ITERATIVE SOLUTION OF NONLINEAR EQUATIONS
WITH STRONGLY ACCRETIVE OPERATORS

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Let $E$ be a real Banach space with a uniformly convex dual, and let $K$ be a nonempty closed convex and bounded subset of $E$. Suppose $T : K \rightarrow K$ is a strongly accretive map such that for each $x, y$ in $K$ the equation $Tx = f$ has a solution in $K$. It is proved that each of the two well known fixed point iteration methods (the Mann and Ishikawa iteration methods) converges strongly to a solution of the equation $Tx = f$. Furthermore, our method shows that such a solution is necessarily unique. Explicit error estimates are given. Our results resolve in the affirmative two open problems (J. Math. Anal. Appl. Vol 151 (2) (1990), p. 460) and generalize important known results.

**ABSTRACT**

Let $E$ be a real Banach space with a uniformly convex dual, and let $K$ be a nonempty closed convex and bounded subset of $E$. Suppose $T : K \rightarrow K$ is a strongly accretive map such that for each $x, y$ in $K$ the equation $Tx = f$ has a solution in $K$. It is proved that each of the two well known fixed point iteration methods (the Mann and Ishikawa iteration methods) converges strongly to a solution of the equation $Tx = f$. Furthermore, our method shows that such a solution is necessarily unique. Explicit error estimates are given. Our results resolve in the affirmative two open problems (J. Math. Anal. Appl. Vol 151 (2) (1990), p. 460) and generalize important known results.

**1. INTRODUCTION**

Let $E$ be a real normed linear space. A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called accretive [2] if for each $x, y$ in $D(T)$ and all $t \geq 0$, the following inequality is satisfied:

$$\|x - y\| \leq \|x - y + t(Tx - Tp)\|$$

(1)

If $E$ is a Hilbert space, the accretive condition (1) reduces to

$$\text{Re}(Tx - Ty, x - y) \geq 0,
$$

for all $x, y \in E$. The accretive operators were introduced independently in 1967 by F.E. Browder [2] and T. Kato [19]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem:

$$\frac{du}{dt} + Tu = 0, \quad u(0) = u_0$$

is solvable if $T$ is locally Lipschitzian and accretive on $E$. Browder also proved that if $T : E \rightarrow E$ is locally Lipschitzian and accretive then $T$ is $m$-accretive, i.e., the map $(I + T)$, where $I$ denotes the identity map of $E$, is surjective. This result was subsequently generalized by R.H. Martin [22] to continuous accretive operators.

For a Banach space $X$ we shall denote by $J$ the normalized duality map from $X$ to $2^X$ given by

$$Jx = \{ f^* \in X^* : \| f^* \|^2 = \| x \|^2 = (x, f^*) \},$$

where $X^*$ denotes the dual space of $X$ and $(, )$ denotes the generalized duality pairing. It is well known that if $X^*$ is strictly convex then $J$ is single-valued and if $X^*$ is uniformly convex, then $J$ is uniformly continuous on bounded sets (see e.g. [31]). In the sequel we shall denote the single-valued normalized duality map by $j$.

Let $K$ be a nonempty subset of $E$. A mapping $U : K \rightarrow K$ is called strongly accretive if for each $x, y$ in $K$ there exists $\omega \in J(x - y)$ such that

$$\langle Ux - Uy, \omega \rangle \geq k\|x - y\|^2,$$

for some real constant $k > 0$. Without loss of generality we may assume $k \in (0, 1)$. Strongly accretive operators have been studied by various authors (see e.g., [1], [3], [16], [23], [25],[26]). In [23], the following surjectivity result is proved:

**Theorem M.** (Morales, [23]). Let $E$ be a Banach space and $T : E \rightarrow E$ be continuous and strongly accretive. Then $T$ is surjective.

An obvious consequence of Theorem M is that for each given $f \in E$, the equation $Tx = f$ has a solution in $E$. Two well known fixed point iteration methods (the Mann and Ishikawa methods) are...
iteration methods (defined below) have recently been successfully employed to approximate a solution of this equation in \(L_p\) spaces, \(p \geq 2\) (see e.g. [9], [10], [12]).

In [9], the author proved that if \(X = L_p, p \geq 2\), the **Mann iteration process** converges strongly to a solution of \(Tz = f\) when \(T\) is Lipschitzian and strongly accretive. The method of [9] also shows that such a solution is necessarily unique. More recently, following a question posed in [9], it has been proved (see e.g., [10], [12]) that the **Ishikawa iteration process** also converges to the solution of \(Tz = f\) in \(L_p\) spaces, \(p > 2\), again where \(T\) is Lipschitzian and strongly accretive.

