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Noor Mohammad

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S. Somasundaram

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Noor Mohammad * and S. Somasundaram
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We seek the generalization of the Gelfand–Naimark theorems for locally C^*–algebras. Precisely, if A is a unital commutative locally C^*–algebra, then it is shown that A is *–isomorphic (topologically and algebraically) to $C(\Delta)$. Further, if A is any locally C^*–algebra, then it is realized as a closed *–subalgebra of some $L(H)$ upto a topological algebraic *–isomorphism. Also, a brief exposition of the Gelfand–Naimark–Segal construction is given and some of its consequences are discussed.

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* Permanent address: Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan.

1. INTRODUCTION

Since the appearance of the Gelfand–Naimark theorems for C^*–algebras in 1943, this subject has undergone considerable development (see for instance, [5] and the references therein). During the last few decades much progress has been made in the field of non-normed topological algebras in order to seek analogous concepts in a more general framework (see [1], [2], [3], [11], [12], [13]). The notion of a locally m–convex algebra (i.e., a topological algebra whose topology is given by a family of submultiplicative seminorms), being a natural generalization of a Banach algebra first appeared in the literature in 1946 and is due to Arens [1]. The first fundamental results on this type of algebra were published in 1952 by Arens [2] and Michael [13] independently. The term "locally C^*–algebra" was introduced by Inoue [11]. These algebras play an important role in the representation theory and also in quantum physics (cf. [11], [16]). For special kinds of these algebras one can refer to M. Fragoulopoulou [8].

Our objective is to seek the generalization of the Gelfand–Naimark theorems for locally C^*–algebras. In doing so, we put together all the relevant results so as to present an up-to-date and reasonably complete picture of the progress made in this direction. The whole paper is an expository article.

The plan of the paper is as follows. In Section 2 we give a series of basic results on locally multiplicatively convex *–algebras. Section 3 is devoted to the study of commutative complete l.m.c. *–algebras. We develop appropriate details for the proof of the Gelfand–Naimark type theorem for commutative locally C^*–algebras. Finally in Section 4 we give a brief exposition of the Gelfand–Naimark–Segal construction, and discuss some of its consequences. We also show that every locally C^*–algebra is realized up to a topological algebraic *–isomorphism, as a closed *–subalgebra of some $B(H)$, H a locally Hilbert space.

2. LOCALLY MULTIPLICATIVELY CONVEX *–ALGEBRAS

Definition 2.1 A *topological algebra* is a topological vector space A in which the ring multiplication $(x,y) \to xy$ is separately continuous. By a *locally multiplicatively convex* (:l.m.c.) algebra we mean a topological algebra A whose topology is defined by a family $(p_\alpha), \alpha \in I$ (I a directed index set), of submultiplicative seminorms, i.e., $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$ for every $\alpha \in I$ and $x, y \in A$. An l.m.c. algebra A with an involution $*$ such that $p_\alpha(x^*) = p_\alpha(x)$ for all $\alpha \in I$, $x \in A$, is called an *l.m.c.* **–algebra*. If moreover, each $p_\alpha, \alpha \in I$, is a C^*–seminorm, i.e., $p_\alpha(x^*x) = p_\alpha(x)^2$ for every $\alpha \in I$ and $x \in A$, then A is said to be an *l.m.c.* C^**-algebra*. A complete l.m.c. C^* algebra is called a *locally* C^**–algebra*.

The *unitization* of an l.m.c. *–algebra A is the l.m.c. *–algebra $A_e = A \oplus \mathbb{C}$ with involution $(x,\lambda)^* = (x^*, \bar{\lambda})$ and the topology defined by $p_{\alpha,e}(x,\lambda) := p_\alpha(x) + |\lambda|, \alpha \in I, (x,\lambda) \in A_e$.

For an l.m.c. *-algebra A, define an algebra norm $\|\cdot\|_\alpha$ on A/N_α by $\|x_\alpha\|_\alpha := p_\alpha(x)$, $x_\alpha = x + N_\alpha, x \in A, \alpha \in I$, where $N_\alpha = \ker(p_\alpha)$. Then, the completion A_α of the normed *-algebra A/N_α is a Banach *-algebra.

