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THE PROBLEM OF PRINCIPAL CHIRAL FIELD
WITH THE PARAMETERS DEPENDING
ON FREE ARGUMENTS
AND ITS INTEGRATION

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Abstract


A method to determine the solutions for principal chiral field (PCF) equation with the parameters depending on independent arguments for arbitrary semisimple algebra is worked out. Each solution depends on \( N(G) - r/2 \) arbitrary functions of independent arguments \( \psi_1 \) and \( \psi_2 \). Moreover, the number of derivatives of the arbitrary functions appearing in the solution distinguishes them, gathering them into series.

Annotation


Разработан метод решения уравнения главного кирального поля, параметры которого зависят от независимых аргументов, для произвольных полупростых алгебр. Каждое решение зависит от \( N(G) - r/2 \) (\( N(G) \) - размерность полупростой алгебры, \( r \) - ее ранг) произвольных функций независимого аргумента \( \psi_1 \) и \( \psi_2 \), причем количество производных от произвольных функций различают построенные решения, группируя их в серии.
INTRODUCTION

The system of equations of the POP with movable poles arises in various branches of mathematical physics \(^1\text{-}^3/\). The relationship between the system of four-dimensional self-dual equations and the POP problem with moving poles was established in earlier works of one of us \(^3/\). This circumstance allows us to introduce into the POP the topological characteristics (carrying them step-by-step from the theory of self-duality equations respectively) and raise the question of finding a solution of the POP problem with a preassigned value of the topological charge. In the present work we use the homogeneous Hilbert (Riemannian) problem (HHP) for the solution of this problem and develop a constructive method for finding the solutions of the POP problem depending on a sufficient large class of available functions, the definite selection of which brings us to the finite value of the topological charge of the system. This program has been accomplished in its final form only in the case of algebra \(A_1\). In this work we solve the problem of constructing the solutions of POP equations depending on \(N(G)-r\) (\(N(G)\) is the dimension of algebra, \(r\) is its rank) arbitrary functions of independent arguments \(\xi_1\) and \(\xi_2\) respectively, for an arbitrary semi simple algebra. Besides of the fixed number of arbitrary functions, the solution is defined by the order of the derivatives \(\Phi_{\alpha}^k(\xi_1), \Phi_{\alpha}^k(\xi_2)\) entering the solution too. Every solution possesses an equal number of arbitrary functions, one distinguishes them by the order of the derivatives entering into them.
The paper is organized as follows: in section I the PCP problem is connected with HHP, the coefficient function of which has a definite form. In section II technical formulas are defined. The method of solving HHP by reducing it to a system of linear algebraic equations is formulated in section III. In sections IV and V calculations associated with algebra \( A_2 \) are worked out. The general case for the arbitrary semi-simple algebra is shown in section VI. In section VII another general method of the integration of the PCP equation for arbitrary semi-simple algebra is presented, depending on the \( 2r \) arbitrary functions. Conclusive remarks are given in section VIII.

I. Consider the HHP to find \( \Phi^\pm(\lambda) \), where

\[
\Phi^+(\lambda) = \exp(iF_0(\lambda))\Phi^-(\lambda) \quad \text{on } L. \tag{1.1}
\]

The \( \Phi^\pm(\lambda) \) are evaluated in some semi simple group and are analytic functions inside and outside of the circle \( L \) on the complex plane, respectively. The circle \( L \) is chosen in such a way, that points \( \xi_1 \) and \( \xi_2 \) are inside the circle \( L \), the point \( \lambda = \infty \) is outside \( L \), the \( F_0(\lambda) \) belongs to the \( \Phi^-\) algebra of a corresponding group. The boundary condition for the HHP in the neighborhood of the point \( \lambda = \infty \) is given by the representation

\[
\Phi^-(\lambda) = 1 + \frac{F}{\lambda} + \ldots .
\]

The coefficient function of HHP has the form

\[
\exp(iF_0(\lambda)) = \exp\left(\frac{SH}{2}\ln \frac{\lambda-\xi_1}{\lambda-\xi_2}\right) \exp\left(-\frac{SH}{2}\ln \frac{\lambda-\xi_1}{\lambda-\xi_2}\right), \tag{1.2}
\]

where \( \xi_1 \) and \( \xi_2 \) are some parameters, \( s \) is an integer number. The group element \( A \) is independent of \( \xi_1, \xi_2 \) and analytic on \( L \) and it has no singularities, when its matrix elements are extended analytically to the inside of the circle \( L \). \( H \) is some operator of the algebra, diagonal in all the representations of the algebra and its eigenvalues are integer.

Now, we show that this HHP is equivalent to the problem of PCP with moving poles. It is obvious, that the group element \( \tilde{\Phi}^- = \exp(-\frac{SH}{2}\ln \rho)\Phi^- \), where \( \omega = \frac{\lambda-\xi_1}{\lambda-\xi_2} \), is analytical outside the circle \( L \) and has the expansion
\[ \tilde{\Phi} = 1 + \left[ \frac{\text{sh}(\xi_2 - \xi_1)}{2} - \frac{\text{ph}}{\lambda} \right] \lambda = 1 + \frac{\tilde{p}}{\lambda}. \quad (1.3) \]

in the neighborhood of the point \( \lambda = \infty \). The HHP is rewritten in the form

\[ \tilde{A} \tilde{\Phi}^- = \exp \left( - \frac{\text{sh} \ln \rho}{2} \right) \Phi^+. \quad (1.4) \]

Since the element \( A \) is independent of the parameters \( \xi_1 \) and \( \xi_2 \), then we have

\[ (\lambda - \xi_1)(\tilde{\Phi}^-)^{-1}(\tilde{\Phi}^-)_{\xi_1} = (\Phi^+)^{-1} \left[ - \frac{\text{sh}}{2} \Phi^+ + (\lambda - \xi_1) \Phi^+_{\xi_1} \right], \quad (1.5) \]

\[ (\lambda - \lambda_2)(\tilde{\Phi}^-)^{-1}(\tilde{\Phi}^-)_{\xi_2} = (\Phi^+)^{-1} \left[ \frac{\text{sh}}{2} \Phi^+ + (\lambda - \lambda_2) \Phi^+_{\xi_2} \right]. \quad (1.6) \]

