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**LYAPUNOV EQUATION FOR INFINITE-DIMENSIONAL
DISCRETE BILINEAR SYSTEMS***

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Abstract. Mean-square stability for discrete systems requires that uniform convergence is preserved between input and state correlation sequences. Such a convergence preserving property holds for an infinite-dimensional bilinear system if and only if the associate Lyapunov equation has a unique strictly positive solution.

Key words. Discrete bilinear systems; infinite-dimensional systems; operator theory; stability.

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1. NOTATION

Let $B[X]$ be the Banach algebra of all bounded linear operators from a Banach space X into itself, and let $G[X]$ be the group of all invertible operators from $B[X]$. Both the norm in X and the induced uniform norm in $B[X]$ will be denoted by $\|\cdot\|$, and $r_\sigma(\cdot)$ will stand for the spectral radius in $B[X]$. An operator $T \in B[X]$ is said to be similar to a strict contraction if $\|W^{-1}TW\| < 1$ for some $W \in G[X]$. Throughout this paper, H will stand for a separable nontrivial complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and an upper star $*$ will denote adjoint of a Hilbert-space operator, as usual. The Hilbert space $\bigoplus_{j=0}^{\infty} H$ obtained by the direct (orthogonal) sum of countably infinite copies of H will be denoted by $\ell_2(H)$, and its inner product by $\langle \cdot, \cdot \rangle_{\ell_2(H)}$. Let $B^+[H]$ be the weakly closed cone of $B[H]$ made up of all self-adjoint nonnegative operators, let $G^+[H] = B^+[H] \cap G[H]$ be the class of all strictly positive operators, and let $B_{\infty}^+[H]$ be the class of all compact operators from $B^+[H]$. The trace of $T \in B_{\infty}^+[H]$ is defined as $\text{tr}(T) = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle = \sum_{k=1}^{\infty} \lambda_k$, where $\{e_k; k \geq 1\}$ is any orthonormal basis for H , and $\{\lambda_k \geq 0; k \geq 1\}$ is a sequence made up of all eigenvalues of $T \in B_{\infty}^+[H]$, each nonzero one counted according to its multiplicity. Set $B_1^+[H] = \{T \in B_{\infty}^+[H] : \text{tr}(T) < \infty\}$, the class of all nonnegative nuclear operators. Finally, for any $h \in H$, let $(h \circ h) \in B_1^+[H]$ be defined as $(h \circ h)x = \langle x, h \rangle h$ for all $x \in H$.

2. INTRODUCTION

Given an operator sequence $\{R_i \in B[X]; i \geq 0\}$, consider another operator sequence $\{Q_i \in B[X]; i \geq 0\}$ recursively defined by a linear autonomous difference equation

$$(1) \quad Q_{i+1} = \mathcal{F}(Q_i) + R_{i+1}, \quad Q_0 = R_0,$$

where \mathcal{F} stands for a bounded linear transformation from the Banach space $B[H]$ into itself. Recall that $\{Q_i \in B[H]; i \geq 0\}$ converges in $B[H]$ whenever $\{R_i \in B[H]; i \geq 0\}$ converges in $B[H]$ if and only if [6]

$$(2) \quad r_\sigma(\mathcal{F}) < 1.$$

As a background for our further discussion let us consider first a well-known particular case, viz. $\mathcal{F} = \mathcal{F}_F$, where $\mathcal{F}_F \in B[B[H]]$ is defined by

$$\mathcal{F}_F(Q) = F Q F^* \quad \forall Q \in B[H]$$

for some $F \in B[H]$. Since $\|\mathcal{F}_F^i\| = \|F^i\|^2$ for every $i \geq 0$, $r_\sigma(\mathcal{F}_F) = r_\sigma(F)^2$ according to the Gelfand formula for the spectral radius. Hence, $r_\sigma(\mathcal{F}_F) < 1$ if and only if $r_\sigma(F) < 1$. However, as is well known (see e.g. [1, 3, 7, 9, 10]), $r_\sigma(F) < 1$ if and only if

$$(3) \quad \text{for every } V \in G^+[H] \text{ there exists a unique solution } W \in G^+[H] \text{ for the Lyapunov equation } V = W - \mathcal{F}(W).$$

The purpose of this paper is to show that the equivalence between (2) and (3) still holds for a more general case where, instead of setting $\mathcal{F} = \mathcal{F}_F$, we take $\mathcal{F} := \mathcal{F}_F + T$ with T in $B[B[H]]$ defined as follows.

$$T(Q) = \sum_{k,l=1}^{\infty} (C e_l; e_k) A_k Q A_l^* \quad \forall Q \in B[H],$$

where $\{A_k \in B[H]; k \geq 0\}$ is an arbitrary bounded sequence of operators, $C \in B_1^+[H]$ is any nonnegative nuclear operator, and $\{e_k; k \geq 1\}$ is a suitable orthonormal basis for H ensuring convergence for the above infinite series.

