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ABSTRACT: An expression for the U(3) content of the matrix elements of one- and two-body operators in Elliott's basis is obtained. Three alternative ways of evaluating this content with increasing performance in computing time are presented. All of them allow an exact representation of that content in terms of integers, avoiding rounding errors in the computer codes. The role of dual bases in dealing with non-orthogonal bases is also clarified.

1. INTRODUCTION

The use of matrix elements of one- and two-body operators is of fundamental importance in Nuclear Structure. In developing non-relativistic microscopic models of the nucleus one considers a system of A nucleons with space, spin and isospin degrees of freedom subject to a Hamiltonian containing kinetical, Coulomb and nuclear terms, the later expressed as two-body operators representing the N-N forces. The energy levels of the bound states and the corresponding wave functions are obtained by diagonalizing this Hamiltonian in a model subspace of the total Hilbert space of the problem. Besides, many quantities of interest in nuclear spectroscopy can be expressed in terms of matrix elements of one-body operators between initial and final nuclear states.

The total antisymmetry of the wave functions, required by Pauli principle, is accomplished by coupling space basis functions having definite permutational symmetry with spin-isospin basis functions with conjugate permutational symmetry. We will be concerned here only with the space degrees of freedom of the nucleons. A complete treatment of the spin-isospin degrees of freedom for nuclear collective models can be found in reference[1]).

The U(3) group arises quite naturally in Nuclear Physics as a consequence of the 3-dimensional nature of physical space. As we will see below, in evaluating its contribution to the matrix elements of one- and two-body operators the most difficult part to compute is giver by

$$F_1([\bar{E}][\varepsilon][\varepsilon'];[E][E'][E_0];KL,K'L',K_0L_0)=$$

$$\sum_{\beta}\left\{\begin{matrix}[\bar{E}] & [\varepsilon] & [E]_{\beta} \\ [\bar{E}] & [\varepsilon'] & [E'] \\ [0] & [E_0] & [E_0]\end{matrix}\right\} C\begin{matrix}[E_{01}E_{02}E_{03}] & [E'_1E'_2E'_3] & \beta[E_1E_2E_3] \\ K_0\ L_0 & K'\ L' & K\ L\end{matrix} \qquad (1.1)$$

that we will call the U(3) isoscalar block and denote henceforth as U3IB for short.

In (1.1), the C's are isoscalar factors of U(3) Clebsch-Gordan coefficients in the Elliott's basis[2]) and {...} is a U(3) 9-j symbol that, since one of its entry is zero, can be related to an U(3) 6-j symbol or Racah coefficient. By short we wrote $[E]\equiv[E_1E_2E_3]$, etc, to denote U(3) irreducible representations (irreps). The label β is an outer multiplicity label used to distinguish the eventual multiple appearance of the U(3) irrep [E] in the reduction of the Kronecker product of irreps $[E_0]$ and [E'].

Since the U3IB is model independent and independs also on the shape of the interaction, it worths to compute it once for all and this will be the aim of this work. Taking a look in the literature of the U(3) group theoretical quantities (References[3-25]) to quote some of them) one faces two main problems: most of the isoscalar factors are given in the U(3) canonical chain and the outer multiplicity is hard to work. Besides, in applications to nuclear systems, when the complete matrix elements of one- and two-body interactions are needed, for fixed [E] and [$\bar{E}$] allowed[27]) by Pauli principle, the U3IB will be multiplied by model dependent terms and summed in [$\bar{E}$], and also multiplied by factors depending on the shape of the interactions and summed in [ε], [ε'] and [E_o]. One sees that a long calculation envolving many sums is needed, so it is necessary to develop formulas for the U3IB very efficient regarding computing time. Another problem that arises begining with A ~ 40, due to the great number of sums and the extension of their ranges, is that of rounding errors when the calculations are performed using floating point arithmetic.

For these reasons we preferred to follow the line of work started in reference[28]) that we henceforth refer to as (I). To enhance the computer efficience we develop three different ways of computing the U3IB. Two of them do not deal with the outer multiplicity at all, a result that can be antecipated from (1.1) since the outer multiplicity label is summed over. The last one uses particular labellings of the outer multiplicity[28]) ease to work with.

Many quantities in group theory have the property of being expressed as a polynomial with integer coefficients multiplied by the square root of rational factors. In (I) we named the expressions with such an structure of polynomial-type expressions. Those expressions have the nice property of being expressed in terms of intergers $n_1, \ldots, n_4$ as $(n_1/n_2)(n_3/n_4)^{1/2}$. Therefore by using appropriate computer codes for dealing with integers of arbitrary size, the polynomial-type expressions can be computed exactly. We insisted in building our formulas out of polynomial-type expressions and the resulting formulas turned out to be also of polynomial-type. In this way we avoided completely the rounding errors. Unfortunately the final polynomial-type expressions are too long and improper to be printed. [This printing problem also arose in (I) but, since the group-theoretical quantities described there were much simpler, it could be solved by the use of an appropriate notation.] Nevertheless we give in all three cases all the ingredients out of which the final polynomial-type expression can be constructed.

In section 2 we present a derivation of the analogue of the U3IB for the U(3) canonical basis. In section 3 we derive eq. (1.1) for the U3IB in Elliott's basis and obtain an alternative expression for it using its analogue obtained in section 2. In section 4 a third alternative expression is obtained by computing separately the U(3) 9-j symbol and the isoscalar factors using particular classification index β. In section 5 a few finishing remarks are made and finally, in the Appendix we set up the portion of the theory of dual bases needed in this work.

