bundles of degree d.

Lemma 3:Let L be a line bundle, $F = (F_1, ..., F_n)$ an n-dimensional indecomposable vector bundle, if $deg(L) > deg(F_i), i = 1, ..., n$, and if $e \in H^0(X, L \otimes F^*)$ with $e \neq 0$, then the extension E of F by L which is determined by e is indecomposable.

Proof:If $E = H_1 \oplus H_2$, assume $L \mapsto E \mapsto H_1$ is non-zero, then L must be a line subbundle of H_1 and $E/L = H_1/L \oplus H_2$. But $F = E/L$ is indecomposable so $H_1 = L, H_2 = F, E = L \oplus F$, a trivial extension, and then e=0, a contradiction.

BY consider E^*, then from Lemma3, we have

Lemma4:Let L be a line bundle, $F = (F_1, ..., F_n)$ an n-dimensional indecomposable vector bundle, if $deg(L) < deg(F_i)$, $i = 1, ..., n$, $e \in H^0(X, F \otimes L^*)$ with $e \neq 0$, then the extension of L by f which is determined by e is indecomposable.

Now we give the construction of E_d, by the Kodaira-Serre duality, here we may assume $d \leq n(g-1)$.

If n=1, let E_d be a line bundle of degree d with $dimH^0(X, E_d)$ is the maximal among all line bundles of degree d.

Now assume for n-1, $d(mod(n-1)) = t$, we have E_d with 1) E_d is an (n-1)-dimensional indecomposable vector of degree d; 2) E_d have a splitting $E_d = (L_1, ..., L_{n-1})$ with $deg(L_i) = (d-t)/(n-1) + (n-2i+1)(g-1)$, if $1 \leq i \leq n-t$; $deg(L_i) = (d-t)/(n-1) + (n-2i+1)(g-1) + 1$, if $n-t+1 \leq i \leq n$;3). For each i, $dimH^0(X, L_i)$ is the maximal among all line bundles of the same degree.

For n and d with d(mod n)=t, let $d_1 = (n-1)(d-t)/n + (n-1)(g-1) + t - 1$, $d_2 = (d-t)/n - (n-1)(g-1) + 1$, if $t \neq 0$; $d_1 = (n-1)(d-t)/n + (n-1)(g-1)$, $d_2 = (d-t)/n - (n-1)(g-1)$ if t=0. $E_{d_1} = (L_1, ..., L_{n-1})$ be an (n-1)-dimensional vector bundle of degree d_1 which satisfies 1) ,2) and 3) above. Since $deg(L_{n-1}) - d_2 \leq 2(g-1)$, one can choose a line bundle L_n with $deg(L_n) = d_2$, $H^1(X, L_{n-1} \otimes L_n^*) \neq 0$ (In case n=2, $d_2 = 0$, just let $L_2 = I$). Let $e \in H^1(X, L_{n-1} \otimes L_n^*) \subseteq H^1(X, F_{d_1} \otimes L_n^*)$ with $e \neq 0$, E_d be the extension of L_n by E_{d_1} which is determined by e. Since $deg(L_n) < deg(L_i)$,

$i = 1, ..., n-1$, by Lemma 4, E_d is indecomposable. Property 2) comes directly from the construction. Furthermore, if $n > 2$, since $d \leq n(g-1)$, we have $d_2 < 0$, if n=2, $d_2 = 0$, we have $L_2 = I$, so E_d do satisfy 3).

Lemma 5:Let $E_d = (L_1, ..., L_n)$ be constructed above, if there exists an i, $2 \leq i \leq n$, such that $L_i = I$ or $L_i = [p]$ with $L_{i-1} = K$, then $dimH^0(X, E_d) = \sum_{i=1}^n dimH^0(X, L_i) - 1$. Otherwise, $dimH^0(X, E_d) = \sum_{i=1}^n dimH^0(X, L_i)$. Here [p] is the line bundle which determined by the divisor $p \in X$, K is the canonical line bundle.

Proof: By the Kodaira-Serre duality, we may assume $d \leq n(g-1)$.

If n=2, $d_2 \geq 0$, then $L_2 = I$, let $e \in H^1(X, L_1 \otimes L_2^*)$ be the element which determines the extension $E_d = (L_1, L_2)$. A non-zero section $c \in H^0(X, L_2)$ is a constant, c can be left, by Lemma 2, c e=0, and then e=0, a contradiction. The case $L_1 = K$, $L_2 = [p]$ follows from the case $L_2 = I$ by taking the Kodaira-Serre duality.

If $n > 2$, then $deg(L_n) < 0$, so $dimH^0(X, E_d) = dimH^0(X, (L_1, ..., L_{n-1}))$, we get the Lemma by induction.

E_d will play a key role in the proof of Theorem 1 and Theorem 2, in fact, we have

Theorem 3:Assume X is a general curve and $g > 3$, then for any n-dimensional indecomposable vector bundle E on X with deg(E)=d, we have $dimH^0(X, E) \leq dimH^0(X, E_d)$.

Before proving this Theorem, we will first give some Lemmas.

Lemma 6:Let E be an n-dimensional indecomposable vector bundle, $E = (F_1, ..., F_n)$ a maximal splitting, $E_d = (L_1, ..., L_n)$ the vector bundle constructed above, then $deg(F_n) \geq deg(L_n)$. Furthermore, if $deg(F_n) > deg(L_n)$,then for i, $2 \leq i \leq n-1$, $\sum_{j=i}^n deg(F_j) \geq \sum_{j=i}^n deg(L_j)$ Furthermore if $\sum_{j=i}^n deg(F_j) = \sum_{j=i}^n deg(L_j)$, then we have $d(modn) = s \neq 0$, and $i \geq n-s+1$.