Such a deterministic linear setup, concerning the convergence preserving property of the linear model (1), naturally arises in the mean-square stability problem for infinite-dimensional discrete (either linear or bilinear) systems operating in a stochastic environment. Actually, mean-square stability for an infinite-dimensional discrete (either linear or bilinear) system requires that the state correlation sequence, say $\{Q_i \in B_1^+[H]; i \geq 0\}$, converges in $B[H]$ whenever the input correlation sequence, say $\{R_i \in B_1^+[H]; i \geq 0\}$, converges in $B[H]$; and such a state correlation sequence evolves according to the linear model (1), with $\mathcal{F} = \mathcal{F}_F$ for the case of a linear system (see e.g. [4, 11]), or with $\mathcal{F} = \mathcal{F}_F + T$ for the case of a bilinear system (see e.g. [5] and, for the continuous-time case, [2, 12]).

Our main result is presented in section 4 by using the auxiliary results developed in section 3. The present approach was motivated by the earlier works on finite-dimensional stochastic bilinear systems in [8] and on infinite-dimensional deterministic linear systems in [7], extending the former to an infinite-dimensional setting and the latter to the underlying class of operators $\mathcal{F} = \mathcal{F}_F + T$.

3. PRELIMINARIES

Consider a bounded sequence of operators $\{A_k \in B[H]; k \geq 1\}$, and let $\{e_k; k \geq 1\}$ be an orthonormal basis for H . Given $C \in B_1^+[H]$, set $T_n \in B[B[H]]$ as follows. For each integer $n \geq 1$,

$$T_n(Q) = \sum_{k,l=1}^n (C e_l; e_k) A_k Q A_l^* \quad \forall Q \in B[H].$$

Assumption. $T_n \rightarrow T \in B[B[H]]$ as $n \rightarrow \infty$ in $B[B[H]]$.

Under the above assumption, write

$$T(Q) = \sum_{k,l=1}^{\infty} (C e_l; e_k) A_k Q A_l^* \quad \forall Q \in B[H]$$

and, given $F \in B[H]$, let $\mathcal{F} \in B[B[H]]$ be defined as follows.

$$\mathcal{F}(Q) = F Q F^* + T(Q) \quad \forall Q \in B[H].$$

Remark. Note that the very definition of $\{T_n \in B[B[H]]; n \geq 1\}$ in terms of a bounded sequence $\{A_k \in B[H]; k \geq 1\}$ and a nonnegative nuclear operator $C \in B_1^+[H]$ is not enough to ensure its convergence as assumed above. For instance, take $c \in H$ such that $|\sum_{k=1}^n \langle c; e_k \rangle| \rightarrow \infty$ as $n \rightarrow \infty$ (e.g. with $H = \ell_2$, take $c = (1, 1/2, 1/3, \dots) \in \ell_2$ and let $\{e_k; k \geq 1\}$ be the standard orthonormal basis for ℓ_2), and set $C = (c \circ c) \in B_1^+[H]$. Thus, $\sum_{k,\ell=1}^n \langle C e_\ell; e_k \rangle = \sum_{k,\ell=1}^n \langle e_\ell; c \rangle \langle c; e_k \rangle = |\sum_{k=1}^n \langle c; e_k \rangle|^2$ for each $n \geq 1$. Therefore, with $A_k = I$ for every $k \geq 1$, $T_n(I) = |\sum_{k=1}^n \langle c; e_k \rangle|^2 I$ for each $n \geq 1$, so that $\|T_n(I)\| \rightarrow \infty$ as $n \rightarrow \infty$. On the other hand, the assumed uniform convergence of $\{T_n \in B[B[H]]; n \geq 1\}$ can always be achieved by choosing a suitable orthonormal basis for H . For instance, take any $s \in H$ and set $S = C + (s \circ s)$. Since $S \in B_1^+[H]$, let $\{e_k; k \geq 1\}$ be an orthonormal basis for H made up of eigenvectors of S (whose existence is ensured by the spectral theorem for compact normal operators). Let $\lambda_k \geq 0$ be the eigenvalue of S associated with the eigenvector e_k for each $k \geq 1$, and set $\mathcal{L}_n \in B[B[H]]$ for every $n \geq 1$ as follows.