These equations are obtained from equation (1.4) by differentiating with respect to \( \xi_1 \) and \( \xi_2 \), respectively. Hence, the r.h.s. of the above equations can be analytically continued to the whole \( \lambda \)-plane and it has a finite pole at \( \lambda = \infty \), i.e. they are polynomial. If one uses the expansion at the point \( \lambda = \infty \), we conclude, that these polynomials have a zero order, and are equal to \( \tilde{p}_{\xi_1} \) and \( \tilde{p}_{\xi_2} \), respectively. Carrying out similar calculations in the neighborhood of the point \( \lambda = 0 \) lying inside circle \( \lambda \) (for definiteness) we obtain

\[ \tilde{p}_{\xi_1} = \xi_1 (\Phi_o^+)^{-1} \left[ \frac{\text{sh} \Phi_o^+ - \Phi_o^+}{2 \xi_1} \right] = \xi_1 (\tilde{\Phi}_o^+)^{-1} \Phi_o^+ \xi_1, \quad (1.7) \]

\[ \tilde{p}_{\xi_1} = \xi_2 (\Phi_o^+)^{-1} \left[ \frac{\text{sh} \Phi_o^+ - \Phi_o^+}{2 \xi_2} \right] = \xi_2 (\tilde{\Phi}_o^+)^{-1} \Phi_o^+ \xi_2, \quad (1.8) \]

where

\[ \Phi_o^+ = \Phi^+(\lambda = 0), \quad \tilde{\Phi}_o^+ = \Phi_o^+ \exp \left( \frac{\text{sh}}{2} \ln \frac{\xi_1}{\xi_2} \right), \]
and thus we arrive at the equation
\[(\xi_2 - \xi_1) \tilde{F}^i_{\xi_1 \xi_2} = \left[ \tilde{F}^i_{\xi_1}, \tilde{F}^i_{\xi_2} \right],\] (1.9)

which is called the PCF equation with moving poles. It can be derived by crossing differentiation of equations (1.7) and (1.8).

The relationship between eq. (1.9) with four-dimensional self-duality equations consists in the following. The self-duality equations in the Yang form:\(^2\)
\[
\frac{(G_{-G^{-1}})}{y} + \frac{(G_{-G^{-1}})}{z} = 0 \quad (1.10)
\]

after introducing the element \(f\) lying in algebra,
\[
G_{-G^{-1}} = f_z, \quad G_{-G^{-1}} = -f_y \quad (1.11)
\]

transform into
\[
f_y + f_z + [f_y, f_z] = 0. \quad (1.12)
\]

If we seek for the solution of eq. (1.12) in the form
\[
f = \frac{1}{y} f(\xi_1, \xi_2), \quad (1.13)
\]

where \(\xi_1 - r+it, \xi_2 = -r+it, \quad r = \sqrt{x^2 + y^2 + z^2}\), then eq. (1.12) readily reduces to eq. (1.9). If we substitute \(f\) from eq. (1.13) into the solution for the instanton charge of four-dimensional self-duality equations and then pass to a two-dimensional integral, then we obtain
\[
Q = \int \int_{\xi_1 - \xi_2 > 0} d\xi_1 d\xi_2 \left\{ \frac{\partial^2}{\partial \xi_1 \partial \xi_2} \frac{(\xi_1 - \xi_2)^2}{2} + 1 \right\} \frac{1}{(\xi_1 - \xi_2)^2} \text{Sp} \left[ \frac{\partial F}{\partial \xi_1} - \frac{\partial F}{\partial \xi_2} \right]^2. \quad (1.14)
\]

The last relation means that the PCF equation possess proper topological characteristics and the value of the topological charge is defined by (1.14) if it is finite (without reference to the four-dimensional self-duality equations).

II. All the solutions are constructed with the help of the following elements:
1) the group element is a lower triangle matrix on the basis of the function \( \phi \) and its derivatives up to the \( k \)-th order according to the rule

\[
\Phi_k = \begin{bmatrix}
\phi, & 0, & \ldots, & 0 \\
0, & \phi, & \ldots, & 0 \\
\frac{\partial^{k-1} \phi^{(k-1)}}{(k-1)!}, & \ldots, & \phi, & 0 \\
\frac{\partial^k \phi^{(k)}}{k!}, & \ldots, & 0, & \phi
\end{bmatrix},
\]  

(2.1)

where \( \delta = \xi_2 - \xi_1 \). This yields the following properties

\[
\Phi_k^- + f_k^- = (\phi + f)_k^-, \\
\Phi_k^- \cdot f_k^- = (\phi f)_k^-,
\]

(2.2) \hspace{1cm} (2.3)

i.e. the elements \( \Phi_k^- \) behave as ordinary c-numbers w.r.t. the operation of addition and multiplication, in particular

\[
(\Phi_k^-)^{-1} = (\Phi^{-1})_k^-.
\]

(2.3')

Further, we use the matrix of binomial coefficients

\[
C_k(x) = \begin{bmatrix}
1, & -x, & x^2 & (-1)^k x^k \\
0, & 1, & -2x, & \ldots, & (-1)^{k+1} C_k^l x^{k-l} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0, & 0 & \ldots & \ldots & 1
\end{bmatrix},
\]

(2.4)

which satisfies the group property

\[
C_k(x)C_k(y) = C_k(x+y).
\]

(2.5)

Further construction is determined by \( C_k(\pm 1), \Phi_k^- \). We need a formula for finding the residues on the poles of an arbitrary order at the points \( \xi_1 \) and \( \xi_2 \). With the aim to obtain them, consider the expression

\[
\phi(\lambda) \begin{bmatrix} \lambda - \xi_2 \\ \lambda - \xi_1 \end{bmatrix}^s \sum_{\alpha = p_1}^{p_2} \frac{f_{\alpha}}{(\lambda - \xi_2)^{\alpha}}, \quad \alpha > 0, \quad p_2 \geq p_1 > 0, \quad s < 0,
\]

(2.6)

where the parameters \( s, \alpha, p_2, p_1 \) are integers.
Introduce the so-called single residue vector $\mathbf{F}$. Its first $s$ components are the residues of the $s$-th order on the point $\lambda = \xi_1$. Next, successive components are the residues of (2.6) on the point $\lambda = \xi_2$ from the highest degree to the lowest one.