Proof:Let $(F_1, ..., F_{n-1}) = G_1 \oplus ... \oplus G_l$ be the directe sum decomposition of $(F_1, ..., F_{n-1})$ into indecomposable components. Same as in the proof of Vanishing Theorem, for each i, we have

$$deg(G_i \otimes L_n) = deg(G_i) - r(G_i)deg(L_n) \leq (r(G_i)+1)r(G_i)(g-1)$$

we get

$$deg((F_1,...,F_{n-1})) - (n-1)deg(L_n) \le \sum_{i=1}^{l}(r(G_i)+1)r(G_i)(g-1)$$

$$\le n(n-1)(g-1).$$

but we know

$$deg(F_1,...,F_{n-1}) + deg(F_n) = d,$$

so

$$deg(F_n) \ge \frac{d}{n} - (n-1)(g-1) = \frac{(d-s)}{n} - (n-1)(g-1) + \frac{s}{n} = deg(L_n) - \frac{(n-s)}{n}$$

($s \neq 0$), and if $deg(F_n) = deg(L_n) - \frac{(d-s)}{n}$, then $(F_1,...,F_{n-1})$ is indecomposable.

Now assume $deg(F_n) > deg(L_n)$ and there exists a t, $1 \le t \le n-1$, such that if $i > t$, then $\sum_{j=i}^{n} deg(F_j) > \sum_{j=i}^{n} deg(L_j)$, but $\sum_{j=t}^{n} deg(F_j) \le \sum_{j=t}^{n} deg(L_j)$, then $deg(F_t) < deg(L_t)$.

First suppose $H^1(X,(F_1,...,F_{t-1}) \otimes F_t^*) = 0$, let $(F_1,...,F_{t-1}) = G_1 \oplus ... \oplus G_l$ be the directe sum decomposition of $(F_1,...,F_{t-1})$ into indecomposable components. Since E is indecomposable, so for each i, $H^1(X, G_i \otimes (F_{t+1},...,F_n)^*) \neq 0$, and then there exists an j(i), $t+1 \le j(i) \le n$, such that $H^1(X, G_i \otimes L_{j(i)}) \neq 0$, we get

$$deg(G_i) - r(G_i)deg(L_{j(i)}) \le (r(G_i)+1)r(G_i)(g-1)$$

Now choose $F_m \in \{F_{j(1)},...,F_{j(l)}\}$ with $deg(F_m) = max\{deg(F_{j(i)}), i = 1,...,l\}$, then

$$deg((F_1,...,F_{t-1})) - (t-1)deg(F_m) \le \sum_{i=1}^{l}(r(G_i)+1)r(G_i)(g-1)$$

$$\le t(t-1)(g-1).$$

but $deg((F_1,...,F_{t-1})) \ge deg((L_1,...,L_{t-1})) = (t-1)(d-s)/n + (t-1)(n-t+1)(g-1) + max\{0, s-n+t-1\}$, we must have

$$deg(F_m) \ge \frac{(d-s)}{n} + (n-2t+1)(g-1) + \frac{1}{t-1}max\{0, s-n+t-1\}.$$

Since $(F_1,...,F_n)$ is a maximal splitting, from Lemma 1, we know that

$$deg(F_{m-1}) \ge deg(F_m) - g, ..., deg(F_t) \ge deg(F_m) - (m-t)g,$$

we get $\sum_{j=t}^{m} deg(F_j) \ge \sum_{j=t}^{m} deg(F_j)$. But we know $\sum_{j=m+1}^{n} deg(F_j) > \sum_{j=m+1}^{n} deg(L_j)$, we get $\sum_{j=t}^{n} deg(F_j) \ge \sum_{j=t}^{n} deg(L_j)$, a contradiction. Then $H^1(X,(F_1,...,F_{t-1}) \otimes F_t^*) \neq 0$, and from the proof, we get that $deg(F_t) = max\{deg(F_{j(i)}), i = 1,...,l\}$, and

$$deg(F_t) \ge \frac{(d-s)}{n} + (n-2t+1)(g-1) + \frac{1}{t-1}max\{0, s-n+t-1\}.$$

But we have assumed that

$$deg(F_t) < deg(L_t) = \frac{(d-s)}{n} + (n-2t+1)(g-1) + \begin{cases} 1, & \text{if } s \ge n-t+1, \\ 0, & \text{if } s \le n-t \end{cases}$$

so we must have $s = n-t+1 \neq 0$, $deg(F_t) = deg(L_t) - 1$, and $\sum_{j=t}^{n} deg(F_j) = \sum_{j=t}^{n} deg(L_j)$.

Now assume that if $i > u$, we have $\sum_{j=i}^{n} deg(F_j) \ge \sum_{j=i}^{n} deg(L_j)$, but $\sum_{j=u}^{n} deg(F_j) < \sum_{j=u}^{n} deg(L_j)$, then we have $u > n-s+1$, and with the same argument as above, we get

$$deg(F_u) \ge \frac{(d-s)}{n} + (n-2u+1)(g-1),$$

$$deg(F_u) < deg(L_u) = \frac{(d-s)}{n} + (n-2u+1)(g-1).$$

a contradiction. This then completes the proof of the Lemma.

From this Lemma, we have

Lemma 7:Let E be an n-dimensional indecomposable vector bundle of degree d, $E = (F_1,...,F_n)$ a maximal splitting of E, if $deg(F_n) > deg(L_n)$, then for $1 \le i \le n-1$, we have $\sum_{j=1}^{i} deg(F_j) \le \sum_{j=1}^{i} deg(L_j)$ and if the equality hold, then $d(mod n) = s \neq 0$, and $i \ge n-s+1$.

To compare E with E_d, we need some other Lemmas.