$$\mathcal{L}_n(Q) = \sum_{k,\ell=1}^n \langle S e_\ell; e_k \rangle A_k Q A_\ell^* = \sum_{k=1}^n \lambda_k A_k Q A_k^* \quad \forall Q \in B[H].$$

Since $\sum_{k=1}^{\infty} \lambda_k = \sum_{k=1}^{\infty} \langle S e_k; e_k \rangle = \text{tr}(S) < \infty$,

$$\sup_{\nu \geq 1} \|\mathcal{L}_{n+\nu} - \mathcal{L}_n\| \leq \left(\sup_{k \geq 1} \|A_k\|^2 \right) \sum_{k=n+1}^{\infty} \lambda_k \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

so that $\{\mathcal{L}_n \in B[B[H]]; n \geq 1\}$ converges in the Banach space $B[B[H]]$. Now set $\mathcal{K}_n \in B[B[H]]$ such that

$$\mathcal{K}_n(Q) = \sum_{k,\ell=1}^n \langle (s \circ s) e_\ell; e_k \rangle A_k Q A_\ell^* = M_n Q M_n^* \quad \forall Q \in B[H],$$

where $M_n := \sum_{k=1}^n \langle s; e_k \rangle A_k \in B[H]$, for every $n \geq 1$; and note that $(\sum_{k=m}^p \langle s; e_k \rangle)^2 = \sum_{k,\ell=m}^p \langle (s \circ s) e_\ell; e_k \rangle \leq \sum_{k,\ell=m}^p \langle S e_\ell; e_k \rangle = \sum_{k=m}^p \lambda_k$ for every $m, p \geq 1$, since $(s \circ s) \leq S$. Thus,

$$\sup_{\nu \geq 1} \|M_{n+\nu} - M_n\| \leq \left(\sup_{k \geq 1} \|A_k\| \right) \left(\sum_{k=n+1}^{\infty} \lambda_k \right)^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and hence $\{M_n \in B[H]; n \geq 1\}$ converges in the Banach space $B[H]$. By letting $M \in B[H]$ be its limit, it is readily verified that

$$\sup_{\|Q\|=1} \|\mathcal{K}_n(Q) - M Q M^*\| \leq \|M_n - M\|^2 + 2\|M\| \|M_n - M\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

so that $\{K_n \in B[B[H]]; n \geq 1\}$ also converges in $B[B[H]]$. Therefore, $\{T_n \in B[B[H]]; n \geq 1\}$ converges in $B[B[H]]$, since $T_n = L_n - K_n$ for every $n \geq 1$.

Finally, for any $W \in G^+[H]$ let $\mathcal{W} \in G[B[H]]$ be defined by

$$\mathcal{W}(Q) = W Q W \quad \forall Q \in B[H],$$

so that $\mathcal{W}^{-1} \in G[B[H]]$ is such that

$$\mathcal{W}^{-1}(Q) = W^{-1} Q W^{-1} \quad \forall Q \in B[H].$$

Propositions. For every $W \in G^+[H]$,

$$(P_1) \quad \mathcal{W}^{-1} \mathcal{F} \mathcal{W}(B^+[H]) \subseteq B^+[H],$$

$$(P_2) \quad \|\mathcal{W}^{-1} \mathcal{F} \mathcal{W}\| = \|\mathcal{W}^{-1} \mathcal{F} \mathcal{W}(I)\|.$$