The following theorem is fundamental in the theory of l.m.c. algebras. It has various applications. It shows how one can extend to l.m.c. algebras certain properties of Banach algebras (see Theorem 2.4 below).

Theorem 2.2 (Michael [13; Theorem 5.1]). Each complete l.m.c. algebra is isomorphic with the projective limit of Banach algebras. Precisely,

$$A \approx \lim_{\leftarrow} A_\alpha .$$

Definition 2.3 Let A be an algebra. An element $x \in A$ is said to be *left* or right quasi-regular if there is $y \in A$. s.t $yx + x + y = 0$ or $xy + x + y = 0$. If x is both left and right quasi-regular, then it is called *quasi-regular*. If moreover, A is unital then quasi-regularity of an element $x \in A$ is equivalent to invertibility of $e + x$. The set of all quasi-regular elements of A is denoted by $G(A)$. A topological algebra is called a *Q-algebra* if $G(A)$ is open. A sequentially complete Q l.m.c. *-algebra is said to be an MQ *-algebra. Note that each Banach *-algebra is an MQ *-algebra. There are several other examples of such objects (see for instance, [9; pp.16–56]). Recall that the notions of completeness and sequential completeness coincide on Fréchet spaces (i.e., complete and metrizable).

Let A be a unital algebra. For each $x \in A$, the set $sp_A(x) = \{\lambda \in \mathbb{C} : \lambda e - x$ is not invertible in $A\}$ is called the *spectrum of* x. The number $r_A(x) = \sup\{|\lambda| : \lambda \in sp_A(x)\}$ is called the *spectral radius* of x. Recall that if A is a Banach algebra, then $sp_A(x)$ is a non-empty compact subset of $\mathbb{C}$, for each $x \in A$. Furthermore, $r_A(x) = \lim_{n\to\infty} \|x^n\|^{1/n}, r_A(x) \le \|x\|$ (see [5; pp.308-311]).

Theorem 2.4 Let A be a unital complete l.m.c. algebra. Then

(i) x is invertible in A iff x_α is invertible in A_α for each $\alpha \in I$.

(ii) Every A_α is unital.

(iii) $sp_A(x) \neq \phi, sp_A(x) = \bigcup_{\alpha \in I} sp_{A_\alpha}(x_\alpha)$.

(iv) $r_A(x) = \sup_\alpha r_{A_\alpha}(x_\alpha) = \sup_\alpha \lim_{n\to\infty} [p_\alpha(x^n)]^{1/n}$.

Proof We prove only (iii) and (iv). The rest can be found in [13; Theorem 5.2]. Let $\lambda \in sp_A(x)$. Then $\lambda e - x$ is not invertible in A iff $\lambda e_\alpha - x_\alpha$ is not invertible in A_α which means that $\lambda \in sp_{A_\alpha}(x_\alpha)$.
(iv) The first equality follows from (iii), while the second uses mainly spectral radius formula in Banach algebra.

Theorem 2.5 (Gelfand–Mazur). Let A be a unital complete l.m.c. algebra such that each non-zero element is invertible. Then A is isomorphic (algebraically and topologically) to the field of complex numbers.

Proof Let $\lambda \in sp_A(x)$ (since $sp_A(x)$ is non-empty), then $\lambda e - x$ is not invertible in A. Hence $\lambda e - x = 0$. The mapping $\lambda e = x \mapsto \lambda$ is clearly an isomorphism.

3. COMMUTATIVE l.m.c. *-ALGEBRAS

Denote by Δ the set of all continuous multiplicative linear functionals on a unital commutative l.m.c. algebra A and by $\mathcal{M}$ the set of all maximal ideals of A. It is well known that for any closed maximal ideal M of A, A/M is an l.m.c. algebra. Moroever, if A is complete, then so is A/M; A/M is a division algebra, and hence by the Gelfand–Mazur theorem it is isomorphic to $\mathbb{C}$. If $f \in \Delta$, then clearly $\ker(f) = \{x \in A : f(x) = 0\}$ is a maximal ideal of A. Further, if $f, g \in \Delta$ such that $\ker(f) = \ker(g)$, then $f = g$. Thus there is 1-1 correspondence between Δ and $\mathcal{M}$. Precisely, every maximal ideal M in A is the kernel of a unique $f \in \Delta$ and vice versa.