Lemma 8:Assume X is a general curve and $g > 3, F_1, F_2$ are line bundles on X with $d_1 = deg(F_1), d_2 = deg(L_2)$, $h_1 = dim H^0(X, L_1)$, $h_2 = dim H^0(X, L_2)$ and $d_1 \ge d_2$. If $d_1 - d_2 \le 2(g-1)$, then one can choose line bundles $\tilde{F}_1, \tilde{F}_2$ with $deg(\tilde{F}_1) = (d_1+d_2)/2 + g - 1$, $deg(\tilde{F}_2) = (d_1+d_2)/2 - (g-1)$ if $d_1 + d_2$ is even; $deg(\tilde{F}_1) = (d_1+d_2-1)/2 + (g-1)$, $deg(\tilde{F}_2) = (d_1+d_2+1)/2 - (g-1)$ if d_1+d_2 is odd, such that $h_1 + h_2 \le dim H^0(X, \tilde{F}_1) + dim H^0(X, \tilde{F}_2)$.

Proof: By Kodaria-Serre duality, we may assume that $d_1 + d_2 \le 2(g-1)$.

First suppose $d_1 + d_2$ is even. If $d_2 < 0$, from $d_1 - d_2 \le 2(g-1)$, we have $d_1 \le (d_1+d_2)/2 + (g-1)$, the Lemma is trivial.

Now assume $h_2 \neq 0$, and $d_1 + d_2 < 2(g-1)$. By the existence Theorem of special line bundles, we have

$$g - h_1((g-1) - d_1 + h_1) \ge 0$$

$$g - h_2((g-1) - d_2 + h_2) \ge 0$$

To prove that there exists a line bundle $\tilde{F}_1$ with $deg(\tilde{F}_1) = (d_1+d_2)/2 + (g-1)$, $dim H^0(X, \tilde{F}_1) \ge h_1 + h_2$, we need only to show that

$$g - (h_1+h_2)((g-1) - (\frac{(d_1+d_2)}{2} + (g-1)) + h_1 + h_2) \ge 0.$$

but

$$2[g - (h_1+h_2)(-\frac{(d_1+d_2)}{2} + h_1 + h_2)] = [g - h_1(g-1-d_1+h_1)]$$
$$+ [g - h_2(g-1-d_2+h_2)] + [*]$$

where

$$[*] = 2h_1h_2 + (h_1+h_2)^2 - h_1d_2 - h_2d_1 - (h_1+h_2)(g-1).$$

So we need only to show that $[*] \le 0$.

First if $d_2 > 0$, since X is not a hyperelliptic curve, by the Clifford Theorem, we have $h_1 + h_2 \le (d_1+d_2)/2 + 1 \le g-1$, $(d_1 + d_2 < 2(g-1))$, and $h_1 \le d_1$, $h_2 \le d_2$, so $[*] \le 0$. If $d_2 = 0$, but $h_1 + h_2 < g-1$, we will still have $[*] \le 0$.

Now suppose $d_2 = 0, h_2 = 1, h_1 + h_2 \ge g-1$. Then $h_1 \ge g-2$. Since here we assume $g > 3$, so $d_1 \ge 2g-4$, and then $[*] \le 0$.

If $d_1 + d_2 = 2(g-1)$, just let $\tilde{F}_1 = K, \tilde{F}_2 = I$, we get the Lemma.

The proof of the case $d_1 + d_2$ is odd can be given exactly the same.

From the proof, we have

Corollary:Same notations as in Lemma 8, if either a) $\tilde{F}_2 = I$ or b) $\tilde{F}_1 = I$, then $dim H^0(X, \tilde{F}_1) + dim H^0(X, \tilde{F}_2) \ge h_1 + h_2 - 1$, unless $dim H^0(X, \tilde{F}_1) = dim H^0(X, F_1)$ $\tilde{F}_2 = F_2 = I$ in a) or $\tilde{F}_1 = F_1 = I$ in b).

Now let E be a given n-dimensional indecomposable vector bundle, $E = (F_1, ..., F_n)$ a maximal splitting. To compare $dim H^0(X, E)$ with $dim H^0(X, E_d)$, Based on Lemma 7, we first compare $E = (F_1, ..., F_n)$ with $E_d = (L_1, ..., L_n)$, by using of Lemma 8, we wish to get from $(F_1, ..., F_n)$ a new sequence of line bundles $\{\tilde{F}_1, ..., \tilde{F}_n\}$ such that $\sum_{i=1}^{n} dim H^0(X, F_i) \le \sum_{i=1}^{n} dim H^0(X, \tilde{F}_i)$, but $deg(\tilde{F}_i) \le deg(L_i)$. To do this, first consider (F_1, F_2), if $deg(F_1) - deg(F_2) > 2(g-1)$, just let $\tilde{F}_1 = F_1$, $\tilde{F}_2 = F_2$ and then go on next step. If $deg(F_1) - deg(F_2) \le 2(g-1)$, since $(F_1, ..., F_n)$ is a maximal splitting, so $deg(F_2) - deg(F_1) \le g$, and then we can always use Lemma 8 to (F_1, F_2) to get a new paire of line bundles $\{\tilde{F}_1, \tilde{F}_2\}$ (In the following, we will use $\{F_1, ..., F_n\}$ to denote a sequence of line bundles and $(F_1, ..., F_n)$ to denote a splitting of a vector bundle).

Now suppose by using Lemma 8, we get from $(F_1, ..., F_{i-1})$ a new sequence of line bundles $\{\tilde{F}_1, ..., \tilde{F}_{i-1}\}$ with a) $\sum_{j=1}^{i-1} deg(\tilde{F}_j) = \sum_{j=1}^{i-1} deg(F_j)$; b) $deg(\tilde{F}_j) - deg(\tilde{F}_{j+1}) \ge 2g-3$, $1 \le j \le i-2$; c) $\sum_{j=1}^{i-1} dim H^0(X, \tilde{F}_j) \ge \sum_{j=1}^{i-1} dim H^0(X, F_j)$.