Proof. Consider the direct (orthogonal) sum $H = \mathcal{C} \oplus H$, which is a Hilbert space with inner product given by

$$\langle x; y \rangle = \xi \bar{v} + \langle x; y \rangle$$

for all $x = \xi \oplus x$ and $y = v \oplus y$ in H ($\xi, v \in \mathcal{C}$, the upper bar denoting complex conjugate, and $x, y \in H$). Given the orthonormal basis $\{e_k; k \geq 1\}$ for H set, for each $k \geq 0$,

$$e_k = \begin{cases} 1 \oplus 0 & \text{if } k = 0, \\ 0 \oplus e_k & \text{if } k \geq 1; \end{cases}$$

so that $\{e_k; k \geq 0\}$ is an orthonormal basis for H . Let $C = (1 \oplus C) \in B_1^+[H]$ be the direct sum of the identity on \mathcal{C} with $C \in B_1^+[H]$, so that

$$\langle C x; y \rangle = \langle \xi \oplus Cx; v \oplus y \rangle = \xi \bar{v} + \langle Cx; y \rangle$$

for all $x = \xi \oplus x \in H$ and $y = v \oplus y \in H$. In particular,

$$\langle C e_\ell; e_k \rangle = \begin{cases} 1 & \text{if } k = \ell = 0, \\ \langle C e_\ell; e_k \rangle & \text{if } k, \ell \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for each $n \geq 1$, set $\mathcal{F}_n \in B[B[H]]$ as follows.

$$\mathcal{F}_n(Q) = F Q F^* + T_n(Q) = \sum_{k, \ell=0}^n \langle C e_\ell; e_k \rangle F_k Q F_\ell^* \quad \forall Q \in B[H],$$

with $F_0 := F \in B[H]$ and $F_k := A_k \in B[H]$ for every $k \geq 1$, so that $\mathcal{F}_n \rightarrow \mathcal{F} \in B[B[H]]$ as $n \rightarrow \infty$ in $B[B[H]]$, where

$$\mathcal{F}(Q) = FQF^* + T(Q) = \sum_{k,l=0}^{\infty} (Ce_l; e_k) F_k Q F_l^* \quad \forall Q \in B[H],$$

according to the convergence assumption on $\{T_n \in B[B[H]]; n \geq 1\}$. Now, since $C \in B_1^+[H]$, the spectral theorem says that

$$Cx = \sum_{j=0}^{\infty} \gamma_j (x; f_j) f_j \quad \forall x \in H$$

for some orthonormal basis $\{f_j; j \geq 0\}$ for H , where $\gamma_j \geq 0$ for every $j \geq 0$ (actually, $\sum_{j=0}^{\infty} \gamma_j = \text{tr}(C) < \infty$). Then, by continuity of the inner product,

$$(Ce_l; e_k) = \sum_{j=0}^{\infty} \gamma_j (e_l; f_j) (f_j; e_k)$$

for every $k, l \geq 0$. Hence, for each $n \geq 1$,

$$\begin{aligned} (\mathcal{F}_n(Q)x; y) &= \sum_{k,l=0}^n \sum_{j=0}^{\infty} \gamma_j (e_l; f_j) (f_j; e_k) (Q F_l^* x; F_k^* y) \\ &= \sum_{j=0}^{\infty} \gamma_j \sum_{k,l=0}^n (Q(e_l; f_j) F_l^* x; (e_k; f_j) F_k^* y) \\ &= \sum_{j=0}^{\infty} \gamma_j \left\langle Q \sum_{l=0}^n (e_l; f_j) F_l^* x; \sum_{k=0}^n (e_k; f_j) F_k^* y \right\rangle \end{aligned}$$

for all $x, y \in H$, since addition is continuous. Thus, if $Q \in B^+[H]$, then

$$(\mathcal{F}_n(Q)x; x) = \sum_{j=0}^{\infty} \gamma_j \left\| Q^{1/2} \sum_{k=0}^n (e_k; f_j) F_k^* x \right\|^2 \quad \forall x \in H,$$

which implies that $\mathcal{F}_n(Q) \in B^+[H]$, for every $n \geq 1$. Therefore,

$$(P'_1) \quad \mathcal{F}(B^+[H]) \subseteq B^+[H],$$

because $B^+[H]$ is closed in $B[H]$. Now, by using the Schwarz inequality twice, and recalling that $\|\mathcal{F}_n(I)^{1/2}\| = \|\mathcal{F}_n(I)\|^{1/2}$ since $\mathcal{F}_n(I) \in B^+[H]$ for every $n \geq 1$, we get