2. CALCULATION OF THE CORRESPONDING U3IB IN CANONICAL BASIS

Instead of the A position vectors r_i of the nucleons one introduces the orthogonal linear combinations of them known as Jacobi vectors:

$$\vec{\rho}_i = \frac{1}{\sqrt{i(i+1)}}\left(\sum_{j=1}^{i} r_j - i\, r_{i+1}\right)\ ; \qquad i=1,2,\ldots,A-1$$

$$\vec{\rho}_A = \frac{1}{\sqrt{A}}\sum_{j=1}^{A} r_j = \sqrt{A}\, R \tag{2.1}$$

The $\vec{\rho}_i$ for $i=1$ to $(A-1)$, being translacional invariant, describe the relative motion. Sometimes it is convenient to replace $\vec{\rho}_{A-2}$ and $\vec{\rho}_{A-1}$ by

$$\vec{\rho}_s = \frac{1}{\sqrt{2(A-1)}}\left(\sqrt{A}\,\vec{\rho}_{A-2} + \sqrt{A-2}\,\vec{\rho}_{A-1}\right),$$

$$\vec{\rho}_a = \frac{1}{\sqrt{2(A-1)}}\left(-\sqrt{A-2}\,\vec{\rho}_{A-2} + \sqrt{A}\,\vec{\rho}_{A-1}\right) = \frac{1}{\sqrt{2}}\left(r_{A-1} - r_A\right) \tag{2.2}$$

The use of translacional invariant two-body forces allows us to split the relative from the center of mass motion. In this way the center of mass motion will be the one of a free particle and we can concentrate our attention only to the relative motion.

As a basis for the Hilbert space for the relative motion one usually uses 3(A-1)-dimensional isotropic harmonic oscillator wave functions with definite oscillator quanta E , A-dimensional permutational symmetry [f], 3-dimensional unitary symmetry [E]rst and other symmetries to fully characterize the basis functions.

The so-called unitary scheme[29,30]) uses the basis functions

$$\left\langle \vec{\rho}_1,\ldots,\vec{\rho}_{A-1} \middle| \begin{matrix}[E]rst\\ \delta(\omega)\alpha[f]g\end{matrix} \right\rangle \qquad (2.3)$$

whose labels are related to the following chain of subgroups

$$\begin{matrix} U(3A-1) & \supset & U(3) & \times & U(A-1) & \supset & O(A-1) & \supset & S(A) \\ [E0\ldots0] & & [E_1E_2E_3] & & [E_1E_2E_30\ldots0] & \delta & (\omega) & \alpha & [f]g \end{matrix} \qquad (2.4)$$

In the first line of (2.4), U, O and S denote unitary, orthogonal and symmetric groups. In the second line, below each group is its irreducible representation (irrep) and below the inclusion symbol $\supset$,the indices that label the multiple appearance of a given irrep of the subgroup. Each of those labels in fact represents a set of labels that will be explicited only when needed.

The group U(A-1) deals with the (quasi)particle degrees of freedom while the space degrees of freedom are dealt by U(3), in this section reduced to the canonical chain

$$\begin{matrix} U(3) & \supset & U(2) & \supset & U(1) \\ [E_1E_2E_3] & & [rs] & & [t] \end{matrix} \quad . \qquad (2.5)$$

The use of totally antisymmetric functions and symmetric one- and two-body operators

$$\hat{O}_1 = \sum_{i=1}^{A} \hat{W}_1(r_i)\, \hat{U}_1(\vec{\sigma}_i \vec{\tau}_i); \quad \hat{O}_2 = \sum_{i<j=2}^{A} \hat{W}_2(r_i - r_j)\, \hat{U}_2(\vec{\sigma}_i - \vec{\sigma}_j, \vec{\tau}_i - \vec{\tau}_j) \tag{2.6}$$

leads to the well known result that in the calculation of the matrix elements of those operators the one- and two-body operators can be replaced by $A\, W_1(r_i)\, U_1(\vec{\sigma}_i \vec{\tau}_i)$ and $(1/2)A(A-1)\, W_2(\vec{r}_j - \vec{r}_k)\cdot U_2(\vec{\sigma}_j - \vec{\sigma}_k, \vec{\tau}_j - \vec{\tau}_k)$ with $i, j \neq k$ arbitrary. So we many replace

$$\hat{O}_1 \rightarrow A\, \hat{W}_1(r_A)\, \hat{U}_1(\vec{\sigma}_A \vec{\tau}_A) = A\, \hat{W}_1(-\sqrt{(A-1)/A}\, \vec{\rho}_{A-1} + R)\, \hat{U}_1(\vec{\sigma}_A \vec{\tau}_A),$$

$$\hat{O}_2 \rightarrow \frac{1}{2} A(A-1)\, \hat{W}_1(r_{A-1} - r_A)\, \hat{U}_2(\vec{\sigma}_{A-1} - \vec{\sigma}_A, \vec{\tau}_{A-1} - \vec{\tau}_A) =$$

$$\frac{1}{2} A(A-1)\, \hat{W}_2(\sqrt{2}\, \vec{\rho}_a)\, \hat{U}_2(\vec{\sigma}_{A-1} - \vec{\sigma}_A, \vec{\tau}_{A-1} - \vec{\tau}_A)\,. \tag{2.7}$$