There arise some cases:

1°. If $s > 0$, $s^p_2 > 2$, then there are no poles on the point $\lambda = \xi_2$ and the matrix connecting $\mathbf{F}$ with $\mathbf{f}$ has the form

$$ L_{\alpha_1} = \delta_{2-s} \left[ \Phi(-1)^{s-p_2} \delta^{s-p_2} \right]_{\alpha_1}, \quad 1 \leq \alpha \leq s, \quad p_2 < j < p_1. \quad (2.7) $$

2°. If $p_1 < s < p_2$, then

$$ L_{\alpha_1} = \begin{cases} \delta_{2-s} \left[ \Phi(-1)^{p_2-s} \delta^{s-p_2} \right]_{\alpha_1} C(1), & 1 \leq \alpha \leq s, \quad p_2 < j < p_1 \\ \delta^{s} (\Phi^{s})_{\alpha_1}, & \text{else} \end{cases} \quad (2.8) $$

3°. If $s < 0$, then there are no poles on the point $\lambda = \xi_1$, and

$$ L_{\alpha_1} = \delta^{-s} (\Phi^{s})_{\alpha_1}, \quad s+1 \leq \alpha \leq p_2 + s, \quad p_2 \leq j \leq p_1 \quad (2.9) $$

(\text{the lower and upper sign "$\cdot$" means that function $\Phi$ depends on $\xi_1$ and $\xi_2$, respectively}).

III. The group is thought to be semisimple. The operator $H$ is an element of its Cartan subalgebra, i.e. the $H$ is a linear combination of the basic elements of Cartan subalgebra with integer coefficients. For definiteness, let the operator $H$ coincide with the Cartan's of the principal embedding of 3d (three dimensional) subalgebra to the initial one and acquire a constant value 2 on the positive roots of the initial algebra,

$$ [H, X^+_\alpha] = 2X^+_\alpha, \quad (3.1) $$
where $H = \sum_{\alpha=1}^{r} \delta_{\alpha} h_{\alpha}$, $\delta_{\alpha} = 2 \sum_{\beta=1}^{r} (K^{-1})_{\alpha \beta}$, $k$ is the Cartan matrix of semi-simple algebra, $\{h_{\alpha}\}_{\alpha=1}^{r}$ is the basis of Cartan subalgebra.

Any finite-dimensional representation $(l_1, l_2, ..., l_r)$ of the semisimple algebra can be constructed on the basis of the highest vector $|M\rangle$, satisfying the conditions

$$X_{\alpha}^+ |M\rangle = 0, \quad H \cdot |M\rangle = \sum_{\alpha=1}^{r} \delta_{\alpha} h_{\alpha} |M\rangle = \sum_{\alpha=1}^{r} \delta_{\alpha} l_{\alpha} |M\rangle = 2J_i |M\rangle.$$  

(3.2)

The representation is constructed by multiple application to $|M\rangle$ of the generators of the simple negative roots and a subsequent diagonalization of the arising basis vectors. The action of $H$ onto the $n$-th basis vector is given by

$$H |M_n\rangle = H X_{\alpha_1}^{-} ... X_{\alpha_n}^{-} |M_n\rangle = \left(\sum_{\alpha=1}^{r} \delta_{\alpha} l_{\alpha} - 2n\right) |M_n\rangle = H(M_n) |M_n\rangle.$$  

(3.3)

Now, we consider the matrix elements of the group element $g$

$$g = \exp\left[\frac{\text{sh} \ln \left(\frac{\lambda^{-\xi_1}}{\lambda^{-\xi_2}}\right)}{2}\right] A(\lambda) \exp\left[\frac{-\text{sh} \ln \left(\frac{\lambda^{-\xi_1}}{\lambda^{-\xi_2}}\right)}{2}\right]$$

between states $M$ and $M'$

$$<M' \mid g \mid M> = <M \mid \exp\left[\text{sh} \ln \left(\frac{\lambda^{-\xi_1}}{\lambda^{-\xi_2}}\right)\right] A(\lambda) \exp\left[- \text{sh} \ln \left(\frac{\lambda^{-\xi_1}}{\lambda^{-\xi_2}}\right)\right] \mid M' > =$$

$$= \left(\frac{\lambda^{-\xi_2}}{\lambda^{-\xi_1}}\right)^{S/2 (H(M') - H(M))} A_{MM'}(\lambda).$$  

(3.4)

The matrix of the operator $g$ has the following structure w.r.t. the parameter $\mu = \left(\frac{\lambda^{-\xi_2}}{\lambda^{-\xi_1}}\right)^{S}$; the matrix elements do not have parameter $\mu$ on the diagonal, the elements above the diagonal have only a positive integer degree of the parameter $\mu$, the elements below the diagonal have a negative integer degree of the parameter $\mu$. 