Theorem 3.1 Let A be a unital commutative Q l.m.c. algebra. Then

(i) Each maximal ideal is closed.

(ii) Each multiplicative linear functional is continuous.

Proof Let M be a maximal ideal in A. Then $\bar{M}$ is an ideal and $M \subset \bar{M} \subset A$. By the maximality of M we see that either $M = \bar{M}$ or $\bar{M} = A$. We rule out the second possibility and show that $M = \bar{M}$. Assume the contrary, i.e., $\bar{M} = A$. Since $G(A)$ is open (because A is a Q-algebra) and $e \in G(A)$, there is some $x \in M \cap G(A)$ such that $x \neq e$. Thus x is invertible and hence $e = xx^{-1} \in M$ which gives a contradiction.
(ii) Let f be a multiplicative linear functional on A. Then $\ker(f)$ is a maximal ideal and therefore closed by (i). The continuity of f now follows from the fact that f is continuous iff $\ker(f)$ is closed (see for instance, [9; Proposition 2.5]).

The relation between elements in A and points in the spectra of elements in A is given in the following.

Theorem 3.2 Let A be a unital commutative complete l.m.c. algebra. Then $\lambda \in sp_A(x)$ iff there is some $f \in \Delta$ such that $f(x) = \lambda$.

Proof First note that an element $x \in A$ is invertible iff $f(x) \neq 0$ for every $f \in \Delta$. Let $\lambda \in sp_A(x)$, then $\lambda e - x$ is not invertible iff $f(\lambda e - x) = 0$ for some $f \in \Delta$. Hence $\lambda = f(x)$ for some $f \in \Delta$.

Corollary 3.3 Let A be as in Theorem 3.2. Then

$$sp_A(x) = \{f(x) : f \in \Delta\};\ r_A(x) = \sup\{|f(x)| : f \in \Delta\} .$$

Moreover, $sp_A(x+y) \subset sp_A(x) + sp_A(y)$, $sp_A(xy) \subseteq sp_A(x)sp_A(y)$ and $sp_A(x^n) = [sp_A(x)]^n$ for any positive integer n.

Theorem 3.4 Let A be a unital Fréchet Q l.m.c. algebra. Then Δ is weakly compact.

Proof By Corollary 3.3, we have

$$sp_A(x) = \{f(x) : f \in \Delta\} \quad and \quad r_A(x) = \sup\{|f(x)| : f \in \Delta\}.$$

Let $U = \{x \in A : r_A(x) \leq 1\}$, then U is a neighbourhood of 0 (since A is a Q–algebra). By the Banach–Alaoglu theorem, the polar of $U, U^0 = \{f \in \Delta : |f(x)| \leq 1$, for every $x \in U\}$ is weakly compact. Further,

$$U = \bigcap_{f \in \Delta} \{x \in A : |f(x)| \leq 1\} = \Delta^0, \Delta \subset \Delta^{00} = U^0.$$

We now show that Δ is weakly closed in A'. Note that $\Delta = \bigcap_{x,y \in A} \{f \in A' : f(xy) - f(x)f(y) = 0\}$. Since the maps: $f \to f(x), f \to f(y), f \to f(xy)$ are continuous on A', it follows that Δ being the intersection of closed sets is closed. Hence Δ is weakly compact.

Definition 3.5 Let A be a unital commutative complete l.m.c. algebra. For each x in A define a mapping $\hat{x} : \Delta \to \mathbb{C}$ by $\hat{x}(f) = f(x)$. The function $\hat{x}$ is called the *Gelfand transform* of x. Denote by $\hat{A} = \{\hat{x} : x \in A\}$. Clearly, $\hat{A}$ is a separating algebra. If moreover, A is an l.m.c. *–algebra then $\hat{A}$ is called *self–adjoint* if $\widehat{x^*} = \bar{\hat{x}}$.