Therefore, in the calculation of matrix elements of one- and two-body operators it is convenient to isolate the $\vec{\rho}_{A-1}$ or $\vec{\rho}_a$ dependence in the basis functions that span the Hilbert space of the wave functions. This is accomplished by eqs. (17.3) and (17.5) of reference[30] as

$$\left\langle \vec{\rho}_1 \ldots \vec{\rho}_{A-1} \middle| \begin{matrix} [E]rst \\ \delta(\omega)\alpha[f][\bar{f}]\bar{g} \end{matrix} \right\rangle =$$

$$\sum_{[\bar{E}][E_{A-1}]\bar{\delta}(\bar{\omega})\bar{\alpha}} A^{([E][\bar{f}])}_{[\bar{E}]\bar{\delta}(\bar{\omega})\bar{\alpha}[E_{A-1}],[E_{A-1}]\delta(\omega)\alpha[f]} \times$$

$$\left[\sum_{\substack{\bar{r}\,\bar{s}\,\bar{t} \\ r_{A-1}s_{A-1}t_{A-1}}} C^{[\bar{E}]\;[E_{A-1}]\qquad [E]}_{\bar{r}\bar{s}\bar{t}\;r_{A-1}s_{A-1}t_{A-1}\;rst} \times \left\langle \vec{\rho}_1 \cdots \vec{\rho}_{A-2} \middle| \begin{matrix} [\bar{E}]\bar{r}\;\bar{s}\;\bar{t} \\ \bar{\delta}(\bar{\omega})\bar{\alpha}[\bar{f}][\bar{f}]\bar{g} \end{matrix} \right\rangle \times \left\langle \vec{\rho}_{A-1} \middle| [E_{A-1}]\; r_{A-1}s_{A-1}t_{A-1} \right\rangle \right] \qquad (2.8)$$

$$\left\langle \vec{\rho}_1 \cdots \vec{\rho}_{A-1} \middle| \begin{matrix} [E]rst \\ \delta(\omega)\alpha[f][\bar{\bar{f}}]\bar{\bar{g}}f_{12} \end{matrix} \right\rangle = \sum_{[E_a][\bar{\bar{E}}][\bar{\bar{E}}]\bar{\bar{\delta}}(\bar{\bar{\omega}})\bar{\bar{\alpha}}} A^{([E][f][f_{12}])}_{[\bar{\bar{E}}][\bar{\bar{E}}]\bar{\bar{\delta}}(\bar{\bar{\omega}})\bar{\bar{\alpha}}[E_a],\,\delta(\omega)\alpha[f]} \left[\sum_{\bar{r}\,\bar{s}\,\bar{t}\;r_a s_a t_a} C^{[\bar{E}]\;[E_a]\quad [E]}_{\bar{r}\bar{s}\bar{t}\;r_a s_a t_a\;rst} \times \right.$$

$$\left. \left\langle \vec{\rho}_a \middle| [E_a] r_a s_a t_a f_{12} \right\rangle \left\langle \vec{\rho}_1, \vec{\rho}_2 \cdots \vec{\rho}_{A-3}, \vec{\rho}_a \middle| \begin{matrix} [\bar{E}]\bar{r}\;\bar{s}\;\bar{t} \\ [\bar{\bar{E}}]\bar{\bar{\beta}}(\bar{\bar{\omega}})\bar{\bar{\alpha}}[\bar{\bar{f}}]\bar{\bar{g}} \end{matrix} \right\rangle \right] \qquad (2.9)$$

where the expansion coefficients A's are related to fractional parentage coefficients and the C's are U(3) Clebsch-Gordan coefficients. Each bar over some subgroup label means that it refers to a subgroup of the same kind of the parent group with one dimension less. The bases in $\vec{\rho}_a$ and $\vec{\rho}_{A-1}$ are simply 3-dimensional harmonic oscillators bases (what implies $[E_{A-1}] \equiv [E_{A-1}00]$ and $[E_a] \equiv [E_a 00]$) while the bases in the other $\vec{\rho}$ variables are of general type (2.3) with the subgroup chain of S(A) specialized to

$$\begin{matrix} S(A) \supset S(A-1) \\ [f] \qquad [\bar{f}]\bar{g} \end{matrix} \quad , \qquad \text{for one-body operators}$$

$$\begin{matrix} S(A) \supset (S(\bar{\bar{A-2}}) \dotplus S(2) \\ [f] \qquad [\bar{\bar{f}}]\bar{\bar{g}} \qquad f_{12} \end{matrix} \qquad \text{for two-body operators.} \qquad (2.10)$$

In computing the matrix elements of one- and two-body operators (2.7) in the bases (2.8) or (2.9) one sees that the integrations in the variables other than $\vec{\rho}_{A-1}$ or $\vec{\rho}_a$ are immediately performed by orthogonality since the operators envolve only $\vec{\rho}_{A-1}$ or $\vec{\rho}_a$. We have then to evaluate only the U(3) part of the matrix elements.