It is our purpose in this paper to prove that if \(E\) is any real Banach space with a uniformly convex dual space \(E^*\), \(K\) is a nonempty closed convex and bounded subset of \(E\) and \(T : K \rightarrow K\) is a strongly accretive map with a nonempty fixed point set then both the Mann and the Ishikawa iteration schemes can be used to approximate the unique solution of the equation \(Tx = f\) in \(L_p\) spaces, \(p > 2\), again where \(T\) is Lipschitzian and strongly accretive.

In [29, p. 89], Reich proved that if \(E^*\) is uniformly convex then there exists a continuous nondecresing function \(\delta : [0, \infty) \rightarrow [0, \infty)\) such that

\[
\delta(t) < t \quad \text{for all } t > 1
\]

where \(\delta(0) = 0\) and \(\delta(\infty) = \infty\).

Remark 1. In [29, p. 89], Reich proved that if \(E^*\) is uniformly convex then there exists a continuous nondecreasing function

\[
b : [0, \infty) \rightarrow [0, \infty)
\]

such that

\[
b(0) = 0, \quad b(ct) \leq cb(t) \quad \text{for all } c \geq 1
\]

and

\[
\|x + y\|^2 \leq \|x\|^2 + 2\langle x, f(y) \rangle + \max\{\|x\|, 1\}\|y\|\|b(\|y\|)\| \tag{5}
\]

for all \(x, y \in E\).

Remark 2. Nevanlinna and Reich [25] have shown that for any given continuous nondecreasing function \(b(t)\) with \(b(0) = 0\), sequences \(\{\lambda_n\}_{n=0}^\infty\) always exist satisfying:

(i) \(0 < \lambda_n < 1\) for all \(n \geq 0\);

(ii) \(\sum_{n=0}^\infty \lambda_n = \infty\); and

(iii) \(\sum_{n=0}^\infty \lambda_n b(\lambda_n) < \infty\).

If \(E = L_p, 1 < p < \infty\) we can choose any sequence \(\{\lambda_n\}_{n=0}^\infty\) in \(\ell^1\) with \(s = p\) if \(1 < p \leq 2\) and \(s = 2\), if \(p > 2\).

For the remainder of this paper, the Lipschitz constant of \(T\) will be denoted by \(L \geq 1\) and the constant appearing in the definition of a strongly accretive map will be denoted by \(k \in (0, 1)\).

### 3. MAIN RESULTS

We prove the following theorems:

**Theorem 1.** Let \(E\) be a real Banach space with a uniformly convex dual space, \(E^*\), and let \(K\) be a nonempty closed convex and bounded subset of \(E\). Suppose \(T : K \rightarrow K\) is a continuous
strongly accretive map. For a given \( f \in K \), define \( S : K \to K \) by \( Sx = f - Tx + x \) for each \( x \in K \). Define the sequence \( \{x_n\}_{n=0}^{\infty} \) iteratively by \( x_0 \in K \),

\[
x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Sx_n,
\]

for \( n \geq 0 \), where \( \{\lambda_n\}_{n=0}^{\infty} \) is a real sequence satisfying:

(i) \( 0 < \lambda_n \leq 1 \) for all \( n \geq 0 \)

(ii) \( \sum_{n=0}^{\infty} \lambda_n = \infty \), and

(iii) \( \sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty \).

Then the sequence \( \{x_n\}_{n=0}^{\infty} \) converges strongly to the solution of \( Tx = f \).

Proof. The existence of a solution to \( Tx = f \) follows from Morales \[23\]. Let \( x^* \) denote a solution.

Clearly, \( x^* \) is a fixed point of \( S \) and for arbitrary \( x, y \in K \) we have:

\[
(Sx - Sy, f(x - y)) \leq (1 - k)||x - y||^2
\]

Using (5), (6) and (7) we obtain:

\[
||x_{n+1} - x^*||^2 \leq (1 - \lambda_n)(x_n - x^*) + \lambda_n b \sum_{n=0}^{\infty} \lambda_n b(\lambda_n)(x_n - x^*)^2
\]

Since \( K \) is bounded. Hence we have,

\[
||x_{n+1} - x^*||^2 \leq [(1 - \lambda_n)(x_n - x^*) + \lambda_n b \sum_{n=0}^{\infty} \lambda_n b(\lambda_n)]
\]

which implies

\[
||x_{n+1} - x^*||^2 \leq [(1 - \lambda_n)(1 - k\lambda_n) + \lambda_n b \sum_{n=0}^{\infty} \lambda_n b(\lambda_n)]||x_n - x^*||^2 + M\lambda_n b(\lambda_n)
\]

The rest of the argument now follows as in \[8\], p. 39 to give that \( x_n \to x^* \) as \( n \to \infty \). For completeness, however, and since this argument will also be needed in our next theorem, we give the details here.

