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ABSTRACT

A set of exact solutions of Einstein equations in vacuum is obtained. Taking this set of solutions as seed solutions and making use of the Belinsky-Zakharov generation technique a set of generated solutions is constructed. Both set of exact solutions and a set of generated solutions describe two solition waves, which propagate in opposite directions and collide with each other, and then recover their original shapes. The singularities of the two set of solutions are analyzed. The relationship between our solutions and other solutions is also discussed.

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1. Introduction

Einstein field equations are a system of nonlinear differential equations. Many authors have searched for exact wave solutions. In 1916, Einstein[1] first studied weak field wave solution. Then Einstein and Rosen[2,3] derived some exact cylindrical wave solutions in 1937. Bondi plane wave solution was found in 1959[4]. In the meantime, Peres wave solutions were published too[5]. On the other hand, in 1978, Belinsky and Zakharov[6] developed a generation technique, which is based on the inverse scattering method (soliton technique). This technique can be applied to Einstein equations in vacuum if one assumes the existence of two commuting Killing vectors. By means of this technique they found N-soliton solutions of Einstein field equations[7]. Belinsky and Fargion[8] constructed two soliton wave solutions in 1980. And then, Ibanez and Verdaguer[9,10] obtained four-soliton solutions and multisoliton solutions, which had many similarities to the classical soliton because of their localization and properties under the collision. The energy problem of gravitational soliton was discussed in Ref[11]. These soliton wave solutions stated above are generated from Kasner solution, the seed solutions. They do not possess the form of travelling wave, and the velocity of the propagation of the soliton either is not equal to the light speed or depends on the associated frame.

In this paper, we search for exact two soliton wave solutions of Einstein equations and study the properties of the obtained solutions. The paper is divided as follows. In section 2, a set of exact two soliton wave solutions is obtained by straightforwardly solving the field equations. In section 3, we discuss the relation between our solutions and some other solutions. In section 4, Using the generation technique and taking the obtained solutions as the seed solution derives the generated solutions. Section 5 is devoted to an analysis of singularties of the exact solutions and the generated solutions.

2. The exact two soliton wave solutions

We start with the following diagonal differential interval

$$-ds^2 = e^{-f}(-dt^2+dz^2) + L^2(e^{2h}dx^2 + e^{-2h}dy^2) \qquad (1)$$
$$= e^{-f}du\,dv + L^2(e^{2h}dx^2 + e^{-2h}dy^2).$$

where u=z-t and v=z+t are null coordinates. f, L and h are functions of u and v alone. From (1) we have the nonvanishing components of $\Gamma^{\alpha}_{\beta\lambda}$

$$\Gamma^1_{13} = \frac{\partial \ln L}{\partial u} + \frac{\partial h}{\partial u}, \qquad \Gamma^1_{14} = \frac{\partial \ln L}{\partial v} + \frac{\partial h}{\partial v},$$
$$\Gamma^2_{23} = \frac{\partial \ln L}{\partial u} - \frac{\partial h}{\partial u}, \qquad \Gamma^2_{24} = \frac{\partial \ln L}{\partial v} - \frac{\partial h}{\partial v},$$
$$\Gamma^3_{11} = -\frac{1}{2} e^{f} L^2 e^{2h}\left(\frac{\partial \ln L}{\partial v} + \frac{\partial h}{\partial v}\right),$$
$$\Gamma^3_{22} = -\frac{1}{2} e^{f} L^2 e^{-2h}\left(\frac{\partial \ln L}{\partial v} - \frac{\partial h}{\partial v}\right),$$
$$\Gamma^4_{11} = -\frac{1}{2} e^{f} L^2 e^{2h}\left(\frac{\partial \ln L}{\partial u} + \frac{\partial h}{\partial u}\right), \qquad (2)$$
$$\Gamma^4_{22} = -\frac{1}{2} e^{f} L^2 e^{-2h}\left(\frac{\partial \ln L}{\partial u} - \frac{\partial h}{\partial u}\right),$$
$$\Gamma^3_{33} = \frac{-\partial f}{\partial u}, \qquad \Gamma^4_{44} = -\frac{\partial f}{\partial v}.$$