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In the first row (from the left to the right) the first stands the element \( A_{\mathbf{MM}} \), next comes matrix \( A_{\mathbf{MM}'} \), their number amounts to the number of \( X^{-}\mathbf{M}> \) linear independent vectors. These vectors come out from a single application of the simple negative roots \( X^{-} \) to the highest vector \( \mathbf{M}> \). Next, there is a matrix element \( A_{\mathbf{MM}'} \), whose number is equal to the number of \( X^{-}X^{-}\mathbf{M}> \) vectors, which come out from a double-reiterated application of the simple negative roots \( X^{-} \) to the highest vector \( \mathbf{M}> \) and so on. The maximal degree of \( \mathbf{M} \) arises when \( \mathbf{M}' \) coincides with the maximal negative value of \( \mathbf{M} \), so that \( A_{\mathbf{MM}'} \). It is easy to notice that if \( \Phi^{-}\) is chosen in the form

\[
\Phi^{-} = \delta_{ij} + \sum_{\alpha=1}^{s/2(2m+H_1)} \frac{a_i^{\alpha}}{(\lambda-\xi_1)^{\alpha}} + \sum_{\beta=1}^{s/2(2m-H_1)} \frac{b_i^{\beta}}{(\lambda-\xi_2)^{\beta}}, \quad (3.5)
\]

then the multiplication of the \( \Phi^{-} \) by the element \( g \) on the left dose not change the structure of the poles \( \Phi^{-} \), i.e., new poles do not arise in the matrix elements \( (g\Phi^{-})_{ij} \) and their total number coincides. Here, \( H_1 \) are the eigenvalues of the operator \( H \), \( H_1 = \mathcal{H}(\mathbf{M}_1) \). We suppose, that the points \( \xi_1 \) and \( \xi_2 \) lying inside the integration contour and all singularities of matrix elements \( A_{\mathbf{MM}}(\lambda) \) are lying outside the contour (beforehand one accomplishes their analytic extension). Then the element \( g\Phi^{-} \) has singularities right on the points \( \xi_1 \) and \( \xi_2 \), and if the residues of all degrees vanish, then \( (g\Phi^{-}) \) is analytic inside the integration contour. This requirement yields a linear algebraic system of equations to determinate the coefficients \( a_i^{\alpha}, b_i^{\beta} \) on the poles of in arbitrary degree on the points \( \xi_1 \) and \( \xi_2 \); the number of the unknowns equals the number of the equations. The number of arbitrary functions will coincide with the group dimension \( N(G) \) minus its rank, i.e., the solution has \((N(G)-r)/2\) arbitrary functions of the independent argument \( \xi_1 \) and the same number of the functions of the argument \( \xi_2 \).
To illustrate our approach we shall start with a simple example of $A_2$ and we shall end with the examples of a general character.

IV. The HHP coefficient has the form

$$
\exp(iF_0(\lambda)) = \begin{pmatrix}
A_{11}(\lambda), & A_{12}(\lambda)\left(\frac{\lambda - \xi_1}{\lambda - \xi_2}\right)^s, & A_{13}(\lambda)\left(\frac{\lambda - \xi_2}{\lambda - \xi_1}\right)^s \\
A_{21}(\lambda)\left(\frac{\lambda - \xi_1}{\lambda - \xi_2}\right)^s, & A_{22}(\lambda), & A_{23}(\lambda)\left(\frac{\lambda - \xi_2}{\lambda - \xi_1}\right)^s \\
A_{31}(\lambda)\left(\frac{\lambda - \xi_1}{\lambda - \xi_2}\right)^s, & A_{32}(\lambda)\left(\frac{\lambda - \xi_2}{\lambda - \xi_1}\right)^s, & A_{33}(\lambda)
\end{pmatrix} (4.1)
$$

$$
\det \exp(iF_0(\lambda)) = 1.
$$

The group element $\Phi^-$ is chosen in agreement with formula (3.5) as

$$
\Phi^- = \begin{pmatrix}
1 + \sum_{\alpha=1}^{2s} \frac{s_{1\alpha}}{(\lambda - \xi_1)^\alpha}, & \sum_{\alpha=1}^{2s} \frac{s_{12}}{(\lambda - \xi_1)^\alpha}, & \sum_{\alpha=1}^{2s} \frac{s_{13}}{(\lambda - \xi_1)^\alpha} \\
\sum_{\alpha=1}^{2s} \frac{s_{21}}{(\lambda - \xi_2)^\alpha} + \sum_{\beta=1}^{2s} \frac{s_{21}}{(\lambda - \xi_2)^\beta}, & 1 + \sum_{\alpha=1}^{2s} \frac{s_{22}}{(\lambda - \xi_2)^\alpha} + \sum_{\beta=1}^{2s} \frac{s_{22}}{(\lambda - \xi_2)^\beta}, & \sum_{\alpha=1}^{2s} \frac{s_{23}}{(\lambda - \xi_2)^\alpha} + \sum_{\beta=1}^{2s} \frac{s_{23}}{(\lambda - \xi_2)^\beta}, \\
\sum_{\beta=1}^{2s} \frac{s_{31}}{(\lambda - \xi_2)^\beta}, & \sum_{\beta=1}^{2s} \frac{s_{32}}{(\lambda - \xi_2)^\beta}, & 1 + \sum_{\beta=1}^{2s} \frac{s_{33}}{(\lambda - \xi_2)^\beta}
\end{pmatrix} (4.2)
$$

This case corresponds of the first fundamental representation of algebra $A_2$, the Cartan matrix is $\mathbf{a} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, the eigenvalues of $H$ are $H_1 = 2, H_2 = 0, H_3 = -2$, the operator $H$ is $H = h_1 + h_2$, where $h_1, h_2$ - are the basis of the Cartan subalgebra, and the basis vectors are $|1\rangle, X_1^1, X_2X_1^1|1\rangle$.