Set \( \rho_n = ||x_n - x^*||^2 + M\lambda_n b(\lambda_n) = \sigma_n \). Then inequality (8) becomes:

\[
\rho_{n+1} = (1 - \lambda_n)\rho_n + \sigma_n
\]

A simple induction on inequality (9) yields:

\[
0 \leq \rho_{n+1} \leq \prod_{j=1}^{n}(1 - \lambda_j)\rho_1 + \beta_{n+1},
\]

where

\[
\beta_{n+1} = \{\sigma_1, \ldots, \sigma_n + \sum_{i=1}^{n} \sigma_i \prod_{j=i+1}^{n}(1 - \lambda_j), n \geq 1
\]

For any fixed integer \( k \) with \( 1 < k \leq n - 1 \), we obtain

\[
\beta_{n+1} = \sigma_n + \sum_{i=1}^{k} \sigma_i \prod_{j=i+1}^{n}(1 - \lambda_j) + \sum_{i=k+1}^{n} \sigma_i \prod_{j=i+1}^{n}(1 - \lambda_j)
\]

Since \( \delta_n \in [0, 1] \), the above inequality yields,

\[
0 \leq \beta_{n+1} \leq \prod_{j=1}^{n}(1 - \lambda_j) + \sum_{i=1}^{n} \sigma_i
\]

Condition (ii) now implies

\[
\lim_{n \to \infty} \prod_{j=1}^{n}(1 - \lambda_j) = 0, \quad k \geq 1,
\]

so that

\[
0 \leq \lim inf_{n \to \infty} \beta_n \leq \lim sup_{n \to \infty} \beta_n \leq \sum_{i=1}^{\infty} \sigma_i
\]

Since inequality (11) holds for arbitrary \( k \geq 1 \), and since condition (iii) implies \( \lim_{n \to \infty} \sum_{i=1}^{\infty} \sigma_i = 0 \), it follows that

\[
\lim inf_{n \to \infty} \beta_n = \lim sup_{n \to \infty} \beta_n = \lim_{n \to \infty} \beta_n = 0
\]

From inequality (10) and equation (12), we obtain \( \rho_n \to 0 \) as \( n \to \infty \) so that \( x_n \to x^* \) as \( n \to \infty \). Uniqueness follows as in \[9\], completing proof of Theorem 1.

If \( E = L_p, \quad 1 < p < \infty \), Theorem 1 can be stated more simply as follows:

Corollary 1. Let \( E = L_p \) (or \( L_p \)), \( 1 < p \leq 2 \) and \( K, T \) and \( S \) be as in Theorem 1. Define the sequence \( \{x_n\}_{n=0}^{\infty} \) iteratively by \( x_0 \in K \),

\[
x_{n+1} = (1 - \lambda_n)x_n + \lambda_n Sx_n
\]

for \( n \geq 0 \), where \( \{\lambda_n\}_{n=0}^{\infty} \) is a real sequence satisfying:

(i) \( 0 < \lambda_n < 1 \) for all \( n \geq 0 \);
Then for any given \( f \) in \( K \), the sequence \( \{x_n\}_{n=0}^{\infty} \) converges strongly to the unique solution of \( Tx = f \).

**Proof.** Remark 2 and conditions (ii) and (iii) imply \( \sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty \). The result follows from Theorem 1.

**Corollary 2.** Let \( E \sim L_p \) (or \( \ell_p \)), \( 2 \leq p < \infty \), and let \( K, T \) and \( S \) be as in Theorem 1. Define the sequence \( \{x_n\}_{n=0}^{\infty} \) iteratively by \( x_0 \in K \),

\[
x_{n+1} = (1 - \lambda_n) x_n + \lambda_n Sx_n ,
\]

for \( n \geq 0 \), where \( \{\lambda_n\}_{n=0}^{\infty} \) is a real sequence satisfying:

(i) \( 0 < \lambda_n < 1 \) for all \( n \geq 0 \);
(ii) \( \sum_{n=0}^{\infty} \lambda_n = \infty \); and
(iii) \( \sum_{n=0}^{\infty} \lambda_n^2 < \infty \).

Then for any given \( f \in K \) the sequence \( \{x_n\}_{n=0}^{\infty} \) converges strongly to the unique solution of \( Tx = f \).

**Proof.** The proof follows exactly as in the proof of Corollary 1.

**Remark 3.** Theorem 1 resolves in the affirmative Problem 1 of [9], p. 460, for the much larger class of continuous strongly accretive maps and in the real Banach spaces \( E \) (with uniformly convex dual \( E^* \)) much more general than the \( L_p \) spaces considered in [9].