In the calculations, we have used the coordinates (x,x,x,x) = (x,y,u,v). The straightforward calculations give the nonvanishing components of $R_{\mu\nu}$

$$R_{11} = -2e^{f}L^2e^{2h}\left(\frac{L_{,u}L_{,v}}{L^2} + h_{,uv} + \frac{L_{,uv}}{L} + \frac{L_{,u}h_{,v}}{L} + \frac{L_{,v}h_{,u}}{L}\right),$$
$$R_{22} = -2e^{f}L^2e^{-2h}\left(\frac{L_{,u}L_{,v}}{L^2} - h_{,uv} + \frac{L_{,uv}}{L} - \frac{L_{,u}h_{,v}}{L} - \frac{L_{,v}h_{,u}}{L}\right),$$
$$R_{33} = -2\left[\frac{L_{,uu}}{L} + (h_{,u})^2 + \frac{f_{,u}L_{,u}}{L}\right], \qquad (3)$$
$$R_{44} = -2\left[\frac{L_{,vv}}{L} + (h_{,v})^2 + \frac{f_{,v}L_{,v}}{L}\right],$$
$$R_{34} = -2\left(\frac{L_{,uv}}{L} + h_{,u}h_{,v} - \frac{1}{2}f_{,uv}\right).$$

Setting $R_{\mu\nu}$ in (3) equal to zero leads to a system of field equations in vacumm

$$\frac{L_{,u}L_{,v}}{L^2} + h_{,uv} + \frac{L_{,uv}}{L} + \frac{L_{,u}h_{,v}}{L} + \frac{L_{,v}h_{,u}}{L} = 0, \qquad (4)$$
$$\frac{L_{,u}L_{,v}}{L^2} - h_{,uv} + \frac{L_{,uv}}{L} - \frac{L_{,u}h_{,v}}{L} - \frac{L_{,v}h_{,u}}{L} = 0, \qquad (5)$$
$$\frac{L_{,uu}}{L} + (h_{,u})^2 + \frac{f_{,u}L_{,u}}{L} = 0, \qquad (6)$$
$$\frac{L_{,vv}}{L} + (h_{,v})^2 + \frac{f_{,v}L_{,v}}{L} = 0, \qquad (7)$$
$$\frac{L_{,uv}}{L} + h_{,u}h_{,v} - \frac{1}{2}f_{,uv} = 0. \qquad (8)$$

From the field equations (4) and (5) we can easily obtain

$$\frac{L_{,u}h_{,v}}{L} + \frac{L_{,v}h_{,u}}{L} + h_{,uv} = 0, \qquad (9)$$

and show that L^2 satisfies the wave equation

$$(L^2)_{,uv} = 0 \qquad (10)$$

The general solution for L^2 can be written as

$$L^2 = J_1(u) + J_2(v) \qquad (11)$$

where J_1 and J_2 are arbitrary functions of u and v respectively. In order to solve the system of field equations we may suppose that the function f(u,v) takes the following form

$$f(u,v) = K_1(u) + K_2(v). \qquad (12)$$

This means that the metric function $e^{-f(u,v)}$ is separable. It is obvious that (12) leads to

$$f_{,uv} = 0 \quad , \tag{13}$$

then $k_1(u)$ and $k_2(v)$ can also be **arbitrary**. Substituting (11) and (13) into the field equation (8) results in

$$\frac{J_{1,u}\, J_{2,v}}{4[\,J_1(u) + J_2(v)\,]^2} - h_{,u}h_{,v} = 0 \quad . \tag{14}$$

A particular solution of (14) is

$$h = \frac{1}{2} \ln [\, J_1(u) + J_2(v)\,] \quad . \tag{15}$$

Moreover, it can be demonstrated that (15) and (11) satisfy (9), which is derived from the field equations (4) and (5). Now the field equation (6) can be reduced to

$$J_{1,uu} + J_{1,u} K_{1,u} = 0 \tag{16}$$

which has a **particular** solution

$$J_1(u) = \int e^{-K_1(u)} du \quad .$$

Similarly,

$$J_2(v) = \int e^{-K_2(v)} dv \quad . \tag{17}$$

They can be rewritten in the form

$$e^{-K_1(u)} = J_{1,u} \quad ,$$

$$e^{-K_2(v)} = J_{2,v} \tag{17'}$$

Thus this relation **indicates** that the field equation (6) imposes a restriction on the arbitrary functions J_i and k_i $(i = 1,2)$. From (12) and (17) we obtain the metric function