Recall that the *radical* of A, denoted by $R(A)$, is the set

$$R(A) = \{x \in A : f(x) = 0 \quad \text{for every} \quad f \in \Delta\}$$
$$= \bigcap_{M \in \mathcal{M}} M.$$

If $R(A) = \{0\}$, then A is said to be *semisimple*. We now have the following characterization theorem for locally C^*–algebras (see also [13] and [15]).

Theorem 3.6 Let A be a unital commutative locally C^*– algebra. Then the Gelfand representation $x \mapsto \hat{x}$ is a continuous *–isomorphism of A onto $C(\Delta)$ with the topology of uniform convergence on the equicontinuous subsets of A. If A is moreover Fréchet, then $C(\Delta)$ has compact–open topology and $A = C(\Delta)$, i.e., A is full (Definition 8.3 [13]).

Proof Since A is a locally C^*–algebra, each A_α is a C^*–algebra and hence A_α is semisimple [5; Proposition 24.1, p.82]. Therefore, A is semisimple by Proposition 7.3 [13]. Further, A is symmetric [13; Proposition 7.4], since A_α is symmetric for each $\alpha \in I$ (see also [7; Proposition 4.1]). Also Δ is non–empty [12; Lemma V, 6.3]. Clearly, $x \mapsto \hat{x}$ is a homomorphism, and semisimplicity of A implies that $x \mapsto \hat{x}$ is in fact an isomorphism. By symmetry of A we get $\widehat{x^*} = \bar{\hat{x}}$, i.e., the Gelfand representation is a *–isomorphism. Note that each $\hat{x}$ is clearly continuous. The rest follows from Theorem 8.4 [13].

4. POSITIVE FUNCTIONALS AND *–REPRESENTATIONS

In this section we study positive functionals and *–representations of l.m.c. *–algebras. The Gelfand–Naimark–Segal construction will be considered and thus a relationship between these two concepts is established.

Let A be a unital l.m.c. *–algebra and f a linear functional on A. Then f is said to be *positive* if $f(x^*x) \geq 0$ for every $x \in A$. It is well known that each positive functional on A satisfies the following properties:

(1) $f(y^*x) = \overline{f(x^*y)}$;

(2) $|f(x^*y)|^2 \leq f(x^*x)f(y^*y)$ for every $x, y \in A$.

In particular, $f(x^*) = \overline{f(x)}$, i.e., f is hermitian, and $|f(x)|^2 \leq f(e)\, f(x^*x)$.
Denote by $P(A)$ the set of all continuous positive functionals on A. A positive functional f on A is said to be *pure* or *indecomposable* if every $g \in P(A)$ which is dominated by f (i.e., $g \leq f \Leftrightarrow g(x^*x) \leq f(x^*x)\ \forall\, x \in A$) is of the form $g = \lambda f$ for some $\lambda \in [0, 1]$.

Theorem 4.1 [10; Theorem 2]. Every positive functional f on a unital MQ *–algebra is continuous.

Proof Note that $|f(x)| \leq f(e)\, r_A(x)$ for all $x \in A$ with $x^* = x$. Assume that $f(x) \neq 0$. Then, clearly $f(e) > 0$. For each $\varepsilon > 0$, the set $W = \{x \in A : r_A(x) < \varepsilon f(e)\}$ is a neighbourhood of 0 because A is a Q–algebra. Hence for all $x \in W \cap H(A)$, $|f(x)| < \varepsilon$. This proves that f is continuous at 0. Here $H(A) = \{x \in A : x^* = x\}$. Therefore by the linearity of f we get that it is continuous everywhere.

Let A'_s be the weak topological dual of A. Then $A'_s = \bigcup_{\alpha \in I} U^0_\alpha$, where U^0_α is the polar of the neighbourhood $U_\alpha = \{x \in A : p_\alpha(x) \leq 1\}$. Further, $P(A) = \bigcup_\alpha P_\alpha(A)$ with $P_\alpha(A) = \{f \in P(A) : |f(x)| \leq 1$ for every $x \in U_\alpha\}$.