**ERROR ESTIMATES.** Following the method of [8], setting \( \lambda_n = a(n+1)^{-1} \) we obtain that the error estimate in Theorem 1 is given by

\[
||x_{n+1} - x^*|| = O(n^{-(p-1)/2})
\]

If \( E \sim L_p \) (or \( \ell_p \)) then,

\[
||x_n - x^*|| = O(n^{-(p-1)/2}), \quad \text{if } 1 < p \leq 2
\]

and,

\[
||x_n - x^*|| = O(n^{-1/2}), \quad \text{if } 2 < p < \infty
\]

**Theorem 2.** Let \( E \) be a real Banach space with a uniformly convex dual space, \( E^* \) and let \( K \) be a nonempty closed convex and bounded subset of \( E \).

Suppose \( T : K \to K \) is a Lipschitzian strongly accretive map. For a given \( f \in K \), define \( S : K \to K \) by \( Sx = f - Tx + x \) for each \( x \in K \). Let \( \{\lambda_n\}_{n=0}^{\infty} \) and \( \{\beta_n\}_{n=0}^{\infty} \) be real sequences satisfying:

(i) \( 0 \leq \lambda_n \leq \beta_n < 1 \) for all \( n \geq 0 \);
(ii) \( \sum_{n=0}^{\infty} \lambda_n = \infty \);
(iii) \( \lim_n \beta_n = 0 \);
(iv) \( \sum_{n=0}^{\infty} \lambda_n b(\lambda_n) < \infty \).

For arbitrary \( x_0 \in K \) define the sequence \( \{x_n\}_{n=0}^{\infty} \) in \( K \) by

\[
x_{n+1} = (1 - \lambda_n) x_n + \lambda_n Sx_n , \quad \text{for } n \geq 0.
\]

Then \( \{x_n\}_{n=0}^{\infty} \) converges strongly to the unique solution of \( Tx = f \).

**Proof.** The existence of a solution follows from Morales [23]. Let \( x^* \) be a solution. Clearly \( x^* \) is a fixed point of \( S \) and \( S \) is Lipschitzian with Lipschitz constant \( L^* = (1 + L) \). Moreover for each \( x, y \in K \) we have:

\[
\langle Sx - Sy, f(x - y) \rangle \leq (1 - k) ||x - y||^2
\]

Using (13) and (5) we obtain:

\[
||x_{n+1} - x^*||^2 = ||(1 - \lambda_n)(x_n - x^*) + \lambda_n(Sx_n - Sx^*)||^2
\]

\[
\leq (1 - \lambda_n)^2 ||x_n - x^*||^2 + 2\lambda_n(1 - \lambda_n)\langle Sx_n - Sx^*, f(x_n - x^*) \rangle
\]

\[
+ \max_k (1 - \lambda_k) ||x_k - x^*||^2 + L^*(1 + L) ||Sx_n - Sx^*||^2
\]

\[
\leq (1 - \lambda_n)^2 ||x_n - x^*||^2 + 2\lambda_n(1 - \lambda_n)\langle Sx_n - Sx^*, f(x_n - x^*) \rangle + M\lambda_n b(\lambda_n) ,
\]

for some constant \( M \), since \( K \) is bounded. Using inequality (15) and condition (iii) we have for sufficiently large \( n \), \( L^*(1 + L^*) \beta_n \leq k^2 \) so that:

\[
\langle Sx_n - Sx^*, f(x_n - x^*) \rangle = \langle Sx_n - Sx_n, f(x_n - x^*) \rangle + \langle Sx_n - Sx^*, f(x_n - x^*) \rangle
\]

\[
\leq [L^*(1 + L^*) \beta_n + (1 - k)^2] ||x_n - x^*||^2
\]

\[
\leq [1 - (1 - k)^2] ||x_n - x^*||^2
\]

Substitution of this inequality in (16) yields:

\[
||x_{n+1} - x^*||^2 \leq ((1 - \lambda_n)^2 + 2(1 - k)(1 - k))\lambda_n (1 - \lambda_n + \lambda^2 (1 - k)(1 - k)^2) ||x_n - z^*||^2
\]

\[
+ M\lambda_n b(\lambda_n)
\]

\[
\leq (1 - \lambda_n k(1 - k)) ||x_n - x^*||^2 + M\lambda_n b(\lambda_n) .
\]
The remainder of the argument now follows exactly as in the proof of Theorem 1 to give that 
\( x_n \rightarrow x^* \) as \( n \rightarrow \infty \), completing proof of Theorem 2.

Remark 4. Theorem 2 extends the results of [10] and [12] from \( L_p \) spaces, \( p \geq 2 \) to the more 
general Banach spaces considered here.

Remark 5. It follows from Theorems 1 and 2 that either the Mann iteration process or the 
Ishikawa iteration process can be used to approximate the solution of the equation \( Tx = f \) in any 
real Banach space with a uniformly convex dual, if \( T \) is Lipschitzian and strongly accretive. However, 
since the error estimates for the two methods for the class of operators under consideration 
are of the same order the Mann iteration process may be preferred due to its simplicity.

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