The element $\Phi^-$ is analytic outside the contour $L$ and has asymptotic

$$
\Phi^- = \left(\begin{array}{ccc}
a^1_{11} & a^1_{12} & a^1_{13} \\
a^1_{21} + b^1_{21}, & a^1_{22} + b^1_{22}, & a^1_{23} + b^1_{23} \\
b^1_{31} & b^1_{32} & b^1_{33}
\end{array}\right). (4.3)
$$
The group element $\Phi^-$ has been calculated by formula (1.1), it has singularities on the points $\xi_1$, $\xi_2$. This singularities are absent if the residues of all the degrees amount to zero on the points $\xi_1$, $\xi_2$. Thus we obtain a linear system of algebraic equations to define the unknown parameters $a^\alpha_{ij}$ and $b^\beta_{ij}$. For instance, the element $\Phi^+_{11}$ occurs to be $\Phi^+_{11}$:

$$
\Phi^+_{11} = A_{11}(\lambda) \left[ 1 + \sum_{\alpha=1}^{2s} \frac{a^\alpha_{11}}{(\lambda-\xi_1)^\alpha} \right] + A_{12}(\lambda) \left[ \frac{\lambda-\xi_2}{\lambda-\xi_1} \right]^s \times \left[ \sum_{\alpha=1}^{s} \frac{a^\alpha_{21}}{(\lambda+\xi_1)^\alpha} + \sum_{\beta=1}^{2s} \frac{b^\beta_{21}}{(\lambda-\xi_2)^\beta} \right] + A_{13}(\lambda) \left[ \frac{\lambda-\xi_2}{\lambda-\xi_1} \right] \sum_{\beta=1}^{2s} \frac{b^\beta_{31}}{(\lambda-\xi_2)^\beta}.
$$

(4.4)

Formulas (2.7)-(2.9) take place for each term of the sum in formula (4.4) and we conclude that the condition of analyticity is the vanishing residues of the $2s$-th degree on the points $\xi_1$ and $\xi_2$.

$$
\delta^{2s} (A_{11} \delta^{-2s})^{-1}_{2s \times 2s} \begin{bmatrix}
a^s_{11} \\
\vdots \\
a^{s+1}_{11} \\
\vdots \\
a^s_{21} \\
\vdots \\
\vdots \\
\vdots \\
b^1_{21}
\end{bmatrix} + \delta^{2s} (A_{12} \delta^{-2s})^{-1}_{2s \times 2s} \begin{bmatrix}
a^s_{11} \\
\vdots \\
a^{s+1}_{11} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
b^1_{21}
\end{bmatrix} = \begin{bmatrix}
(-1)^s A_{12} \delta^{-s} \\
\vdots \\
0_{s \times s}
\end{bmatrix}.
$$

(4.5)

The subscript stands for the matrix dimension). Using properties (2.2)-(2.3').
\[
\begin{pmatrix}
\alpha^{2s} \\
\vdots \\
\alpha^{s+1} \\
\alpha^s \\
\vdots \\
\alpha \\
\end{pmatrix}
+ \begin{pmatrix}
\delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{2s \times s}^{-} \\
\vdots \\
\delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{2s \times s}^{-} \\
\delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{2s \times s}^{-} \\
\end{pmatrix} = 0.
\]

\[
(\Phi_{13})_{2s \times 2s}^{-} C_{2s}(1) = 0.
\]

(4.5')

where \( \Phi_{12} = \frac{A_{12}}{A_{11}} \), \( \Phi_{13} = \frac{A_{13}}{A_{11}} \).

From the elements \( \Phi_{21}^{+}, \Phi_{31}^{+} \), we obtain the remaining equations to determine the unknowns \( (a_{11}^{2s}, \ldots, a_{11}^{s+1}, a_{11}^{s}, \ldots, a_{11}^{1}) \), \( (a_{21}^{s}, \ldots, a_{21}^{1}, b_{31}^{s}, \ldots, b_{31}^{1}) \). The full system of equation is obtained by considering all matrix elements of the matrix \( \Phi^{+} \).

Let us introduce the matrices \( M, N, P \):

\[
M = \begin{pmatrix}
I_{2s} & \delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{2s \times s}^{-} \\
\delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{s \times s}^{-} & I_{2s}
\end{pmatrix},
\]

\[
N = \begin{pmatrix}
(\Phi_{13})_{2s \times 2s}^{-} C_{2s}(1) & 0 \\
0 & (\Phi_{13})_{2s \times 2s}^{-} C_{2s}(1)
\end{pmatrix},
\]

\[
P = \begin{pmatrix}
\delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{s \times s}^{-} & \delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{s \times s}^{-} \\
\delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{s \times s}^{-} & \delta^{-s}((-1)^{s}\Phi_{12}\delta^{s})_{s \times s}^{-}
\end{pmatrix}.
\]
The symbol \((\cdots)^{-}\)\(_{s\times1}\) stands for the first column of the matrix \((\cdots)^{-}\), where \(S\) denotes the raw number. Here, we again introduce the notation

\[
A_{21} = \Phi_{21}, \quad A_{23} = \Phi_{23}, \quad \Phi_{31} = \frac{A_{31}}{A_{33}}, \quad \Phi_{32} = \frac{A_{32}}{A_{33}}
\]

more over \(\Phi_{21}(\xi_{2}) = \Phi_{21}(\xi_{1}=\xi_{2}), \quad \Phi_{23}(\xi_{2}) = \Phi_{23}(\xi_{1}=\xi_{2})\).
The system of defining equations is rewritten in an elegant form
\[ M \cdot N = -P \]  \hspace{1cm} (4.6)

Statement. The system of equations for the 18S unknowns reduced to the system for the 12S unknowns.

Proof. Taking the first S equations containing the unknowns \( (a_1^{2s}, \ldots, a_1^{s+1}) \) and the S equations containing \( (a_2^{2s}, \ldots, a_2^{s+1}) \) and similarly as above the system of equations for \( (b_1^{2s}, \ldots, b_1^{s+1}) \) and \( (b_2^{s}, \ldots, b_2^{1}) \). We obtain two systems of equations

\[
\begin{align*}
\bar{a}_{11} + \delta^{-s} (\Phi_{12} \delta^s)^- \cdot \bar{a}_{21} + (\Phi_{12})^- \cdot \bar{b}_{31} &= 0 \\
\delta^s (\Phi_{21} \delta^{-s})^- \cdot \bar{a}_{11} + a_{21} + \delta^s (\Phi_{23} \delta^{-s})^- \cdot \bar{b}_{31} &= 0 \\
\delta^{-s} (\Phi_{21} \delta^s)^- \cdot \bar{a}_{11} + b_{21} + \delta^{-s} (\Phi_{23} \delta^s)^- \cdot \bar{b}_{31} &= \delta^{-s} (\Phi_{21} \delta^s)^- \\
(\Phi_{31}) ^{-} \cdot \bar{a}_{11} + \delta^2 (\Phi_{32} \delta^{-s})^- \cdot \bar{b}_{31} + \bar{b}_{31} &= 0
\end{align*}
\]  \hspace{1cm} (4.7)