Considering an U(3) decomposition of the two-body interaction

$$\hat{W}_2(\sqrt{2}\,\vec{\rho}_a) = \sum_{[E_0]r_0s_0t_0} Q^{[E_0]}_{r_0s_0t_0}(W)\, \hat{W}^{[E_0]}_{r_0s_0t_0}(\sqrt{2}\,\vec{\rho}_a) \qquad (2.11)$$

and using the U(3) Wigner-Eckart theorem

$$\langle [E_a] r_a s_a t_a | W^{[E_0]}_{r_0s_0t_0} | [E_a'] r_a' s_a' t_a' \rangle = C^{[E_0]\ [E']\ [E_a]}_{r_0s_0t_0\ r_a's_a't_a\ r_as_at_a} \langle [E_a] \| W^{[E_0]} \| [E'_a] \rangle. \qquad (2.12)$$

the U(3) part of the matrix element of the two-body interaction is

$$\sum_{[E_0]r_0s_0t_0} Q^{[E_0]}_{r_0s_0t_0}(w) \langle [E] \| W^{[E_0]} \| [E'] \rangle \times$$

$$F([\bar{E}][\varepsilon][\varepsilon'][E][E'][E_0]; rst, r's't', r_0s_0t_0) \qquad (2.13)$$

with

$$F([\bar{E}][\varepsilon][\varepsilon'][E][E'][E_0]; rst, r's't', r_0s_0t_0) = \sum_{\substack{\bar{r}\,\bar{s}\,\bar{t} \\ r_\varepsilon s_\varepsilon t_\varepsilon r_{\varepsilon'} s_{\varepsilon'} t_{\varepsilon'}}}$$

$$C^{[\bar{E}]\ [\varepsilon]\ \ [E]}_{\bar{r}\bar{s}\bar{t}\ r_\varepsilon s_\varepsilon t_\varepsilon\ rst}\ C^{[\bar{E}]\ [\varepsilon']\ \ [E']}_{\bar{r}\bar{s}\bar{t}\ r_{\varepsilon'} s_{\varepsilon'} t_{\varepsilon'}\ r's't'}\ C^{[E_0]\ [\varepsilon']\ \ [\varepsilon]}_{r_0 s_0 t_0\ r_{\varepsilon'} s_{\varepsilon'} t_{\varepsilon'}\ r_\varepsilon s_\varepsilon t_\varepsilon}$$

(2.14)

In (2.13) and (2.14) we made, by convenience, the following relabeling of the summation indices

$[E_a] r_a s_a t_a \rightarrow [\varepsilon] r_\varepsilon s_\varepsilon t_\varepsilon$; $[E'_a] r'_a s'_a t'_a \rightarrow [\varepsilon'] r_{\varepsilon'} s_{\varepsilon'} t_{\varepsilon'}$.

The same treatment for the one-body operators yields to the same result (2.13) and (2.14) now $[\varepsilon] r_\varepsilon s_\varepsilon t_\varepsilon$ meaning $[E_{A-1}] r_{A-1} s_{A-1} t_{A-1}$ and mutatis mutatis for $[\varepsilon'] r's't'$.

One can see by (2.14) that in computing the U(3) content of one- and two-body operators given by (2.13) the most difficult part to compute is the F(...). Fortunately it is model independent and depends on the operators only through their irreducible character $[E_0]\ r_0 s_0 t_0$ and not on their shapes. Since all the allowed values of $[E_0] r_0 s_0 t_0$ are uniquely defined by [E] and [E'] (see eq. (2.15) below) one can compute the set of F(...) for all allowed $[E_0] r_0 s_0 t_0$ that suit for any microscopic model and any interaction. For this reason one focus now our attention to the calculation of the F(...)'s.

The RHS of eq. (2.14) can be identified with the LHS of eq. (9.5) of[29]) with $\chi = [0]$. Therefore, taking into account that the U(3) Kronecker product involving one symmetric representation ([ε] and [ε'] in our case) is multiplicity free one obtains

$$F([\bar{E}][\epsilon][\epsilon'][E][E'][E_0];rst,r's't',r_0s_0t_0) =$$

$$\sum_\beta \left\{ \begin{matrix} [\bar{E}] & [\epsilon] & [E] \\ & & \beta \\ [\bar{E}] & [\epsilon'] & [E'] \\ [0] & [E_0] & [E_0] \end{matrix} \right\} C \begin{matrix} [E_0] & [E'] & \beta[E] \\ r_0s_0t_0 & r's't' & rst \end{matrix} \quad (2.15)$$

where

$$\sum_\beta \left\{ \begin{matrix} [\bar{E}] & [\epsilon] & [E] \\ & & \beta \\ [\bar{E}] & [\epsilon'] & [E'] \\ [0] & [E_0] & [E_0] \end{matrix} \right\} = \langle([0],[\bar{E}])[\bar{E}],([E_0],[\epsilon'])[\epsilon])[E]| \times$$
$$|(([0],[E_0])[E_0],([\bar{E}],[\epsilon'])[E'])\beta[E]\rangle \quad (2.16)$$

is a U(3) 9-j symbol.

The isoscalar factor of U(3) Clebsch-Gordan coefficient in canonical chain is defined by

$$C \begin{matrix} [E'] & [E''] & \beta[E] \\ r's't' & r''s''t'' & rst \end{matrix} = C \begin{matrix} [E'] & [E''] & \beta[E] \\ [r's'] & [r''s''] & [rs] \end{matrix} \times$$

$$C \begin{matrix} (r'-s')/2 & (r''-s'')/2 & (r-s)/2 \\ (r'+s'-2t')/2 & (r''+s''-2t'')/2 & (r+s-2t) \end{matrix} ? \quad (2.17)$$

where the first C in the RHS is an U(3) isoscalar factor and the second an usual SU(2) Clebsch-Gordan coefficient.