$$e^{-f(u,v)} = e^{-K_1(u) - K_2(v)} = J_{1,u} J_{2,v} \quad . \tag{18}$$

We have applied the equations (4),(5),(6) and (8) to search for the **two** soliton wave solutions . We have not used the field (7) yet. But we have **noticed** that when we permutate the differential indexes u and v , the form of (4),(5) and (8) are invariant, while (6) and (7) interchange because they have the **same** form. Therefore, the utilization of (7) enables **us** to put $J_1(u)$ and $J_2(v)$ in the same form, namely $J_1(x) = J_2(x) = J(x)$. Up to now, we have exhausted all of the field equations and attained a family of exact solutions of Einstein field equations

$$g_{11} = L^2 e^{2h} = [\, J_1(u) + J_2(v)\,]^2 \,,$$

$$g_{22} = L^2 e^{-2h} = 1 \quad ,$$

$$g_{33} = \left| J_{1,u} J_{2,v} \right| \quad , \tag{19}$$

$$g_{44} = -\left| J_{1,u} J_{2,v} \right| \quad ,$$

$$J_1(x) = J_2(x) = J(x) \;.$$

In writing g_{33} we have taken **an** absolute value so that the change in spacelike (or timelike) property of the coordinate z (or t) can be avoided when $J_{,u}$ or $J_{,v}$ change their signs. This family of exact solutions is dependent on the function J(u) or J(v). If we properly choose the function J, we can search for exact **two** soliton wave solutions. For example, we can give a set of them as follows

Solution 1

$$J(x) = e^{-x^2} \quad ,$$

$$g_{11} = (\,e^{-u^2} + e^{-v^2}) \quad ,$$

$$g_{22} = 1 \quad , \tag{20}$$

$$g_{33} = -g_{44} = 4|uv|\, e^{-u^2-v^2}$$

Solution 2

$$J(x) = |th\,x| ,$$
$$g_{11} = [\,|th\,u| + |th\,v|\,]^2 ,$$
$$g_{22} = 1 , \tag{21}$$
$$g_{33} = -g_{44} = sech^2 u\, sech^2 v .$$

Solution 3

$$J(x) = sech^2 x ,$$
$$g_{11} = (sech^2 u + sech^2)^2 ,$$
$$g_{22} = 1 \tag{22}$$
$$g_{33} = -g_{44} = 4\,|th\,u\; th\,v\; sech^2 u\; sech^2 v|$$

Solution 4

$$J(x) = tg^{-1} e^{x^2} ,$$
$$g_{11} = (tg^{-1} e^{u^2} + tg^{-1} e^{v^2})^2 ,$$
$$g_{22} = 1 , \tag{23}$$
$$g_{33} = -g_{44} = \frac{4\,|uv|}{(e^{u^2} + e^{-u^2})(e^{v^2} + e^{-v^2})}$$

From **figures** 1 and 2 it can be seen that the set of the exact solutions **describe two** soliton waves which propagate in **opposite** directions and collide with each other, and the shapes of the **two** soliton waves are recovered after the collision. In **figures** 1 and 2, we only show **solutions** 1 and 2. The situation of **solutions** 3 and 4 is similar to that of **solutions** 1 and 2 **shown** in the figures.

3. The relation to other two soliton solutions

In this section, we discuss the relation **between** our family of exact solutions and other **two** soliton waves solutions. According to (10) we can also take $J^2(u,v)$ as

$$L^2 = J_1(u) - J_2(v) \tag{24}$$

But f(u,v) is still expressed by (12). (8) can be reduced to

$$\frac{J_{1,u} J_{2,v}}{4(J_1 - J_2)^2} + h_{,u} h_{,v} = 0 \tag{25}$$

(25) can be solved , a particular solution is

$$h = \frac{1}{2} \ln \frac{J_1^{1/2} + J_2^{1/2}}{J_1^{1/2} - J_2^{1/2}} \tag{26}$$