where \( \bar{a}_{11} = (a_1^{2s}, \ldots, a_1^{s+1}), \bar{a}_{21} = (a_2^{2s}, \ldots, a_2^{s+1}), \bar{b}_{31} = (b_3^{2s}, \ldots, b_3^{s+1}), \bar{b}_{21} = (b_2^{s}, \ldots, b_2^{1}) \)

\[
\tilde{b}_{31} = C(1) \begin{bmatrix} b_{31}^{2s} \\ \vdots \\ b_{31}^{s+1} \end{bmatrix}, \quad \tilde{a}_{11} = C(1) \begin{bmatrix} a_{11}^{2s} \\ \vdots \\ a_{11}^{s+1} \end{bmatrix}
\]

The relationship between \( \tilde{a}_{11} \) and \( \tilde{a}_{21}, \tilde{b}_{31} \) and \( \tilde{b}_{21} \) is easily found by eliminating \( \tilde{b}_{31} \) and \( \tilde{a}_{11} \) from (4.7), (4.8) respectively; they are given explicitly by

\[
\begin{align*}
\bar{a}_{11} &= \delta^{-s} \left[ \begin{bmatrix} \Phi_{13} - \Phi_{23} \Phi_{12} \\ \Phi_{23} - \Phi_{13} \Phi_{21} \end{bmatrix} \cdot \delta^s \right] \cdot \bar{a}_{21}, \quad \hspace{1cm} (4.9) \\
\bar{b}_{31} &= \delta^s \left[ \begin{bmatrix} \Phi_{21} \Phi_{32} - \Phi_{31} \\ \Phi_{23} \Phi_{31} - \Phi_{21} \end{bmatrix} \cdot \delta^{-s} \right] \cdot \bar{b}_{21}.
\end{align*}
\]  \hspace{1cm} (4.10)
I.e., we have shown that the system of linear algebraic equations with respect to 6S unknowns is reduced to the system of linear algebraic equation with respect to the 4S unknowns. Then, it is apparent that the full system for the 18S unknowns is reduced for 12S of the unknowns.

To accomplish the reduction to the algebra $A_t$, it is necessary to assume, that the group element $A$ with the matrix elements $A_{ik}$ belongs to the representation $l = 1$ of the group $SL(2, R)$ i.e., the matrix of dimension $3 \times 3$ has the form

$$A = \exp(\alpha X^+).\exp(\tau H).\exp(\beta X^-),$$

(4.11)

where

$$X^+ = \begin{bmatrix} 0, & \sqrt{2}, & 0 \\ 0, & 0, & \sqrt{2} \\ 0, & 0, & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 2, & 0, & 0 \\ 0, & 0, & 0 \\ 0, & 0, & -2 \end{bmatrix}, \quad X^- = \begin{bmatrix} 0, & 0, & 0 \\ \sqrt{2}, & 0, & 0 \\ 0, & \sqrt{2}, & 0 \end{bmatrix}.$$

Taking into account this fact and denoting $\Phi_{12} = \Phi, \quad \Phi_{32} = \Phi$, we obtain

$$\Phi_{12} = \frac{\Phi^2}{2}, \quad C = -\frac{\Phi^2}{2}, \quad \Phi_{31} = \frac{\Phi^2}{2}, \quad d = -\frac{\Phi}{2},$$

where

$$C = \frac{\Phi_{13} - \Phi_{23}\Phi_{12}}{\Phi_{23} - \Phi_{13}\Phi_{21}}, \quad d = \frac{\Phi_{21}\Phi_{32} - \Phi_{31}}{\Phi_{23}\Phi_{31} - \Phi_{21}}$$

therefore, the obtained solution is defined by the functions $\Phi$ and $\Phi$, and coincides with the solution previously obtained for the algebra $A_t$ ([3]).

V. Let us consider now the case $H = 2h_1 + 3h_2$, and for simplicity $S$ is equal to the unity, $S = 1$. The coefficient function of HHF is given by

$$\exp(iF_0(\lambda)) = \begin{bmatrix}
A_{11}(\lambda), & A_{12}(\lambda)\left(\frac{\lambda - \xi_2}{\lambda - \xi_1}\right), & A_{13}(\lambda)\left(\frac{\lambda - \xi_2}{\lambda - \xi_1}\right)^5 \\
A_{21}(\lambda)\left(\frac{\lambda - \xi_1}{\lambda - \xi_2}\right), & A_{22}(\lambda), & A_{23}(\lambda)\left(\frac{\lambda - \xi_2}{\lambda - \xi_1}\right)^4 \\
A_{31}(\lambda)\left(\frac{\lambda - \xi_1}{\lambda - \xi_2}\right)^5, & A_{32}(\lambda)\left(\frac{\lambda - \xi_1}{\lambda - \xi_2}\right)^4, & A_{33}(\lambda)
\end{bmatrix}$$

(5.1)

$$\det \exp(iF_0(\lambda)) = 1.$$
The element $\Phi^-$ is chosen in the form of

$$
\Phi = \begin{bmatrix}
1 + \sum_{a=1}^{4} \frac{a^{\alpha}}{(\lambda - \xi_1)^{\alpha}} + \frac{b_1}{\lambda - \xi_2} , & 1 + \sum_{a=1}^{4} \frac{a^{\beta}}{(\lambda - \xi_2)^{\beta}} , & 1 + \sum_{a=1}^{4} \frac{a^{\gamma}}{(\lambda - \xi_2)^{\gamma}} \\
\sum_{a=1}^{4} \frac{a^{\alpha}}{(\lambda - \xi_1)^{\alpha}} + \frac{b_2}{\lambda - \xi_2} , & 1 + \sum_{a=1}^{4} \frac{a^{\beta}}{(\lambda - \xi_2)^{\beta}} , & 1 + \sum_{a=1}^{4} \frac{a^{\gamma}}{(\lambda - \xi_2)^{\gamma}} \\
\sum_{a=1}^{4} \frac{a^{\alpha}}{(\lambda - \xi_1)^{\alpha}} + \frac{b_3}{\lambda - \xi_2} , & 1 + \sum_{a=1}^{4} \frac{a^{\beta}}{(\lambda - \xi_2)^{\beta}} , & 1 + \sum_{a=1}^{4} \frac{a^{\gamma}}{(\lambda - \xi_2)^{\gamma}}
\end{bmatrix}
$$