Theorem 4.2 [6; Theorem 3.1]. Let A be a unital l.m.c. *– algebra. Then for each $\alpha \in I$

$$P(A/N_\alpha) = P_\alpha(A) = P(A_\alpha)$$

within homeomorphisms.

Proof Let $\alpha \in I$ and $P_\alpha(A)$ the corresponding subspace of $P(A)$. Then for each $f \in P_\alpha(A)$, $\ker(P_\alpha) \subset \ker(f)$, so that we can define $f_\alpha \in P(A/N_\alpha)$ by $f_\alpha(x_\alpha) = f(x), x_\alpha \in A/N_\alpha$; and we denote its extension to A_α also by f_α. Thus the map $P_\alpha(A) \to P(A/N_\alpha)$ (resp. $P(A_\alpha)$) $f \mapsto f_\alpha$ is a homeomorphism, the continuity being a consequence of the equicontinuity of $P(A_\alpha)$.

Thus we get

Proposition 4.3 A positive functional f is an extreme point of $P(A)$ iff it is pure with $\|f_\alpha\| = 1$, where f_α is the positive functional on A_α corresponding to f.

Proof It is easily seen that f pure implies f_α pure, and hence f_α is an extreme point of $P(A_\alpha)$. Let $f_1, f_2 \in P(A)$ with $f = \lambda f_1 + (1-\lambda) f_2, \lambda \in (0,1)$. Then $f(x^*x) \geq f_1(x^*x)$, $f(x^*x) \geq f_2(x^*x)$ for every $x \in A$. Further, by the continuity of f there exists $\alpha \in I$ s.t

$$f_1(x^*x) \leq f(x^*x) \leq p_\alpha(x^*x) \leq p_\alpha(x)^2$$

and $f_2(x^*x) \leq f(x^*x) \leq p_\alpha(x^*x) \leq p_\alpha(x)^2$ for every $x \in A$, so that $f, f_1, f_2 \in P_\alpha(A)$. Moreover, we can write $f_\alpha = \lambda f_{1,\alpha} + (1-\lambda) f_{2,\alpha}, \lambda \in (0,1)$ where $f_{1,\alpha}, f_{2,\alpha}$ are the positive functionals on A_2 corresponding to f_1 and f_2 respectively. Since f_α is an extreme point, we get $f_\alpha = f_{1,\alpha} = f_{2,\alpha}$, hence also $f_1 = f = f_2$.

The converse is easy.

By a **–representation* we mean a *–morphism π of an l.m.c. *–algebra A into the C^*–algebra $B(H)$ of all bounded linear operators on some Hilbert space H. The continuity of π is always considered with respect to the norm topology of $B(H)$. Given a *–representation π of A and $\xi \in H$ with $\xi \neq 0$, the relation $f(x) = <\pi(x)\xi, \xi>$, $x \in A$, (where $<\cdot,\cdot>$ denotes the inner product of H) defines a positive functional on A which is continuous provided π is continuous. The following theorem, usually known as the Gelfand–Naimark–Segal construction, establishes a relationship between positive functionals and *–representations. This result was first proved by Brooks [4; Theorem 6.1] for unital complete l.m.c. *–algebras and later by Fragoulopoulou [6; Theorem 3.4] for the more general situation of l.m.c. *–algebras.

Theorem 4.4 Let A be a unital l.m.c. *–algebra and $f \in P(A)$. Then there exists a continuous *–representation π_f of A and a cyclic vector ξ of π_f such that

$$f(x) = <\pi_f(x)\xi, \xi> \quad \text{for all} \quad x \in A .$$

In particular, $\|\xi\| = \|f_\alpha\|^{1/2}$, where f_α is the continuous positive functional on A_α corresponding to f.