For F_i, if $deg(\tilde{F}_{i-1}) - deg(F_i) > 2(g-1)$, just let $\tilde{F}_i = F_i$ and then go on next step. If $deg(\tilde{F}_{i-1}) - deg(F_i) \le 2(g-1)$, since $deg(F_{i-1}) - deg(F_i) \ge -g$, then there exists a j, such that $deg(F_i) - deg(\tilde{F}_{j+1}) > 2(g-1)$, but $| deg(\tilde{F}_j) - deg(F_i) | < 2(g-1)$. Now we use Lemma8 to $\{\tilde{F}_j, F_i\}$ to get a new pair $\{H_1, H_2\}$ and then use Lemma 8 again to any pair in $\{\tilde{F}_1, ..., \tilde{F}_{j-1}, H_1, H_2, \tilde{F}_{j+1}, ..., \tilde{F}_{i-1}\}$ in which do not satisfy the condition b) above. After finite steps, we will get from $\{\tilde{F}_1, ..., \tilde{F}_{i-1}, F_i\}$ a new sequence of line bundles $\{\tilde{F}_1, ..., \tilde{F}_i\}$ (here we always use $\tilde{F}$ to denote a line bundle which come from applying Lemma 8, but each time they may be different).

Now suppose we get $\{\tilde{F}_1, ..., \tilde{F}_n\}$ from $(F_1, ..., F_n)$ by applying Lemma 8, we then separate $\{\tilde{F}_1, ..., \tilde{F}_n\}$ into several groups $\{\tilde{F}_1, ..., \tilde{F}_{i(1)}; \tilde{F}_{i(1)+1}, ..., \tilde{F}_{i(2)}; ...; \tilde{F}_{i(t)+1}, ..., \tilde{F}_n\}$ such that in each group $\{\tilde{F}_{i(t)+1}, ..., \tilde{F}_{i(t+1)}\}$, we always have $deg(\tilde{F}_i) - deg(\tilde{F}_{i+1}) \leq 2g-3, i(t)+1 \leq i \leq i(t+2)$, but between two groups, we have $deg(\tilde{F}_{i(t)}) - deg(\tilde{F}_{i(t)+1}) > 2(g-1)$, $deg(\tilde{F}_{i(t+1)}) - deg(\tilde{F}_{i(t+1)+1}) > 2(g-1)$.

To compare $\{\tilde{F}_1, ..., \tilde{F}_n\}$ with $\{L_1, ..., L_n\}$ more directly, we need the following Lemma

Lemma 9:Let $\{H_1, ..., H_n\}$ be a sequence of line bundles, such that for each i, $1 \leq i \leq n-1$, $deg(H_i) - deg(H_{i+1}) \geq 2g-3$, let $p \in X$ a point,[p] be the line bundle defined by divisor D=p. Then for any paire i, j, $1 \leq i < j \leq n$, with $j - i \geq 2$, define $\tilde{H}_t = H_t$ if $t \neq i, j$; $\tilde{H}_i = H_i \otimes [-p], \tilde{H}_j = H_j \otimes [p]$, we have $\sum_{t=1}^{n} dim H^0(X, \tilde{H}_t) \geq \sum_{t=1}^{n} dim H^0(X, H_t)$.

Proof:If $deg(H_i) < 0$, the Lemma is trivial. If $deg(H_i) \geq 0$, by the condition $deg(H_j) > 2(g-1)$, so $dim H^0(X, \tilde{H}_j) = dim H^0(X, H_j) + 1$, we also have the Lemma.

Now for each group $\{\tilde{F}_{i(t)+1}, ..., \tilde{F}_{i(t+1)}\}$, there may exist more than one index i such that $deg(\tilde{F}_i) - deg(\tilde{F}_{i+1}) = 2g-3$. In this case, let u be the smallest which $deg(\tilde{F}_u) - deg(\tilde{F}_{u+1}) = 2g-3$, and v be the largest which $deg(\tilde{F}_v) - deg(\tilde{F}_{v+1}) = 2g-3$, then apply Lemma 8 to $\{\tilde{F}_u, \tilde{F}_v\}$, this is we use $\tilde{F}_u \otimes [-p]$ and $\tilde{F}_v \otimes [p]$ instead of $\tilde{F}_u, \tilde{F}_v$ in the sequence, and do the same again, we then may assume with the conditions a), b), c) above we also have d)there may exist at most one index i in each group $\{\tilde{F}_{i(t)+1}, ..., \tilde{F}_{i(t+1)}\}$ such that $deg(\tilde{F}_i) - deg(\tilde{F}_{i+1}) = 2g-3$. Now we have

Lemma 10:Let $E = (F_1, ..., F_n)$ be an n-dimensional indecomposable vector bundle, $\{\tilde{F}_1, ..., \tilde{F}_{i(1)}; ...; \tilde{F}_{i(t)+1}, ..., \tilde{F}_n\}$ comes from $(F_1, ..., F_n)$ by applying Lemma 8 and Lemma 9, $E_d = (L_1, ..., L_n)$ be the vector bundle constructed at the beginning of this section, then $deg(\tilde{F}_i) \leq deg(L_i)$, $i = 1, ..., n$.

Proof:Let $\{\tilde{F}_1, ..., \tilde{F}_{i(1)}\}$ be the first group, we claim that $\{\tilde{F}_1, ..., \tilde{F}_{i(1)}\}$ comes from $(F_1, ..., F_{i(1)})$ by applying Lemma 8 and Lemma 9. In fact, since here $deg(\tilde{F}_{i(1)}) - deg(\tilde{F}_{i(1)+1}) > 2(g-1)$, we must have $deg(\tilde{F}_{i(1)}) - deg(F_{i(1)+1}) \geq 2(g-1)$, but $deg(F_{i(1)+2}) - deg(F_{i(1)+1}) \leq g$, if $deg(F_{i(1)+2}) > deg(F_{i(1)+1})$, then Lemma 8 is applicable to $\{F_{i(1)+1}, F_{i(1)+2}\}$, and the new produced must belong to the same group. The same for $F_{i(1)+3}, ..., F_n$.