The $M$, $N$, $P$ have the following structure

$$
M = \begin{bmatrix}
I_5 & 0_{5 \times 5} & 0_{5 \times 5} \\
\Phi_{21}^{-1} \Phi_{21}, & \Phi_{21}^{-1} \Phi_{21}, & \Phi_{21}^{-1} \Phi_{21}, & \Phi_{21}^{-1} \Phi_{21}, & \Phi_{21}^{-1} \Phi_{21}, & \Phi_{21}^{-1} \Phi_{21}, & \Phi_{21}^{-1} \Phi_{21}, & \Phi_{21}^{-1} \Phi_{21}, & \Phi_{21}^{-1} \Phi_{21}
\end{bmatrix}
$$

$$
N = \begin{bmatrix}
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5} \\
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5} \\
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5} \\
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5} \\
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5} \\
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5} \\
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5} \\
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5} \\
0_{5 \times 5} & 0_{5 \times 5} & 0_{5 \times 5}
\end{bmatrix}
$$

$$
P = \begin{bmatrix}
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1} \\
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1} \\
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1} \\
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1} \\
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1} \\
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1} \\
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1} \\
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1} \\
0_{5 \times 1} & 0_{5 \times 1} & 0_{5 \times 1}
\end{bmatrix}
$$

The unknown parameters are defined as in the previous section.
VI. The general case: the matrix elements $\Phi^+_{ij}$ are

$$
\Phi^+_{ij} = \left\{ \begin{array}{l}
\frac{1}{\lambda - \xi_2} \left( \frac{\xi_2}{\lambda - \xi_1} \right) \\
\delta_{k,j} + \sum_{\alpha=1}^{2J+1} \frac{A^\alpha_{kJ}}{(\lambda - \xi_2)^\alpha} + \sum_{\beta} \frac{B^\beta_{kJ}}{(\lambda - \xi_2)^\beta}
\end{array} \right.
$$

(6.1)

The structure of the matrices $M$, $N$ and $P$ is as follows. Three possible cases can be defined for the element $M_{ji}$:

1) If $i < j$ (i.e., the elements below the diagonal), then $H_j - H_i < 0$. The element $M_{ji}$ is the matrix of the $(2J_e, 2J_s)$ dimension and has a block structure:

$$
\begin{array}{c|c|c}
\frac{1}{2}(2J_e + H_j) & \frac{1}{2}(2J_e - H_j) \\
\hline
\frac{1}{2}(H_j - H_i) & \delta_{ij} \left( -1 \right) & \frac{1}{2}(H_j - H_i) \left( \frac{1}{2} \right) \\
\frac{1}{2}(2J_e + H_j) & \delta & 0 \\
\frac{1}{2}(2J_e - H_i) & 0 & \frac{1}{2}(2J_e - H_i) \\
\frac{1}{2}(H_j - H_i) & \delta & \frac{1}{2}(H_j - H_i) \left( \frac{1}{2} \right)
\end{array}
$$

(6.2)

The dimension of the blocks has been indexed in line (here we used formulas (26)-(29) again).

2) If $i < j$, then $H_j - H_i > 0$ (i.e. the elements above the diagonal. The matrix element $M_{ji}$ has the form...
3) If \( l = j \), \( H_l = H_j \). Thus, \( M_{ij} \) is diagonal

\[
\begin{array}{cc}
\frac{s}{2}(2J_{s}+H_{j}) & \frac{s}{2}(2J_{s}-H_{j}) \\
\left(\frac{s}{2}(H_{l}+H_{j})\right)^{-} & \left(\frac{s}{2}(H_{l}-H_{j})\right)^{-} \\
\end{array}
\]

\[
\begin{array}{cc}
\frac{s}{2}(2J_{s}+H_{j}) & \frac{s}{2}(2J_{s}-H_{j}) \\
\left(\frac{s}{2}(H_{l}+H_{j})\right)^{-} & \left(\frac{s}{2}(H_{l}-H_{j})\right)^{-} \\
\end{array}
\]

It becomes clear that, the dimension of the matrix \( M \) is \((2J_{s}L)\), where \( L \) is the dimension of the corresponding representation.

The unknown parameters are \( a_{ij}^\alpha \) and \( b_{ij}^\beta \) (1 and \( j \) fixed), where \( 1 \leq \alpha \leq \frac{s}{2}(2J_{s}+H_{l}) \), \( 1 \leq \beta \leq \frac{s}{2}(2J_{s}-H_{l}) \). It is useful to introduce the unit index \( \gamma \) running from 1 up to \( 2J_{s} \). Then the unknown parameters are gathered into one matrix \( N \), with matrix elements \( N_{ij} \), which is the \( 2J_{s} \) dimensional column vector.
Let us consider the contribution of the unit matrix appearing in formula (6.1), which we shall denote as the matrix \( \mathbf{P} \). The elements \( P_{ji} \) (\( j \) and \( i \) fixed) has the structure of the \( 2J_eS \) dimension column vector, but its structure depends on the indices \( i \) and \( j \).