It is easy to show that (24) and (26) satisfy (9). Besides, **from (6),(24) and (26) we can obtain**

$$\frac{J_{1,uu}}{2J_1^{1/2}} - \frac{1}{4}\frac{(J_{1,u})^2}{J_1^{3/2}} + \frac{K_{1,u} J_{1,u}}{2J_1^{1/2}} = 0 \tag{27}$$

If we set $Q_1(u) = J_1^{1/2}(u)$, then (27) become

$$Q_{1,uu} + K_{1,u} Q_{1,u} = 0 \tag{28}$$

Solving this equation gives

$$Q_1(u) = \int e^{-K_1(u)}\, du$$

Similarly

$$Q_2(v) = J_2^{1/2}(v) = \int e^{-K_2(v)}\, dv$$

We can turn them into

$$e^{-K_1(u)} = Q_{1,u}\ , \qquad e^{-K_2(v)} = Q_{2,v}\ . \tag{29}$$

For reasons given previously, we can put $J_1(u)$ and $J_2(v)$ in the same form, namely

$$J_1(x) = J_2(x) = J(x)$$

then

$$Q_1(x) = Q_2(x) = Q(x)\ .$$

Consequently, we obtain a family of solutions which depend on the function Q(x).

$$\begin{aligned} g_{11} &= L^2 e^{2h} = (J_1 - J_2)\,\frac{J_1^{1/2} + J_2^{1/2}}{J_1^{1/2} - J_2^{1/2}} \\ &= (J_1^{1/2} + J_2^{1/2})^2 = (Q_1 + Q_2)^2\ , \\ g_{22} &= L^2 e^{-2h} = (Q_1 - Q_2)^2\ , \\ g_{33} &= -g_{44} = Q_{1,u}\, Q_{2,v}\ , \\ Q_1(x) &= Q_2(x) = Q(x)\ . \end{aligned} \tag{30}$$

If we choose

$$Q(u) = 2\,tg^{-1} e^{u} + c_{11}\ , \qquad Q(v) = 2\,tg^{-1} e^{v} + c_{12}$$

then we get a solution

$$\begin{aligned} g_{11} &= (2\,tg^{-1} e^{u} + 2\,tg^{-1} e^{v} + c_1)^2\ , \\ g_{22} &= (2\,tg^{-1} e^{u} - 2\,tg^{-1} e^{v} + c_2)^2\ , \\ g_{33} &= -g_{44} = \operatorname{sech} u \operatorname{sech} v\ ; \end{aligned} \tag{31}$$

where c_{11}, c_{12}, c_1 and c_2 are arbitrary real numbers, and $c_1 = c_{11} + c_{12}$ and $c_2 = c_{11} - c_{12}$. This is the solution 1 in Ref.[11]. The other four solutions in Ref.[11] can also be derived provided we properly choose function Q(x). Therefore, the family (30) contains the set of

two soliton wave solutions given in Ref. [11] .

4. The generated solutions

In this section, we take the family of solutions (19) as seed solutions to construct generated solutions of two soliton waves. According to Belinsky-Zakharov generation technique[6], the general expressions for one pole solution are

$$g = \left(I - \frac{\mu^2 - \alpha^2}{\mu^2} P\right) g_0\ , \tag{32}$$

$$g'_{ph} = \left(\frac{\mu}{\alpha} I - \frac{\mu^2 - \alpha^2}{\alpha\mu} P\right) g_0\ , \tag{33}$$

$$f_{ph} = \frac{C}{\sqrt{\alpha}}\,\frac{Q\,\mu^2}{(\alpha^2 - \mu^2)}\, f_0\ . \tag{34}$$

In the above, and hereafter, we use the notations adopted in Ref. [6] . To write the generated solutions, we give the needed entries relative to the seed solutions

$$\alpha = J(u) + J(v)\ , \quad \beta = J(u) - J(v)\ , \tag{35}$$

$$g_0 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix}\ , \quad f_0 = |\alpha_{,u}\,\alpha_{,v}|\ , \tag{36}$$

$$Q = \frac{m_{01}^2 \alpha^2}{4\mu^2\omega^2} + m_{02}^2\ , \tag{37}$$

$$P = \frac{1}{Q}\begin{pmatrix} \dfrac{m_{01}^2 \alpha^2}{4\mu^2\omega^2} & \dfrac{m_{01} m_{02} \alpha^2}{2\mu^2\omega} \\ \dfrac{m_{01} m_{02}}{2\mu\omega} & m_{02}^2 \end{pmatrix} \tag{38}$$