Proof Consider the set $L_f = \{x \in A : f(x^*x) = 0\}, f \in P(A)$. Clearly, L_f is a closed left ideal of A. It is easy to see that A/L_f is a pre–Hilbert space with the inner product defined by

$$<x + L_f, y + L_f> = f(y^*x) \quad \text{for all} \quad x, y \in A .$$

Denote by H_f the completion of A/L_f. A *–representation π_f of A is then defined by $\pi_f(x)(y + L_f) = xy + L_f, y + L_f \in A/L_f$. Further, by [4; proof of Theorem 6.1] there is $\alpha \in I$ such that

$$\|\pi_f(x)(y + L_f)\|^2 = f(y^*x^*xy) \leq f(y^*y) P_\alpha(x^*x) \leq p_\alpha(x)^2 \|y + L_f\|^2$$

for every $y + L_f \in A/L_f$. Hence π_f is a bounded linear operator on A/L_f and has a continuous extension to H_f which is the completion of A/L_f. Furthermore, $\xi = e + L_f$ is a cyclic vector of π_f such that $f(x) = <\pi_f(x)\xi, \xi>$ for every $x \in A$. This completes the proof.

Recall that a *–representation π of A is said to be irreducible if the only closed subspaces invariant under π are $\{0\}$ and H.

Theorem 4.5 Let A be a unital l.m.c. *–algebra. Then a continuous *–representation is irreducible iff the corresponding positive functional is an extreme point of $P(A)$.

Proof Let π be a continuous *–representation. Then by the theorem [14; p.265] the corresponding positive functional defined by $f(x) = <\pi(x)\xi, \xi>$ is pure for each cyclic vector ξ in H. Hence by Proposition 4.3, f is an extreme point of $P(A)$.

Conversely, assume that f is an extreme point of $P(A)$. The again by the same theorem [14; p.265] the corresponding *–representation π is irreducible.

A family of *–representations of an l.m.c. *–algebra A is said to be *complete* if for each non–zero vector x_0 in A, there exists a *–representation π in the family such that $\pi(x_0) \neq 0$.

Theorem 4.6 Let A be a unital l.m.c. *–algebra. Then the family of all continuous irreducible *–representations is complete iff the set

$$R^*(A) = \{x \in A : f(x^*x) = 0 \quad \text{for each} \quad f \in P(A)\} = \{0\} .$$

Proof Assume the family is complete. If $R^*(A) \neq \{0\}$, then for each $y \in R^*(A)$ such that $y \neq 0$, we have $f(y^*y) = 0$ for all $f \in P(A)$. Therefore by Theorem 4.4, $<\pi(y^*y)\xi, \xi> = 0$. This means that $\|\pi(y)\xi\|^2 = <\pi(y)\xi, \pi(y)\xi> = 0$, i.e., $\pi(y) = 0$, thus contradicting the hypothesis that the family is complete. Hence $R^*(A) = \{0\}$.

To prove the converse, we see that $z \neq 0$ implies that $f(z^*z) \neq 0$ for all $f \in P(A)$. In particular, for an extreme point f of $P(A)$ defined by $f(x) = <\pi(x)\xi, \xi>$, we have $0 \neq f(z^*z) = <\pi(z^*z)\xi, \xi>$. Hence $\|\pi(z)\| \neq 0$ for $z \neq 0$. This completes the proof.

As a consequence of Theorems 4.4, 4.5 and 4.6 we get

Corollary 4.7 Let A be a unital l.m.c. *–algebra. Then the following statements are equivalent:

(1) there is an $f \in P(A)$ such that $f(x) \neq 0$;

(2) there is a $g \in P(A)$ such that $g(x^*x) > 0$;

(3) there is a continuous *–representation π of A such that $\pi(x) \neq 0$;

(4) there is a continuous irreducible *–representation π of A such that $\pi(x) \neq 0$.

Definition 4.8 [11; Section 5]. Let I be a directed index set and $H_\lambda, \lambda \in I$, a family of Hilbert spaces such that

$$\forall \lambda \leq \mu, H_\lambda \subseteq H_\mu \quad \text{and} \quad <\cdot,\cdot>_\lambda = <\cdot,\cdot>_\mu |_{H_\lambda} ,$$