$$\begin{aligned} |(\mathcal{F}_n(Q)x; y)| &\leq \|Q\| \sum_{j=0}^{\infty} \gamma_j \left\| \sum_{l=0}^n (e_l; f_j) F_l^* x \right\| \left\| \sum_{k=0}^n (e_k; f_j) F_k^* y \right\| \\ &\leq \|Q\| \left(\sum_{j=0}^{\infty} \gamma_j \left\| \sum_{l=0}^n (e_l; f_j) F_l^* x \right\|^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} \gamma_j \left\| \sum_{k=0}^n (e_k; f_j) F_k^* y \right\|^2 \right)^{1/2} \\ &= \|Q\| (\mathcal{F}_n(I)x; x)^{1/2} (\mathcal{F}_n(I)y; y)^{1/2} \\ &= \|Q\| \|\mathcal{F}_n(I)^{1/2} x\| \|\mathcal{F}_n(I)^{1/2} y\| \leq \|Q\| \|\mathcal{F}_n(I)\| \|x\| \|y\| \end{aligned}$$

for all $x, y \in H$ and every $n \geq 1$, so that

$$\|\mathcal{F}_n(Q)\| = \sup_{\|x\|=\|y\|=1} |(\mathcal{F}_n(Q)x; y)| \leq \|Q\| \|\mathcal{F}_n(I)\|$$

for all $Q \in B[H]$ and every $n \geq 1$. Hence, $\|\mathcal{F}_n\| = \|\mathcal{F}_n(I)\|$, because

$$\|\mathcal{F}_n(I)\| \leq \|\mathcal{F}_n\| = \sup_{\|Q\|=1} \|\mathcal{F}_n(Q)\| \leq \|\mathcal{F}_n(I)\|,$$

for every $n \geq 1$. Therefore, since $\mathcal{F}_n \rightarrow \mathcal{F}$ as $n \rightarrow \infty$ in $B[B[H]]$,

$$(P'_2) \quad \|\mathcal{F}\| = \|\mathcal{F}(I)\|.$$

Finally, take an arbitrary $W \in G^+[H]$ and set $\tilde{F}_k = W^{-1}F_kW \in B[H]$ for every $k \geq 0$. Thus,

$$W^{-1}\mathcal{F}W(Q) = W^{-1}\mathcal{F}(WQW)W^{-1} = \sum_{k,l=0}^{\infty} (Ce_l; e_k) \tilde{F}_k Q \tilde{F}_l^* \quad \forall Q \in B[H].$$

Hence, (P'_1) implies (P_1) , and (P'_2) implies (P_2) . □

Lemma. For any $W \in G^+[H]$ and any $\alpha \in (0, \infty)$,

$$\begin{aligned} \|W^{-1}\mathcal{F}W\| \leq \alpha &\iff \alpha W^2 - \mathcal{F}(W^2) \in B^+[H], \\ \|W^{-1}\mathcal{F}W\| < \alpha &\iff \alpha W^2 - \mathcal{F}(W^2) \in G^+[H]. \end{aligned}$$

Proof. Take an arbitrary $W \in G^+[H]$ and set $\tilde{\mathcal{F}} = W^{-1}\mathcal{F}W \in B[B[H]]$, so that $\tilde{\mathcal{F}}(I) = W^{-1}\mathcal{F}(W^2)W^{-1} \in B[B[H]]$. Thus, for any $\alpha \in (0, \infty)$,

$$((\alpha W^2 - \mathcal{F}(W^2))x; x) = ((\alpha I - \tilde{\mathcal{F}}(I))Wx; Wx) \quad \forall x \in H.$$

Hence, $\alpha W^2 - \mathcal{F}(W^2) \in B^+[H]$ ($\in G^+[H]$) if and only if $\alpha I - \tilde{\mathcal{F}}(I) \in B^+[H]$ ($\in G^+[H]$), which in turn is equivalent (cf. [7]) to $\|\tilde{\mathcal{F}}(I)^{1/2}\| \leq \alpha^{1/2}$ ($< \alpha^{1/2}$), since $\tilde{\mathcal{F}}(I) \in B^+[H]$ by Proposition (P_1) . However, by Proposition (P_2) , $\|\tilde{\mathcal{F}}\| = \|\tilde{\mathcal{F}}(I)\| = \|\tilde{\mathcal{F}}(I)^{1/2}\|^2$. Thus, the above inequality is equivalent to $\|\tilde{\mathcal{F}}\| \leq \alpha$ ($< \alpha$), which completes the proof. □

4. CONCLUSION

In this final section we shall conclude the announced proof for the equivalence between assertions (2) and (3) with $\mathcal{F} = \mathcal{F}_F$ replaced by $\mathcal{F} = \mathcal{F}_F + \mathcal{T}$. As commented on section 2, this supplies a necessary and sufficient condition for the convergence preserving property between input and state correlation sequences, as required in the mean-square stability problem, for infinite-dimensional discrete bilinear systems.