Now assume $deg(\tilde{F}_1) > deg(L_1)$, from d), we get $deg(\tilde{F}_i) \geq deg(L_i)$, $i = 1, ..., i(1)$, and then $\sum_{j=1}^{i(1)} deg(\tilde{F}_j) > \sum_{j=1}^{i(1)} deg(L_j)$. But since $\{\tilde{F}_1, ..., \tilde{F}_{i(1)}\}$ comes from $(F_1, ..., F_{i(1)})$ by applying Lemma 8 and Lemma 9, From Lemma 7, we have

$$\sum_{j=1}^{i(1)} deg(\tilde{F}_j) = \sum_{j=1}^{i(1)} deg(L_j) \leq \sum_{j=1}^{i(1)} deg(L_j),$$

a contradiction. We get $deg(\tilde{F}_1) \leq deg(L_1)$.

Now suppose $deg(\tilde{F}_1) \leq deg(L_1), ..., deg(\tilde{F}_{t-1}) \leq deg(L_{t-1})$, but $deg(\tilde{F}_t) > deg(L_t), 2 \leq t \leq i(1)$, we must have $deg(\tilde{F}_j) = deg(L_j)$,j=1,...,t-1, and $deg(\tilde{F}_i) \geq deg(L_i)$, $i = t, ..., i(1)$, and then $\sum_{j=1}^{i(1)} deg(\tilde{F}_j) > \sum_{j=1}^{i(1)} deg(L_j)$, we get the same contradiction.

For $\{\tilde{F}_{i(1)+1}, ..., \tilde{F}_{i(2)}\}$, since we have $deg(\tilde{F}_{i(1)}) - deg(\tilde{F}_{i(1)+1}) > 2(g-1)$, and $deg(\tilde{F}_{i(1)}) \leq deg(L_{i(1)})$, we get $deg(\tilde{F}_{i(1)+1}) < deg(L_{i(1)+1})$, and then $deg(\tilde{F}_j) \leq deg(L_j)$, $i(1)+2 \leq j \leq i(2)$. The same for other groups. This then completes the proof of the Lemma.

Proof of Theorem 3:Let $E = (F_1, ..., F_n)$ be an n-dimensional indecomposable vector bundle, $\{\tilde{F}_1, ..., \tilde{F}_n\}$ comes from $(F_1, ..., F_n)$ by applying Lemma 8 and Lemma 9, then we have

$$dim H^0(X, E) \leq \sum_{i=1}^{n} dim H^0(X, F_i) \leq \sum_{i=1}^{n} dim H^0(X, \tilde{F}_i).$$

Since $deg(\tilde{F}_i) \leq deg(L_i)$, by the construction of E_d, we have

$$dim H^0(X, E) \leq \sum_{i=1}^{n} dim H^0(X, L_i).$$

Now if there does not exist an i, $2 \leq i \leq n$, such that $L_i = I$ or $L_i = [p]$ with $L_{i-1} = K$, then by Lemma 5, we get $dim H^0(X, E) \leq dim H^0(X, E_d)$.

If there do exist some i, $2 \leq i \leq n$ with $L_i = I$ or $L_i = [p], L_{i-1} = K$, first by the Kodaira-Serre duality, if it is the case $L_i = [p], L_{i-1} = K$, compare $E^* \otimes K$ with $E_d^* \otimes K$, so we may assume only the first case appears, that is $L_i = I$.

Let $\{H_1, ..., H_n\}$ comes from $(F_1, ..., F_n)$ by only appling Lemma 8 (not

Lemma 9). If $deg(H_i) < 0$, then it is easy to see that

$$\sum_{j=1}^{i-1} dim H^0(X, H_j) \leq \sum_{j=1}^{i-1} dim H^0(X, \hat{F}_j) \leq \sum_{j=1}^{i-1} dim H^0(X, L_j)$$

we get $dim H^0(X, E) \leq dim H^0(X, E_d)$. if $deg(H_i) > 0$, then in appling Lemma 9 to change H_i to become $\hat{F}_i$ with $deg(\hat{F}_i) \leq deg(L_i) = 0$ will certainly have $\sum_{i=1}^{n} dim H^0(X, H_i) < \sum_{i=1}^{n} dim H^0(X, \hat{F}_i)$, and then $dim H^0(X, E) \leq dim H^0(X, E_d)$. So we may assume $H_i = I$. If $F_j \neq I$, $j = 2, ..., n$, then I comes from appling Lemma 8, by the Corollary of Lemma 8, we have $dim H^0(X, E) \leq dim H^0(X, E_d)$. So we may assume for some j, $2 \leq j \leq n$, $F_j = I$, and it become H_i in applying Lemma8, furthermore, we may assume $deg(F_t) \geq 0$, $t = 1, ..., j-1$.(if deg(F_t) < 0, since $(F_1, ..., F_n)$ is a maximal splitting, so $H^0(X, (F_t, ..., F_n)) = 0$, from it is easy to see that $dim H^0(X, E) \leq dim H^0(X, E_d)$.

Now suppose that $\{A_1, ..., A_{j-1}\}$ comes from $(F_1, ..., F_{j-1})$ by applying only Lemma8, consider $\{A_1, ..., A_{j-1}, F_j\}$. Since $F_j = I$ would not change, so either $deg(A_{j-1}) - deg(F_j) \geq 2g-3$, or there exists a t, $1 \leq t \leq j-1$, such that $deg(F_j) - deg(A_{t+1}) \geq 2g-3$ and $deg(A_t) - deg(F_j) \geq 2g-3$. But if it is of the case, then since $deg(A_t) - deg(A_{t-1}) > 2(g-1)$, so $\{A_{t+1}, ..., A_{j-1}\}$ must come from a subset $\{F_{i(t+1)}, ..., F_{I(j-1)}\} \subset \{F_1, ..., F_{j-1}\}$ by applying Lemma8, and then $deg(A_{t+1}) \geq deg(F_{i(t+1)}) \geq 0$, a contradiction. So we must have $deg(A_{j-1}) - deg(F_j) \geq 2g-3$. Now consider $\{A_1, ..., A_{j-1}, F_j, F_{j+1}\}$, since $deg(F_{j+1}) - deg(F_j) \leq g$, so if $deg(F_j) - deg(F_{j+1}) < 2g-3$, then Lemma 8 is applicable to $\{F_j, F_{j+1}\}$, and F_j will change, we get $deg(F_j) - deg(F_{j+1}) \geq 2g-3$. Same if $\hat{F}$ comes from $(F_{j+1}, ..., F_n)$ by applying Lemma8, we will have $deg(F_j) - deg(\hat{F}) \geq 2g-3$, and then $\hat{F}$ cannot never go beyond F_j. We conclude that i=j and $\{H_1, ..., H_{i-1}\}$ comes from $(F_1, ..., F_{i-1})$.