Using (2.17) in (2.15) one obtains that the isoscalar part of F(...) is the U(3) isoscalar block in canonical chain:

$$F_I([\bar{E}][\epsilon][\epsilon'][E][E'][E_0];rs,r's',r_0s_0) =$$

$$\sum_{\beta}\left\{\begin{matrix} [\bar{E}] & [\varepsilon] & [E] \\ & & \beta \\ [\bar{E}] & [\varepsilon'] & [E'] \\ [0] & [E_0] & [E_0] \end{matrix}\right\} C\begin{matrix} [E_0] & [E'] & \beta[E] \\ [r_0 s_0] & [r's'] & [rs] \end{matrix} \quad . \tag{2.18}$$

Using (2.17) in (2.14) one can split the dependence in the magnetic quantum numbers and by the use of eq(3.12) of ref.[31]) the SU(2) Clebsch-Gordan coefficients can be gathered into an usual 6-j symbol times the same SU(2) Clebsch-Gordan coefficient that appeared in (2.15). Therefore, equating the isoscalar part of (2.14) to that of (2.15) one obtain an alternative expresion for the U3IB in canonical chain, namely:

$$\sum_{\beta}\left\{\begin{matrix} [\bar{E}] & [\varepsilon] & [E] \\ & & \beta \\ [\bar{E}] & [\varepsilon'] & [E'] \\ [0] & [E_0] & [E_0] \end{matrix}\right\} C\begin{matrix} [E_0] & [E'] & \beta[E] \\ [r_0 s_0] & [r's'] & [rs] \end{matrix} =$$

$$\sum_{\bar{r}\,\bar{s}\,r_{\varepsilon}s_{\varepsilon'}} (-)^{(r_{\varepsilon}+r'-s'+\bar{r}-\bar{s}+r_0-s_0)/2} \left[(r'-s'+1)(r_{\varepsilon}+1)\right]^{1/2} \times$$

$$C\begin{matrix} [\bar{E}_1\bar{E}_2\bar{E}_3] & [\varepsilon+E_{02},E_{02},E_{02}] & [E_1+E_{02},E_2+E_{02},E_3+E_{02}] \\ [\bar{r}\,\bar{s}] & [r_{\varepsilon}+E_{02},E_{02}] & [r+E_{02},s+E_{02}] \end{matrix} \times$$

$$C\begin{matrix} [\bar{E}_1\bar{E}_2\bar{E}_3] & [\varepsilon'00] & [E'_1E'_2E'_2] \\ [\bar{r}\,\bar{s}] & [r_{\varepsilon},0] & [r's'] \end{matrix} \; C\begin{matrix} [E_{01}E_{02}E_{03}] & [\varepsilon'00] & [\varepsilon+E_{02},E_{02},E_{02}] \\ [r_0 s_0] & [r_{\varepsilon},0] & [r_{\varepsilon}+E_{02},E_{02}] \end{matrix} \times$$

$$\times\left\{\begin{matrix} r_{\varepsilon}/2 & (r-s)/2 & (\bar{r}-\bar{s})/2 \\ (r'-s')/2 & r_{\varepsilon'}/2 & (r_0-s_0)/2 \end{matrix}\right\} \tag{2.19}$$

where we have used equivalent transladed U(3) irreps to assure balance in U(3) irreducible labels.

One sees then that the U3IB in canonical chain can be calculated using U(3) isoscalar factors with one simmetic representation that are multiplicity free.

Taking $n=3$ in eq. (9) of ref.[32]) and rearanging terms into generalized Pochhammer symbols

$$P(n;k;d) = \begin{cases} 1 & \text{for } k = 0 \\ n(n+d)(n+2d)\dots(n+(k-1)d) & \text{for } k=1,2\dots \end{cases} \qquad (2.20)$$

introduced in (I) one has the following explicit polynomial-type expression for the U(3) isoscalar factors with one symmetric representation:

$$C\begin{matrix} [\omega_1\omega_2\omega_3] & [p+\alpha,\alpha,\alpha] & [f_1+\alpha,f_2+\alpha,f_3+\alpha] \\ [n_2\ n_3] & [q+\alpha,\alpha] & [h_2+\alpha,\ h_3+\alpha] \end{matrix} = C\begin{matrix} [\omega_1\omega_2\omega_3] & [p\ 0\ 0] & [f_1f_2f_3] \\ [n_2\ n_3] & [q\ 0] & [h_2\ h_3] \end{matrix}$$

$$= \frac{(-)^{n_2+n_3}}{(h_2-f_2)!(h_2-f_3+1)!(\omega_1-f_2)!(\omega_1-f_3+1)!(\omega_2-f_3)!}\Bigg[\frac{(n_2-n_3+1)(f_1-f_2+1)}{(\omega_1-n_2)!(\omega_1-n_3+1)!}$$

$$\times \frac{(f_1-f_3+2)(f_2-f_3+1)(n_2-\omega_2)!(p-q)!(h_2-n_2)!(h_2-n_3+1)!(h_3-n_3)!}{(\omega_2-n_3)!(\omega_1-f_2)!(\omega_1-f_3+1!)!(\omega_2-f_3)!(f_1-\omega_1)!(f_1-\omega_2+1)!(f_2-\omega_2)!} \times$$

$$\frac{(h_2-f_2)!(h_2-f_3+1)!}{(f_1-\omega_3+2)!(n_2-\omega_3+1)!(h_3-f_3)!(n_3-\omega_3)!(f_1-h_2)!(f_1-h_3+1)!(f_2-\omega_3+1)!} \times$$

$$\frac{(f_2-h_3)!}{(f_3-\omega_3)!(n_2-h_3)!}\Bigg]^{1/2} \mathcal{P}\left(\begin{matrix} [\omega_1\omega_2\omega_3] & p & [f_1f_2f_3] \\ [n_2n_3] & q & [h_2h_3] \end{matrix}\right) \qquad (2.21)$$