Inserting (35) and (36) in (32), (33) and (34), we can construct a new family of the generated solutions

$$g_{11} = \mu\alpha - \frac{\mu^2 - \alpha^2}{\alpha\mu Q}\cdot\frac{m_{01}^2 \alpha^4}{4\mu^2\omega^2}\ , \tag{39}$$

$$g_{22} = \frac{\mu}{\alpha} - \frac{\mu^2-\alpha^2}{\alpha\mu Q} m_{02}^2 , \tag{40}$$

$$g_{12} = -\frac{\mu^2-\alpha^2}{\alpha\mu Q} \cdot \frac{m_{01} m_{02} \alpha^2}{2\mu\omega} , \tag{41}$$

$$f = \left| \frac{c\mu^2 Q}{(\alpha^2-\mu^2)\sqrt{\alpha}} f_0 \right| . \tag{42}$$

The generated solutions are nondiagonal, but still depend on the function J(u) via α . When J(u) takes the forms in (20), (21), (22) and (23), the new family of solutions leads to a new set of two soliton wave solutions. Their detailed expressions are quite complicated, and we will not write them here. Figures 3 and 4 show that the solutions generated from solutions 1 and 2 describe two soliton waves which have the behavior similar to that of their seed solutions, although they are rather complex.

5. Discussion of the singularities

This section is devoted to the discussion of both sets of exact solutions and sets of generated solutions.

We first study the family of solutions (19). From (19) one can see that when $L^2 = 0$ the metric will be morbid, and singularity probably occurs in the $\Gamma^\alpha_{\beta\gamma}$. It can be pointed out that at $u = \infty = v$, g_{11} and g_{33} for solutions 1 and 3 and g_{33} for solutions 2 and 4 are zero; at $u = 0 = v$, g_{33} for solutions 1, 3 and 4 and g_{11} for solution 2 are equal to zero. For solution family (19), the nonvanishing components of $\Gamma^\alpha_{\beta\gamma}$ are

$$\Gamma^1_{13} = \frac{J_{,u}}{L^2} , \qquad \Gamma^1_{14} = \frac{J_{,v}}{L^2} ,$$

$$\Gamma^3_{11} = \frac{-L^2}{2J_{,u}} , \qquad \Gamma^4_{11} = \frac{-L^2}{2J_{,v}} ,$$

$$\Gamma^3_{33} = \frac{J_{,uu}}{J_{,u}} , \qquad \Gamma^4_{44} = \frac{J_{,vv}}{J_{,v}} .$$

The calculations indicate that all of the set of the exact solutions have singular components of $\Gamma^\alpha_{\beta\gamma}$. These singular components will cause incompleteness of some geodesics for the set of two soliton wave solutions (20),(21),(22) and (23). Equation of geodesic deviation indicate that Riemann tensor represents gravitional field. We should further study the singularity of Riemann tensor for the solution set. The analysis shows that at $u = 0 = v$ $u = \infty = v$, $R^\mu{}_{\nu\alpha\lambda}$ has singular components. For example, at $u = \infty = v$ $R^1{}_{314}$ of solutions 1 and 3, $R^3{}_{131}$ of solution 2 and $R^4{}_{141}$ of solution 4 are divergent; at $u = 0 = v$, $R^3{}_{131}$ of solutions 1 and 2 , $R^1{}_{414}$ of solution 3 and $R^4{}_{141}$ of solution 4 are infinitive. Therefore, the set of two soliton solutions (20),(21),(22) and (23) are regular, except at $u = 0 = v$ and $u = \infty = v$.

Now we turn to discussion of singularities for set of generated solutions. From (39),(40),(41) and (42) we can show that at the points where $J(u) = J(v)$ and $\alpha = \beta$, the morbid g_{11} and g_{12} equal to zero; g_{22} and f are divergent. Consequently we have the following results: at $u = \infty = v$, solution 1 and 3 have vanishing g_{12} and f and singular g_{11} and g_{22} ; at $u = 0 = v$, solution 2 has infinitive g_{12} and f, while g_{11} and g_{22} are zero. It can be expected that these generated solutions will be singular at $u = o = v$ and $u = \infty = v$.