where $<\cdot,\cdot>$ denotes the inner product on H_λ, $\lambda \in I$. Now consider $H := \varinjlim_\lambda H_\lambda = \bigcup_\lambda H_\lambda$, equipped with the respective inductive limit topology, i.e., the finest locally convex topology on H with respect to which the natural injections $i_\lambda : H_\lambda \to H$, $\lambda \in I$, are continuous. H thus topologized is called a *locally Hilbert space*. Now define $L(H) := \{T : H \to H$ continuous and linear where $T = \varinjlim_\lambda T_\lambda$ with $T_\lambda \in B(H_\lambda)\}$ (in this regard, see also Theorem 4.9 below). Then, $L(H)$ is an algebra. Moreover, if $(T_\lambda), \lambda \in I$, is an inductive system of bounded linear operators on $H_\lambda, \lambda \in I$, the same is also true for the family $(T_\lambda^*), \lambda \in I$, of their adjoints, so that the map

$$* : L(H) \to L(H), \quad T \mapsto T^* := \varinjlim_\lambda T_\lambda^* ,$$

defines an involution on $L(H)$. If now, $\|\cdot\|_\lambda$ denotes the C^*-norm on $B(H_\lambda), \lambda \in I$, then the function $q_\lambda(T) := \|T_\lambda\|_\lambda, T \in L(H)$, defines a C^*-seminorm on $L(H)$, for every $\lambda \in I$. Thus, $L(H)$ endowed now with the topology of q_λ's becomes a locally C^*-algebra.

We now have

Theorem 4.9 [11; Theorem 5.1]. Every locally C^*-algebra is realized as a closed *-subalgebra of some $L(H)$ upto a topological algebraic *-isomorphism, where H is locally Hilbert space.

Proof Since A is a locally C^*-algebra, each A_α is a C^*-algebra. Therefore A_α has an isometric *-representation φ_α acting on a Hilbert space $H_\alpha, \alpha \in I$. Put $\mathcal{H}_\lambda := \bigoplus_{\alpha\leq\lambda} H_\alpha$ (orthogonal direct sum of H_α's, $\alpha \leq \lambda$). For given $x = (x_\alpha) \in A$ define

$$T_\lambda^x : \mathcal{H}_\lambda \to \mathcal{H}_\lambda \quad \text{by} \quad \xi_\lambda = (\xi_\alpha)_{\alpha\leq\lambda} \mapsto T^x(\xi_\lambda) :=$$
$$= (\varphi_\alpha(x_\alpha)(\xi_\alpha))_{\alpha\leq\lambda}, \quad \text{where} \quad \|\varphi_\alpha(x_\alpha)\| = \|x_\alpha\|_\alpha,$$

$x_\alpha \in A/N_\alpha = A_\alpha$ and $p_\alpha(x) \leq p_\lambda(x), x \in A$ for every $\alpha \leq \lambda$. Hence

$$\|\varphi_\alpha(x_\alpha)(\xi_\alpha)\| \leq \|\varphi_\alpha(x_\alpha)\|\,\|\xi_\alpha\| = p_\alpha(x)\|\xi_\alpha\| \leq p_\lambda(x)\|\xi_\alpha\|, \alpha \leq \lambda .$$

Thus,

$$\|T_\lambda^x(\xi_\lambda)\|^2 = \sum_{\alpha\leq\lambda} \|\varphi_\alpha(x_\alpha)(\xi_\alpha)\|^2 \leq p_\lambda(x)^2\|\xi_\lambda\|^2, \forall\, \xi_\lambda \in \mathcal{H}_\lambda$$

so that one finally gets $T_\lambda^x \in B(\mathcal{H}_\lambda)$. On the other hand, it is easily seen that the family $(T_\lambda^x), \lambda \in I$, forms an inductive system of bounded linear operators on $\mathcal{H}_\lambda$'s. Consider now the locally Hilbert space $H := \varinjlim_\lambda \mathcal{H}_\lambda$ and define $\varphi : A \to L(H)$ by $x \to \varphi(x) := \varinjlim_\lambda T_\lambda^x$. Then, φ is one-to-one since A is Hausdorff. Moreover,

$$q_\lambda(\varphi(x)) := \|T_\lambda^x\|_\lambda = \sup\{\|\varphi_\alpha(x_\alpha)\| : \alpha \leq \lambda\}$$
$$= \sup\{\|x_\alpha\|_\alpha : \alpha \leq \lambda\} = p_\lambda(x) ,$$

for any $\lambda \in I, x \in A$. This completes the proof.

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