with

$$\mathcal{P}\left(\begin{matrix} [\omega_1\omega_2\omega_3] & p & [f_1f_2f_3] \\ [n_2n_3] & q & [h_2h_3] \end{matrix}\right) = \sum_{x=f_2}^{\omega_1}\ \sum_{y=f_2}^{\omega_2} (-)^{x+y}\, P(x-n_2+1;\omega_1-x;1) \times$$

$$P(x-f_3+2;\omega_1-x;1)P(x-h_3+2;\omega_1-x;1)P(x-f_2+1;\omega_1-x;1)P(h_2-x+1;x-f_2;1) \times$$
$$P(\omega_1-x+1;x-f_2;1)P(y-h_3+1;\omega_2-y;1)P(y-f_3+1;\omega_2-y;1)P(h_2-y+2;y-f_3;1) \times$$
$$P(h_3-y+1;y-f_3;1)P(\omega_1-y+2;y-f_3;1)P(\omega_2-y+1;y-f_3;1)P(h_2-\omega_2+1;\omega_2-y;1) \times$$
$$P(f_2-h_3+1;x-f_2;1)(x-y+1)(f_1-x)!(f_1-y+1)!f_2-y)!(x-\omega_2)!(x-\omega_3+1)! \times$$
$$(y-\omega_3)! \; . \tag{2.22}$$

Replacing (2.21) into the RHS of (2.19) and using the polynomial-type expression for the SU(2) 6-j coefficient given in [33]) one sees that after simplifications all the remaining factors under the square root symbol factor out and the terms inside the sums can be arranged into terms involving only integers and without denominators. In this way one obtains a polynomial-type expression for the U3IB in canonical chain.

3. Calculation of U3IB in Elliott's Basis

Up to now we have used U(3) canonical basis since Clebsch-Gordan coefficients, 6-j symbols and all that rely on the orthogonality of the basis functions. However, in physical applications one needs to have good angular momentum and its z projection, therefore it is more convenient to use Elliott's basis functions. Elliott's basis functions $|E_t(\lambda\mu)KLM\rangle$ are labelled by labels of the chain

$$U(3) \supset \begin{matrix} U_1 \\ \times \\ SU(3) \\ (\lambda\mu) \end{matrix} \supset \begin{matrix} O^+(3) \\ K \;\; L \end{matrix} \supset \begin{matrix} O^+(2) \\ M \end{matrix} \tag{3.1}$$

and are not orthogonal in the inner multiplicity label K. Usualy this difficulty is overcome by orthogonalizing the basis states with the same value of L and M and different values of K. We will avoid this orthogonalization process by using the concept of dual bases (See definitions and properties needed in this paper in the appendix) introduced by Alisauskas[34]) .

The basis states of Elliott's basis are related to those of the canonical basis by[35,28])

$$|E_t(\lambda\mu)KLM\rangle = \sum_{rs} A^{(\lambda\mu)\;M}_{KL\;(pq)} \;|[E_1E_2E_3]r\; s\; t=(r+s-M)/2\rangle , \tag{3.2}$$

with the labels related by

$$E_t=E_1+E_2+E_3;\ \lambda=E_1-E_2;\ \mu=E_2-E_3;\ p=r-E_2;\ q=s-E_3;\ M=r+s-2t. \qquad (3.3)$$

The transformation coefficients A 's are given in terms of the matrix elements of some projection operators by

$$A^{(\lambda\mu)\ M}_{KL\ (pq)} = P^{(\lambda\mu)\ M}_{KL\ (pq)}\left[P^{(\lambda\mu)\ K}_{KL\ (00)}\right]^{-1/2} \quad \text{for } \lambda \geq \mu\ ,$$

$$(3.4)$$

$$A^{(\lambda\mu)\ M}_{KL\ (pq)} = (-)^{(\lambda+\mu+p-q-K-M)/2}\, A^{(\mu\lambda)\ M}_{KL\ (\mu-q,\lambda-p)} \quad \text{for } \lambda < \mu\ .$$

The overlap of two basis states with same $(\lambda\mu)LM$ is symmetric in K,K' and given by

$$B^{(\lambda\mu)L}_{K\ K'} = P^{(\lambda\mu)\ K'}_{KL\ (00)}\left[P^{(\lambda\mu)\ K}_{KL\ (00)}\ P^{(\lambda\mu)\ K'}_{K'L\ (00)}\right]^{-1/2}, \quad \text{for } \lambda \geq \mu\ ,$$

$$(3.5)$$

$$B^{(\lambda\mu)L}_{K\ K'} = (-)^{(K-K')/2}\, B^{(\mu\lambda)L}_{K\ K'}, \quad \text{for } \lambda < \mu\ .$$

The quantities P's were introduced in[35]) and in (I) we give an explicit polynomial-type expression for them. Using them we produced in (I) polynomial-type expressions for the matrix elements of the transformation basis matrix A, as well as for the overlap matrix B.