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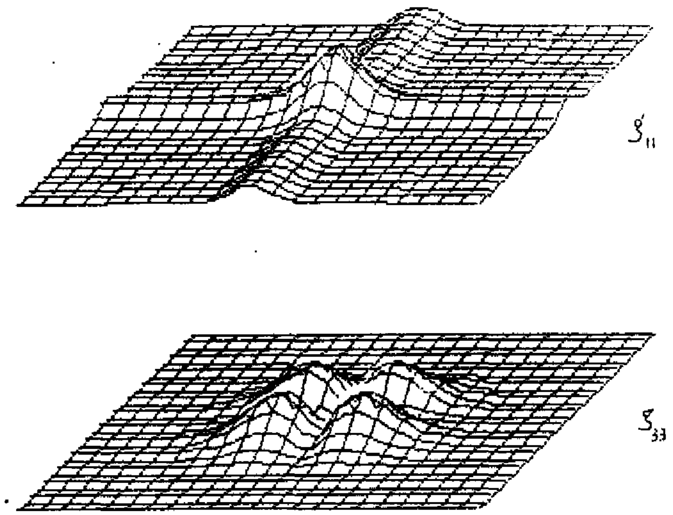

Figure 1. The two soliton wave solution 1 in u-v plane. The region is $-6 \leq u \leq 6$, $-6 \leq v \leq 6$.

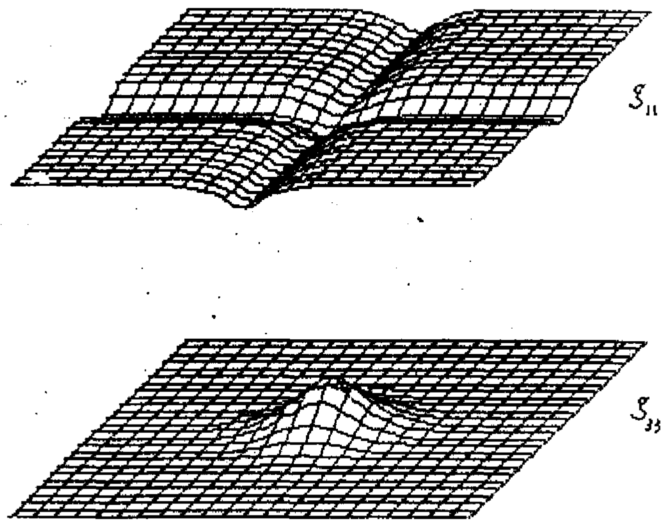

Figure 2. The two soliton wave solution 2 in u-v plane. The region is $-6 \leq u \leq 6$, $-6 \leq v \leq 6$.