1°. If \( j > 1 \)

\[
N_{ij} = \begin{cases} 
\frac{\alpha - \frac{S}{2}(2J_e - H_1)}{a_{ij}}, & 2J_eS \leq \alpha \leq 1 + \frac{S}{2}(2J_e - H_1) \\
\frac{S}{2}(2J_e - H_1) \leq \alpha \leq 1 
\end{cases} 
\]

(6.5)

\[
P_{ji} = \begin{bmatrix}
\text{Zero} \\
\frac{S}{2}(H_j - H_1) \times \\
\times \left[ \left( A_{ji} \frac{S}{2}(H_j - H_1) \right)^{-1} \right] \\
\end{bmatrix} 
\]

(6.6)

2°. If \( j < 1 \)

\[
P_{ji} = \begin{bmatrix}
\frac{S}{2}(H_j - H_1) \left[ (-1)^{\frac{S}{2}(H_j - H_1)} A_{ji} \frac{S}{2}(H_j - H_1) \right]^{-1} \\
\frac{S}{2}(H_j - H_1) \\
\text{Zero} \\
\end{bmatrix} 
\]

(6.7)

Here we are dealing with the first column of the matrix \((\ldots)^{-1}\), defined earlier. The number of rows are indicates the line. We must divide the matrices \( M, N \) and \( P \) into the diagonal elements \( A_{jj} \) so as to arrive at the same expressions of the previous section (after denoting

\[
\frac{A_{ji}}{A_{jj}} = \Phi_{ji}, \quad \frac{\bar{A}_{ji}}{A_{jj}} = \Phi_{ji} \]

).
Then, we finally arrive at an elegant equation
\[ MN = -p \quad (6.8) \]
for the definition of the unknown parameters present in the asymptotic expansion of the element \( \Phi^- \) which, in their turn give rise to the solution of POF problem with moving poles.

VII. The obtained solution is not general enough in the sense, that, the most certain solution of equation (1.9) depends on \( 2N(G) \) arbitrary functions. Now we exhibit a solution depending on \( 2r \) arbitrary functions of independent arguments. This solution was obtained proceeding from the four dimensional self duality equation\(^2/3\). In the present work we show how to immediately obtain this solution.

Let us consider the group element \( g \), which satisfies the system of equations
\[
\begin{align*}
g_{t_1}g^{-1} &= \tilde{w}^0 + w^+ , \\
g_{t_2}g^{-1} &= \tilde{w}^0 + w^- .
\end{align*}
\quad (7.1)\quad (7.2)
\]
It is supposed, that the algebra has a gradation and graduation operator \( H \) is an element of algebra. The elements \( \tilde{w}^0, \tilde{w}^0, w^\pm \) belong to the subspaces \( G_0, G_\pm \) respectively, i.e. \([H, \tilde{w}^0] = [H, \tilde{w}^0] = 0\), \([H, w^\pm] = 2w^\pm \).

The equations for the \( \tilde{w}^0, \tilde{w}^0, w^\pm \) are obtained with the Maurer–Cartan identity and the graduation property of the algebra
\[
\begin{align*}
\tilde{w}_1^0 - \tilde{w}_2^0 &= [\tilde{w}^0, \tilde{w}^0] + [w^-_1, w^+_1] , \\
w^+_2 &= [\tilde{w}^0, w^+_1] , \\
w^-_1 &= [\tilde{w}^0, w^-_1] .
\end{align*}
\quad (7.3)\quad (7.4)\quad (7.5)
\]
Let us consider the element of the algebra \( f = \frac{\xi_1 - \xi_2}{2} g^{-1} H g \). Differentiating \( f \) with respect to \( \xi_1 \) and \( \xi_2 \)

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\[ f_{\xi_1} = g^{-1}Hg + (\xi_1 - \xi_2)\left(-g^{-1}g_{\xi_1}g^{-1}H + g^{-1}Hg_{\xi_1}\right) = \]
\[ = g^{-1}Hg + (\xi_1 - \xi_2)\left(-g^{-1}(W^0 + W_1^+)Hg + g^{-1}H(W^0 + W_1^+)g\right) = \]
\[ = g^{-1}Hg + 2(\xi_1 - \xi_2)g^{-1}W_1^+g; \]
\[ f_{\xi_2} = -g^{-1}Hg - 2(\xi_1 - \xi_2)g^{-1}W_1^-g \]
and evaluating the commutator \( f_{\xi_1}, f_{\xi_2} \)
\[ [f_{\xi_1}, f_{\xi_2}] = [g^{-1}Hg + 2(\xi_1 - \xi_2)g^{-1}W_1^+g, -g^{-1}Hg - 2(\xi_1 - \xi_2)g^{-1}W_1^-g] = \]
\[ = 4g^{-1}(\xi_1 - \xi_2)(W_1^- + W_1^+ - (\xi_1 - \xi_2)[W_1^+, W_1^-])g \]
and then constructing \( f_{\xi_1, \xi_2} : f_{\xi_1, \xi_2} = -4g^{-1}(W_1^- + W_1^+ - (\xi_1 - \xi_2)[W_1^+, W_1^-])g \)
i.e. \( (\xi_2 - \xi_1) = f_{\xi_1, \xi_2} = [f_{\xi_1}, f_{\xi_2}] \). (7.6)

We come to conclusion, that \( f \) is a solution of the PCF equation with moving poles. Since the solution of (7.1), (7.2) depends on \((N(G)+2)\) arbitrary functions of the argument \( \xi_1 \) and \((N(G)-2)\) arbitrary functions of the argument \( \xi_2 \) ([6]), where \((N(G) \pm 2)\) are the dimensions of the subspaces of the graded algebra with the indices \( \pm 2 \), respectively. For the semisimple algebra and its principal imbedding we have \( N-2 = N + 2 = r \).

VIII. The main fundamental result is contained in the formulas of section VI, VII, where the explicit form of the solution for the PCF equation with moving poles has been obtained. These solutions are not general enough. Since the number of arbitrary functions is less than it is needed for the construction of the most general solution.

The limiting transition to the soliton solution of the PCF with constant parameters is unknown for us at the present moment. This limit is not trivial and its realization may be the key to clarify reason of the algebraicity of the solutions stated is the present work.

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Reference


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