For the use with the Elliott's basis, eq. (2.8) is conveniently rewritten as

$$\left\langle \vec{\rho}_1 \ldots \vec{\rho}_{A-1} \right| \left. \begin{matrix} [E]KLM \\ \delta(\omega)\alpha[f][\bar{f}]\bar{g} \end{matrix} \right\rangle = \sum_{[\bar{E}]\bar{\delta}(\bar{\omega})\bar{\alpha}[E_{A-1}]} A^{([E][\bar{f}])}_{[\bar{E}]\bar{\delta}(\bar{\omega})[E_{A-1}],\delta(\omega)\alpha[f]}$$

$$\sum_{\bar{K}\bar{L}\bar{M}K_{A-1}L_{A-1}M_{A-1}} C^{[\bar{E}]\ [E_{A-1}]\ [E]}_{\bar{K}\bar{L}\bar{M}\ K_{A-1}L_{A-1}M_{A-1}\ KLM} \times .$$

$$\langle \vec{\rho}_1\ \vec{\rho}_2 \cdots \vec{\rho}_{A-2} | \begin{matrix}[\bar{E}]\ \bar{K}\ \bar{L}\ \bar{M} \\ \bar{\delta}(\bar{\omega})\bar{a}(\bar{f})\bar{g}\end{matrix} \rangle^D \langle \vec{\rho}_{A-1} | [E_{A-1}]\ K_{A-1}L_{A-1}M_{A-1} \rangle^D \Bigg] \tag{3.6}$$

and similarly for eq. (2.9). In eq. (3.6) the A 's are the same as in eq. (2.8), the superscript D in the basis functions means dual (see Appendix) and the U(3) Clebsch-Gordan coefficients in Elliott's basis are defined as the coefficients of the linear combination that couples two dual Elliot's bases to an usual one, namely,

$$|[E]\beta KLM\rangle = \sum_{K'L'M'\ K''L''M''} C^{[E']\ [E'']\ \beta[E]}_{K'L'M'\ K''L''M''\ KLM} \times$$

$$|[E']K'L'M'\rangle^D\ |[E'']K''L''M''\rangle^D \quad , \tag{3.7}$$

where β is an outer muultiplicity index. Since in eq. (36) $[E_{A-1}]$ $= [E_{A-1}00]$ is a symmetric $U(3)$ irrep, $K_{A-1} = 0$ and no outer multiplicity index is needed.

Again, as in the canonical chain, when one computes the matrix elements of one-body operators between basis states (3.6) (or its analogue in the case of two-body operators) the integrations in other variables than $\vec{\rho}_{A-1}$ (or $\vec{\rho}_A$) are immediately performed by orthogonality and one needs to compute only the U(3) part of the matrix elements. Considering again an U(3) decomposition of the interaction and using the U(3) Wigner-Eckart theorem in the Elliott's basis(See Appendix), one obtains for each component the U(3) part of the one-body interaction

$$\sum_{[E_0]K_0L_0M_0} Q^{[E_0]}_{K_0L_0M_0}(W) \langle [c']\| W^{[E_0]} \| [c] \rangle \times$$

$$F([\bar{E}][c][c'];[E][E'][E_0];KLM,K'L'M',K_0L_0M_0) \qquad (3.8)$$

where

$$F([\bar{E}][c][c'];[E][E'][E_0];KLM,K'L'M',K_0L_0M_0) = \sum_{\bar{K}'\bar{K}\bar{L}\bar{M}\, lml'm'}$$

$$C^{[\bar{E}]\ \ [c]\ \ [E]}_{\bar{K}\bar{L}\bar{M}\ \ 0lm\ \ KLM}\ D^{[\bar{E}]\bar{L}}_{\bar{K}\bar{K}'}\ C^{[\bar{E}]\ \ [c']\ \ [E']}_{\bar{K}'\bar{L}\bar{M}\ \ 0l'm'\ \ K'L'M'}\ C^{[E_0]\ \ [E']\ \ [c]}_{K_0L_0M_0\ \ 0l'm'\ \ 0lm} \qquad (3.9)$$

with the following relabeling of indices

$[E_{A-1}]K_{A-1}L_{A-1}M_{A-1} \rightarrow [c]0lm$; $[E'_{A-1}]K'_{A-1}L'_{A-1}M'_{A-1} \rightarrow [c']0l'm'$, since $[E_{A-1}]$ and $[E'_{A-1}]$, being symmetric, have $K_{A-1} = K'_{A-1} = 0$. Besides, the D's in (3.9) are the matrix elements of the inversed overlap matrix.

The same result is obtained for the U(3) part of the matrix elements of two-body operators with $[c]0lm$ and $[c']0l'm'$ meaning $[E_{A-1}]K_{A-1}L_{A-1}M_{A-1}$ and $[E'_{A-1}]K'_{A-1}L'_{A-1}M'_{A-1}$, respectively.

Using eq. (A.21) of the Appendix to identify the RHS of (3.9) one arives at

$$F([\bar{E}][c][c'];[E][E'][E_0];KLM,K'L'M',K_0L_0M_0) =$$

$$= \sum \begin{Bmatrix} [\bar{E}] & [c] & [E] \\ & & \beta \\ [\bar{E}] & [c'] & [E'] \\ [0] & [E_0] & [E_0] \end{Bmatrix} C^{[E_0]\ \ [E']\ \ \beta[E]}_{K_0L_0M_0\ \ K'L'M'\ \ KLM}, \qquad (3.10)$$

where the U(3) 9-j symbol is defined by (2.16).