Theorem. *The following assertions are equivalent.*

- (a) $r_o(\mathcal{F}) < 1$.
- (b) \mathcal{F} is similar to a strict contraction.
- (c) There exists $W \in G^+[H]$ such that $W - \mathcal{F}(W) \in G^+[H]$.
- (d) For every $V \in G^+[H]$ there exists $W \in G^+[H]$ such that

$$V = W - \mathcal{F}(W).$$

Moreover, if the above holds, then the solution $W \in G^+[H]$ of the Lyapunov-like equation $V = W - \mathcal{F}(W)$, for any $V \in G^+[H]$, is unique and given by

$$W = \sum_{j=0}^{\infty} \mathcal{F}^j(V) = (I - \mathcal{F})^{-1}(V)$$

(with I standing for the identity in $B[B[H]]$), where the above convergence is in the uniform topology of $B[B[H]]$.

Proof. It is trivially verified that (d) \implies (c), and (c) \implies (b) according to the previous Lemma. Since similarity preserves the spectrum, $r_o(\mathcal{F}) = r_o(W^{-1}\mathcal{F}W) \leq \|W^{-1}\mathcal{F}W\|$ for every $W \in G[B[H]]$, so that (b) \implies (a). To verify that (a) \implies (d), proceed as follows. Suppose $r_o(\mathcal{F}) < 1$ (so that $\sum_{j=0}^{\infty} \|\mathcal{F}^j\| < \infty$) and take an arbitrary $V \in G^+[H]$. According to Proposition (P₁) (with $W = I$) we get by induction on j that $\mathcal{F}^j(V) \in B^+[H]$ for every $j \geq 0$. Then, for every $n \geq 0$,

$$\|(V^{1/2})^{-1}\|x\|^2 \leq \|V^{1/2}x\|^2 \leq \sum_{j=0}^n \|\mathcal{F}^j(V)^{1/2}x\|^2 \leq \left(\sum_{j=0}^{\infty} \|\mathcal{F}^j\|\right) \|V\| \|x\|^2$$

for all $x \in H$. Hence, we may define a map $\Phi : H \rightarrow \ell_2(H)$, given by

$$\Phi x = \bigoplus_{j=0}^{\infty} \mathcal{F}^j(V)^{1/2}x \quad \forall x \in H,$$

which is clearly linear and bounded, with

$$\|\Phi x\|_{\ell_2(H)}^2 = \sum_{j=0}^{\infty} \|\mathcal{F}^j(V)^{1/2}x\|^2 \quad \forall x \in H,$$

so that it has a bounded inverse on its (closed) range. Thus, $\Phi^*\Phi \in G^+[H]$. By the continuity of the inner product we get (cf. proof of Proposition (P₁)), for every $x \in H$,

$$\begin{aligned} \langle \mathcal{F}(\Phi^*\Phi)x; x \rangle &= \left\langle \sum_{k,l=0}^{\infty} (C e_l; e_k) F_k \Phi^* \Phi F_l^* x; x \right\rangle \\ &= \sum_{k,l=0}^{\infty} (C e_l; e_k) \langle \Phi F_l^* x; \Phi F_k^* x \rangle_{\ell_2(H)}. \end{aligned}$$

However, for each $k, \ell \geq 0$ and every $x \in H$,

$$\begin{aligned} \langle \Phi F_\ell^* x; \Phi F_k^* x \rangle_{\ell_2(H)} &= \sum_{j=0}^{\infty} \langle \mathcal{F}^j(V)^{1/2} F_\ell^* x; \mathcal{F}^j(V)^{1/2} F_k^* x \rangle \\ &= \sum_{j=0}^{\infty} \langle F_k \mathcal{F}^j(V) F_\ell^* x; x \rangle = \left\langle F_k \left(\sum_{j=0}^{\infty} \mathcal{F}^j(V) \right) F_\ell^* x; x \right\rangle, \end{aligned}$$