If $\sum_{t=1}^{i-1} deg(H_t) = \sum_{t=1}^{i-1} deg(F_t) < \sum_{t=1}^{i-1} deg(L_t)$, from $deg(L_{i-1}) \geq 2g-3$, we will get $\sum_{t=1}^{i-1} dim H^0(X, H_t) < \sum_{t=1}^{i-1} dim H^0(X, L_t)$, and then we will have $dim H^0(X, E) \leq dim H^0(X, E_d)$. So we may assume

$$\sum_{t=1}^{i-1} deg(F_t) = \sum_{t=1}^{i-1} deg(L_t) = i(i-1)(g-1) - s,$$

$$0 \leq s \leq i-1$$

Now let $e \in H^1(X, (F_1, ..., F_{i-1}) \otimes F_i^*)$ be the element which determines the extension of F_i by $(F_1, ..., F_{i-1})$ in E. If e=0, same as in the proof of Lemma 6, there must exist an m, $i+1 \leq m \leq n$, such that $deg(F_m) \geq -1$, and then applying Lemma 8 to $(F_{i+1}, ..., F_n)$ will produce some $\hat{F}$ with $deg(\hat{F}) \geq -1$, Lemma 8 is then applicable to $\{F_i, \hat{F}\}$. F_i will change. So $e \neq 0$, but then the non-zero sections of $H^0(X, F_i) = H^0(X, I)$ cannot be left to be sections of E, we also have $dim H^0(X, E) \leq dim H^0(X, E_d)$. This completes the Proof of Theorem3.

4 Proof of Theorem 1

Since E_d gives the necessary and sufficient condition of the existence of special n-dimensional indecomposable vector bundles in case X is a general curve (from the proof, one easily sees that for any h, $1 \leq h \leq dim H^0(X, E_d)$, there exists an n-dimensional indecomposable vector bundle E with deg(E) =d, $dim H^0(X, E) = h$), so E_d gives necessary and sufficient condition on X if X is assumed to be arbitrary. To prove Theorem 1, we need only to study the relation between $h = dim H^0(X, E_d)$ and d.

Proof of Theorem1: First we assume $d \leq n(g-1)$. If n=1, the theorem is exactly the existence Theorem of special line bundles. Now assume this Theorem for n-1.

For given d with $d(mod n) = t \neq 0$ (the case t=0 can be proved exactly the same), $-(n-l+1)n(g-1) \leq d-t \leq -(n-l)n(g-1)$. Let $d_1 = (n-1)(d-t)/n + (n-1)(g-1) + t - 1$, $d_2 = (d-t)/n - (n-1)(g-1) + 1$, E_d is then an extension of a line bundle L_n of degree d_2 by E_{d_1}. If $n > 2$, then $d_2 < 0$, if n=2 d=2g-3, then $L_2 = I$, the non-zero sections of I cannot be left to be sections of E_d, so in both cases we have $dim H^0(X, E_d) = dim H^0(X, E_{d_1}) = h$. The case $l \leq n$ can be proved directly by induction from the formula of $h = dim H^0(X, E_{d_1})$ and d_1, we here consider only the case l=n+1. In this case, $(n-1)(g-1) \leq d_1 \leq 2(n-1)(g-1)$. Let $d_3 = 2(n-1)(g-1) - d_1$, $E_{d_3} = E_{d_1}^* \otimes K$ then $d_3 = (n-1)(g-1-(n-1)(d-t)/n-1) + n - t$, so $d_3(mod(n-1)) = n-t$, and $0 \leq d_3 - (n-t) \leq (n-1)(g-1)$. By induction, the formula for $h_3 = dim H^0(X, E_{d_3})$ and d_3 is

$$(n-1)^2 g - [-\frac{n(d_3-(n-t))}{2} - \frac{n(n-1)(n-4)(g-1)}{4} -$$

$$(n-1)max\{0, \frac{2(n-t)+n-2(n-1)}{2}\}+(n-1)h_3]$$

$$\times[-\frac{(n-2)(d_3-(n-t))}{2}-\frac{(n-2)^2(n-1)(g-1)}{4}-$$

$$(n-1)max\{0, \frac{2(n-t)+n-2n}{2}\}+(n-1)h_3]\geq 0$$

if n is even;

$$(n-1)^2g-[-\frac{(n-3)(d_3-(n-t))}{2}-\frac{(n-1)^2(n-3)(g-1)}{4}-$$

$$(n-1)max\{0, \frac{2(n-t)+n-2(n-1)-3}{2}\}+(n-1)h_3]$$

$$\times[-\frac{(n-1)(d_3-(n-t))}{2}-\frac{(n-1)^2(n-3)(g-1)}{4}-$$

$$(n-1)max\{0, \frac{2(n-t)+n-2(n-1)-1}{2}\}+(n-1)h_3]\geq 0$$

if n is odd.