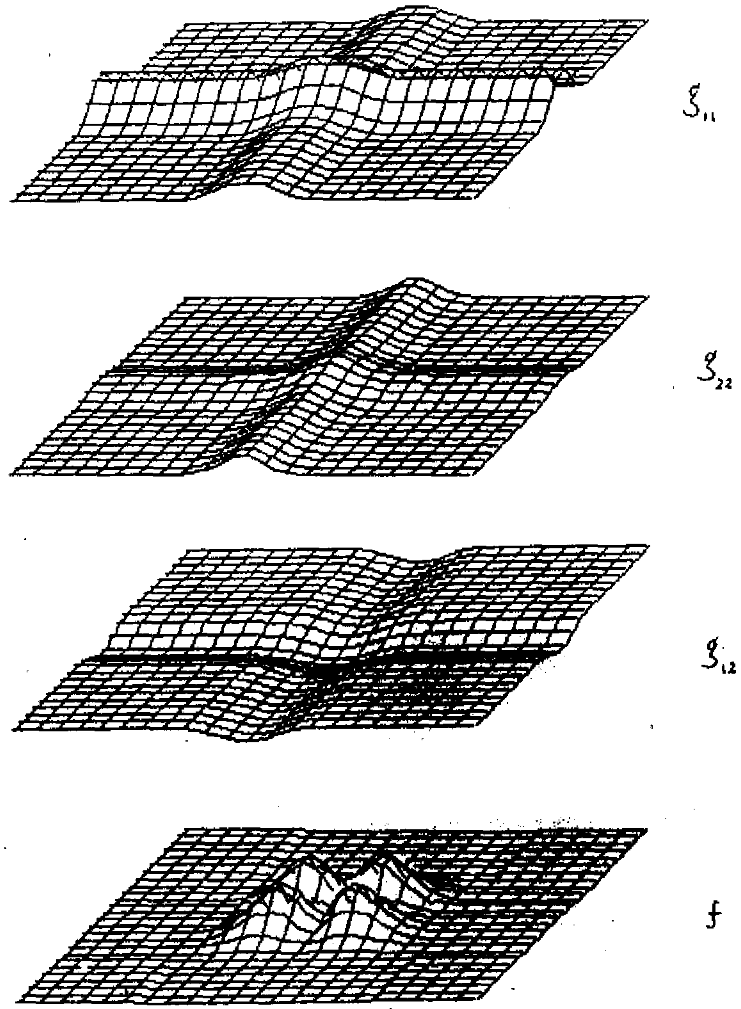

Figure 3 . The generated solution from the two soliton wave solution 1 in u-v plane. The region is $-6 \leq u \leq 6$, $-6 \leq v \leq 6$. The parameters are $m_{01} = 1$, $m_{02} = 1$ and $\omega = 3$.

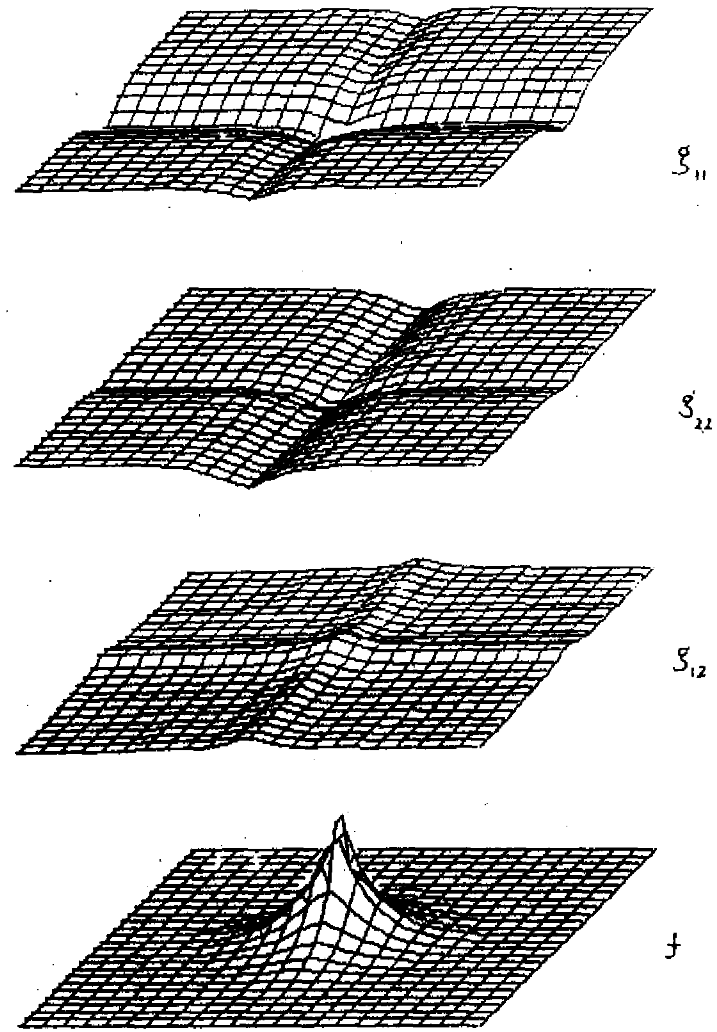

Figure 4. The generated solution from the two soliton wave solution 2 in u-v plane. The region is $-6 \leq u \leq 6$, $-6 \leq v \leq 6$. The parameters are $m_{01} = 1$, $m_{02} = 1$ and $\omega = 3$.