since $\{\sum_{j=0}^n \mathcal{F}^j; n \geq 0\}$ converges in $B[B[H]]$ whenever $r_\sigma(\mathcal{F}) < 1$. Therefore,

$$\begin{aligned} \langle \mathcal{F}(\Phi^* \Phi)x; x \rangle &= \left\langle \sum_{k, \ell=0}^{\infty} \langle C e_\ell; e_k \rangle F_k \left(\sum_{j=0}^{\infty} \mathcal{F}^j(V) \right) F_\ell^* x; x \right\rangle \\ &= \left\langle \mathcal{F} \left(\sum_{j=0}^{\infty} \mathcal{F}^j(V) \right) x; x \right\rangle = \sum_{j=0}^{\infty} \langle \mathcal{F}^{j+1}(V)x; x \rangle \\ &= \sum_{j=1}^{\infty} \|\mathcal{F}^j(V)^{1/2} x\|^2 = \|\Phi x\|_{\ell_2(H)}^2 - \|V^{1/2} x\|^2 \end{aligned}$$

for every $x \in H$, so that

$$\langle (V - \Phi^* \Phi + \mathcal{F}(\Phi^* \Phi))x; x \rangle = \|V^{1/2} x\|^2 - \|\Phi x\|_{\ell_2(H)}^2 + \langle \mathcal{F}(\Phi^* \Phi)x; x \rangle = 0$$

for all $x \in H$. Hence, $V = \Phi^* \Phi - \mathcal{F}(\Phi^* \Phi)$. Thus, (a) \implies (d) with $W = \Phi^* \Phi \in G^+[H]$. Moreover, such an operator is unique. Indeed, if $V = W - \mathcal{F}(W) \in G^+[H]$ for some $W \in G^+[H]$, then

$$\|\mathcal{F}^j(V)^{1/2} x\|^2 = \langle \mathcal{F}^j(V)x; x \rangle = \|\mathcal{F}^j(W)^{1/2} x\|^2 - \|\mathcal{F}^{j+1}(W)^{1/2} x\|^2$$

for all $x \in H$ and every $j \geq 0$. Hence, for every $x \in H$,

$$\|\Phi x\|_{\ell_2(H)}^2 = \lim_{n \rightarrow \infty} \sum_{j=0}^n \|\mathcal{F}^j(V)^{1/2} x\|^2 = \lim_{n \rightarrow \infty} \left(\|W^{1/2} x\|^2 - \|\mathcal{F}^{n+1}(W)^{1/2} x\|^2 \right) = \|W^{1/2} x\|^2,$$

since $\|\mathcal{F}^n(W)^{1/2} x\|^2 \leq \|\mathcal{F}^n\| \|W\| \|x\|^2 \rightarrow 0$ as $n \rightarrow \infty$, because $r_\sigma(\mathcal{F}) < 1$. Therefore,

$$\langle (\Phi^* \Phi - W)x; x \rangle = \|\Phi x\|_{\ell_2(H)}^2 - \|W^{1/2} x\|^2 = 0 \quad \forall x \in H,$$

so that $W = \Phi^* \Phi$, which proves uniqueness. Finally, if $W \in G^+[H]$ is the solution of $V = W - \mathcal{F}(W) \in G^+[H]$, then $\sum_{j=0}^n \mathcal{F}^j(V) = \sum_{j=0}^n (\mathcal{F}^j(W) - \mathcal{F}^{j+1}(W)) = W - \mathcal{F}^{n+1}(W) \in B^+[H]$ for each $n \geq 0$. Thus,

$$\left\| \sum_{j=0}^n \mathcal{F}^j(V) - W \right\| = \|\mathcal{F}^{n+1}(W)\| \leq \|\mathcal{F}^{n+1}\| \|W\| \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

because $r_\sigma(\mathcal{F}) < 1$. Hence, $\sum_{j=0}^n \mathcal{F}^j(V) \rightarrow W$ as $n \rightarrow \infty$ in $B[H]$. However, since $r_\sigma(\mathcal{F}) < 1$ actually implies that $(I - \mathcal{F}) \in G[B[H]]$ and that $\{\sum_{j=0}^n \mathcal{F}^j \in B[B[H]]; n \geq 0\}$ converges in $B[B[H]]$ to $(I - \mathcal{F})^{-1} \in G[B[H]]$, we get the final claim. \square

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