By the Kodaira-Serre duality, we know that $h_3 = dimH^1(X, E_{d_1}) = -d_1 + (n-1)(g-1)+h$, substitute this and $d_3 = (n-1)(g-1-(d-t)/n-1)+(n-t)$ in the formula and multiply both sides by $n^2/(n-1)^2$, then from

$$\frac{n}{(n-1)}\{\frac{n(n-1)}{2}-(n-1)(t-1)-(n-1)max\{0, \frac{(n-2t+2)}{2}\}\}$$

$$= -nmax\{0, \frac{(2t-n-2)}{2}\}$$

and

$$\frac{n}{n-1}\{\frac{(n-2)(n-1)}{2}-(n-1)(t-1)-(n-1)max\{0, \frac{(n-2t)}{2}\}\}$$

$$= -nmax\{0, \frac{(2t-n)}{2}\}$$

we get

$$n^2g-[-\frac{(n-2)(d-t)}{2}-\frac{n^2(n-2)(g-1)}{4}-nmax\{0, \frac{(2t-n-2)}{2}\}+nh]$$

$$\times[-\frac{n(d-t)}{2}-\frac{n^2(n-2)(g-1)}{4}-nmax\{0, \frac{(2t-n)}{2}\}+nh]\geq 0$$

if n is even.
and from

$$\frac{n}{n-1}\{\frac{(n-3)(n-1)}{2}-(n-1)(t-1)-(n-1)max\{0, \frac{(n-2t-1)}{2}\}\}$$

$$= -nmax\{0, \frac{(2t-n+1)}{2}\}$$

$$\frac{n}{n-1}\{\frac{(n-1)^2}{2}-(n-1)(t-1)-(n-1)max\{0, \frac{(n-2t+1)}{2}\}\}$$

$$= -nmax\{0, \frac{(2t-n-1)}{2}\}$$

we get

$$n^2g-[\frac{(n+1)(d-t)}{2}-\frac{(n+1)n(n-3)(g-1)}{4}-nmax\{0, \frac{(n-2t-1)}{2}\}+nh]$$

$$\times[-\frac{(n-1)(d-t)}{2}-\frac{n(n-1)^2(g-1)}{4}-nmax\{0, \frac{(2t-n-1)}{2}\}+nh]\geq 0$$

If n is odd.

The case $l > n+1$ can be proved directly from the Kodaira-Serre duality and the formulas of $l \leq n+1$, we omit its proof here.

5 Proof of Theorem 2

Proof of Theorem2:By the Kodaira-Serre duality, we will assume here $d \leq n(g-1)$.

If n=1, the theorem is the same as the case of special line bundles.

Now assume n=2. If d(mod2)=0, but $2^2 g - 2h(-d+2h) > 0$, then $g - h((g-1) - (d/2 + (g-1)) + h) > 0$, so there are infinite line bundles L with $deg(L) = d/2 + g - 1$, $dim H^0(X, L)$ and $dim H^0(X, L) \geq h$. For each L, by our construction, we get a two-dimensional indecomposable vector bundle E with deg(E)=d, $dim H^0(X, E) \geq h$. So $W^h_{d,2}(X)$ contains infinite elements.

If d(mod2)=1, $2^2 g - 2h(-(d-1)+2h) \geq 0$, we get $g - h((g-1) - (d/2 + g - 1) + h) \geq 0$, so there exists at least a line bundle L with deg(L)=$d/2 + g - 1$, $dim H^0(X, L) \geq h$, But since $deg(L) - ((d+1)/2 - (g-1)) = 2g - 3$, so for L there are infinite line bundles F with $deg(F) = (d+1)/2 - (g-1)$, $H^1(X, L \otimes F^*) \neq 0$. By our construction we get infinite elements in $W^h_{d,2}(X)$.

Now assume d(mod2)=0, $2^2 g - 2h(-d+2h) = 0$, then $g - h((g-1) - (d/2 + g - 1) + h) = 0$. from the case of line bundles, we know that $W^h_{d/2+g-1}(X)$ contains exactly

$$m = g! \prod_{i=0}^{h-1} \frac{i!}{(-d/2 + h + i)!}$$

elements. For each $L \in W^h_{d/2+g-1}(X)$, since $deg(L) - (d/2 + g - 1) = 2(g-1)$, so $F = K^* \otimes L$ is the only line bundle that $deg(F) = d/2 - (g-1)$, $H^1(X, L \otimes F^*) \neq 0$. Let E be the non-trivial extension of F by L (since $dim H^1(X, L \otimes F^*) = dim H^1(X, K) = 1$, so up to isomorphism, there exists only one non-trivial extension of F by L), we get m elements in $W^h_{d,2}(X)$. Now we wish to show that if $E \in W^h_{2,d}(X)$, then E must be constructed in this way.

First write E as extension of line bundles L_2 by L_1 with $d_1 = deg(L_1)$, $d_2 = deg(L_2)$, $h_1 = dim H^0(X, L_1)$, $h_2 = dim H^0(X, L_2)$.

If $h_2 = 0$, from $d_1 - d_2 \leq 2(g-1)$, we have $d_1 \leq d/2 + g - 1$. If $d_1 = d/2 + g - 1, h_1 = h$, then $L_1 \in W^h_{2,d}(X)$. If $d_1 < d/2 + g - 1$, from $g - h((g-1) - (d/2 + g - 1) + h) = 0$, we get $g - h((g-1) - d_1 + h) < 0$, so $h_1 < h, E \in W^h_{2,d}(X)$.

Suppose $h_2 \neq 0, L_2 \neq I$ (otherwise the non-zero sections of L_2 cannot be left). From $g - h_1(g - 1 - d_1 + _1) \geq 0, g - h_2(g - 1 - d_2 + h_2) \geq 0$, we get

$$g - (h_1 + h_2)(-\frac{d}{2} + g - 1) + [2h_1h_2 + (h_1 + h_2)^2 - h_1d_2 - h_2d_1 - (h_1 + h_2)(g-1)] \geq 0$$

so if $(h_1 + h_2) \geq h$, we then have

$$2h_1h_2 + (h_1 + h_2)^2 - h_1d_2 - h_2d_1 - (h_1 + h_2)(g-1) =: [*] \geq 0.$$

But if $(h_1 + h_2) \leq g - 1$, from the Clifford Theorem, it is easy to see that $[*] < 0$. So we may assume $h_1 + h_2 = g, d_1 + d_2 = 2(g-1)$.

If $L_1 = K, L_2 = I$, then E comes from the above construction. Since we can always assume $deg(L_1) \geq 1$, we do not need to consider the case $L_1 = I, L_2 = K$.

Now assume $L_1 \neq K$. From $g - h_2(g - 1 - d_2 + h_2) \geq 0$, we get $d_2 \geq (g - 1 + h_2 - g/h_2)$, so $d_1 = 2(g-1) - d_2 \leq g - 1 - h_2 + g/h_2, h_1 = g - h_2$. But $g - h_1(g - 1 - g_1 + h_1) \geq 0$, we have

$$g - (g - h_2)(g - 1 - (g - 1 - h_2 + \frac{g}{h_2}) + g - h_2) \geq 0$$

This is

$$-gh_2 - h_2^2 + g \geq 0$$

It is easy to see that when $g \geq 5, 2 \leq h_2 \leq g/2$, we will have

$$-gh_2 + h_2^2 + g < 0$$

so for $h_1 + h_2 = g, d_1 + d_2 = 2(g-1)$, we must have $L_2 = [p], p \in X$. But since $d_1 = 2g - 3, h_1 = g - 1$, then for any $s \in H^0(X, L_2), s \neq 0$, the map $H^1(X, L_1 \otimes [-p]) \mapsto H^1(X, L_1)$ which induced by time s is injection, and then $sr \neq 0$, s cannot be left to be a section of E, $E \in W^h_{2,d}(X)$.

If $h_1 \leq h_2$ with $d_1 \geq 2$, we could get the same contradiction. Since here we assume $dim H^0(X, E) = g \geq 5$, we can always find a section $s \in H^0(X, E)$ such that s hase at least two zero points, and then the line subbundle of E which generated by s will have degree at least two, so we do not need to consider the case $d_1 = 1$. This then complete the proof for $g > 4$. The Theorem is still true for g=4, but not for g=2 and g=3, for this we refer to [9].

Now consider the case n=3. If $d(mod 3) \neq 0$, or if

$$9g - 3g(-d - 3(g-1) + 3h) > 0 \qquad (-6(g-1) \le d \le 0)$$

$$9g - (-2d + 3h)(-d - 3(g-1) + 3h) > 0 \qquad (0 \le d \le 3(g-1))$$

Same as in the case of n=2, we get infinite elements in $W^h_{3,d}(X)$. So we assume d(mod3)=0 and

$$9g - 3g(-d - 3(g-1) + 3h) = 0 \qquad (-6(g-1) \le d \le 0)$$

$$9g - (-2d + 3h)(-d - 3(g-1) + 3h) = 0 \qquad (0 \le d \le 3(g-1))$$

Also same as n=2, then from $W^h_{2,2d/3+2(g-1)}(X)$, we can constructe

$$g! \prod_{i=0}^{h-1} \frac{i!}{(-(d/3 + g - 1) + h + i)!}$$

elements in $W^h_{3,d}(X)$, Now we wish to show if $E \in W^h_{3,d}(X)$, then E must come from this construction.

Let $E = (F_1, F_2, F_3)$ be a maximal splitting of E, $E_d = (L_1, L_2, L_3)$ be the vector bundle given in Theorem 3 ($E_d \in W^h_{3,d}(X)$). First assume $deg(L_3) \le 0$, If $deg(F_3) = deg(L_3)$, then it is easy to see that $(F_1, F_2) \in {W_{2d/3+2(g-1)}}^h(X)$, and then E comes from the above construction. So we may assume $deg(F_3) > deg(L_3)$. Let $\{\tilde{F}_1, \dot{F}_2\}$ comes from (F_1, F_2) by applying Lemma 8. Then $deg(\tilde{F}_1) < deg(L_1), deg(\tilde{F}_2) \le deg(L_2)$. If $deg(L_1) \ge 2g$ or $deg(L_1) \le 2(g-1)$ (in this case, we will have $g - dimH^0(X, L_1)(g - 1 - deg(L_1) + dimH^0(X, L_1)) = 0$), we will have $dimH^0(X, \tilde{F}_1) < dimH^0(X, L_1)$. So for $dimH^0(X, \tilde{F}_1) + dimH^0(X, \dot{F}_2) = dimH^0(X, L_1) + dimH^0(X, L_2)$, we must have $deg(L_1) = 2g-1, deg(L_2) = 1$, and $\dot{F}_1 = K, \tilde{F}_2 = I$ or [p]. If in this case, $F_1 \ne \tilde{F}_1$ or $F_2 \ne \tilde{F}_2$, then $\sum_{i=1}^{2} dimH^0(X, F_i) < \sum_{i=1}^{2} dimH^0(X, \tilde{F}_1) = dimH^0(X, E_d), E \in W^h_{3,d}(X)$. If $F_1 = \dot{F}_1, F_2 = \dot{F}_2$ then the non-zero sections of F_2 cannot be left to be sections of E, we still have $E \in W^h_{3,d}(X)$.

Now assume $deg(F_3) \ge 1$. First apply Lemma8 to (F_1, F_2) to get $\{\tilde{F}_1, \tilde{F}_2\}$, if $deg(\tilde{F}_2) \le 0$, we must have $deg(\dot{F}_2) \ge deg(L_3)$, using the same argument as above to compare $\{\tilde{F}_1, F_3\}$ with (L_1, L_2), we will get $dimH^0(X, E) < dimH^0(X, E_d)$. If $deg(\tilde{F}_2) \ge 1$, apply Lemma 8 again to $\{\tilde{F}_2, F_3\}$ to get $\{H_1, H_2\}$, we then have $deg(L_3) < deg(H_2) < 0$. Compare $\{\tilde{F}_1, H_2\}$ with (L_1, L_2), we also get $dimH^0(X, E) < dimH^0(X, E_d)$, $E \in W^h_{3,d}(X)$.

The proof for general n can be obtained inductively as the case n=3, we will not give it here.

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