Introducing isoscalar factors of U(3) Clebsch-Gordan coefficients in Elliott's basis by

$$C^{[E']\ \ [E'']\ \ \beta[E]}_{K'L'M'\ K''L''M''\ KLM} = C^{[E']\ [E'']\ \beta[E]}_{K'L'\ K''L''\ KL}\ C^{K'\ K''\ K}_{L'\ L''\ L} \tag{3.11}$$

one concludes from (3.8) and (3.10) that the isoscalar content of the U(3) part of the matrix elements of one- and two-body operators is given by

$$F_{J}([\bar{E}][\epsilon][\epsilon'];[E][E'][E_0];KL,K'L',K_0L_0)=$$
$$\sum_{\beta}\begin{Bmatrix}[\bar{E}] & [\epsilon] & [E]\\ & & \beta\\ [\bar{E}] & [\epsilon'] & [E']\\ [0] & [E_0] & [E_0]\end{Bmatrix} C^{[E_0]\ \ [E']\ \ \beta[E\ ]}_{K_0\ L_0\ K'L'\ \ K\ L} \tag{3.12}$$

the quantity that we named as U3IB.

Exactly as we did in section 2, one can use equations (3.9)-(3.11) and equation (3.12) of[31] to obtain an alternative expression for the U3IB, namely

$$\sum_{\beta}\begin{Bmatrix}[\bar{E}] & [\epsilon] & [E]\\ [\bar{E}] & [\epsilon'] & [E']\\ [0] & [E_0] & [E_0]\end{Bmatrix} C^{[E_0]\ [E']\ \beta[E]}_{K_0L_0\ K'L'\ KL} = \sum_{\bar{K}'\bar{K}\ \bar{L}\ J\ J'} C^{[\bar{E}]\ \ [\epsilon]\ [E]}_{\bar{K}\ \bar{L}\ \ 0J\ \ K\ L}$$
$$\times\ D^{[E]\ L}_{\bar{K}\ \bar{K}'}\ C^{[\bar{E}]\ \ [\epsilon']\ [E']}_{\bar{K}'\bar{L}\ \ 0J'\ \ K'L'}\ C^{[E_0]\ [\epsilon']\ [\epsilon]}_{K_0L_0\ 0J'\ \ 0J}\ (-)^{J+L'+\bar{L}+L_0}\ \times$$
$$[(2L'+1)(2J+1)]^{1/2}\begin{Bmatrix}J & L & \bar{L}\\ L' & J' & L_0\end{Bmatrix}\ , \tag{3.13}$$

Explicity polynomial-type expressions for the U(3) isoscalar factors needed above are given in (I) and for the SU(2) 6-j symbol in [33]. For the matrix elements of the inversed overlap matrix one has

$$D^{[E_1E_2E_3]L}_{K\,K'} = [\,P^{(\lambda\mu)K'}_{K\,L\,(00)}\,]^{-1}\,[\,P^{(\lambda\mu)K}_{K\,L\,(00)}\;P^{(\lambda\mu)K'}_{K'L\,(00)}\,]^{1/2} \qquad (3.14)$$

where $[\ldots]^{-1}$ is defined by

$$\sum_{K} [\,P^{(\lambda\mu)K'}_{K\,L\,(00)}\,]^{-1}\;P^{(\lambda\mu)K''}_{K\,L\,(00)} = \delta_{K'K''} \;. \qquad (3.15)$$

Combining equations (4.2) and (5.6) of [36] one obtains

$$[\,P^{(\lambda\mu)K'}_{K\,L\,(00)}\,]^{-1} = \frac{(\mu+\delta+\Delta)!!\,(\lambda+L-\Delta+2)!!}{(2L+1)\lambda!\,(\lambda+\mu+1)!} \times$$

$$\frac{(\lambda+\mu+L+\delta+2)!!}{[(\mu+K)!!(\mu-K)!!(\mu+K')!!(\mu-K')!!(L+K)!(L-K)!(L+K')!(L-K')!]^{1/2}} \times$$

$$\sum_{J=\max(\delta+\Delta,\,L-\lambda+\delta)}^{K}{}' \; Z(K,J)\,\frac{(K+J)!!}{(K-J)!!}\;\sum_{J'=\max(\delta+\Delta,\,L-\lambda+\delta)}^{K'}{}' \; Z(K',J')\,\frac{(K'+J')!!}{(K'-J')!!}\;S(J',J) \qquad (3.16)$$

with

$$\delta = \pi(\lambda+\mu-L);\quad \Delta = \pi(\lambda+L)$$

$$\pi(n) = 0 \;\text{ for } n \text{ even and } 1 \text{ for } n \text{ odd} \qquad (3.17)$$

$$Z(K,J) = \frac{2(K)^{1-\delta}}{K+J} \quad \text{for } K > 0 \text{ and } 1 \text{ for } K = 0$$

and

$$S(J',J) = \frac{(-)^{(J'-\Delta-\delta)/2}\,(\mu-J')!!\,(\lambda-L-\delta+J)!!\,(L-J)!}{(J'-\delta-\Delta)!!\,(J-\delta+\Delta-1)!!} \times$$

$$\sum_{J_2=\delta+\Delta}^{\mu+\delta+\Delta-J'} {}' \frac{(\mu-J_2)!!(L+J_2)!!(\lambda+\mu-L-J_2+\Delta)!!(J_2-\Delta+\delta-1)!!}{(J_2-\delta-\Delta)!!(J_2+J')!!(\mu-J'-J_2+\Delta+\delta)!!} \times$$

$$\sum_{t=\max(0,L-\lambda-\Delta)}^{\min(\mu-J_2,J-\Delta-\delta)} {}' \frac{(-)^{t/2}(\mu-\Delta-\delta-t)!!(\lambda+\delta+t+1)!!}{t!!(\mu-J_2-t)!!(J-\Delta-\delta-t)!!(\lambda+L+J_2+\delta+t+2)!!(\lambda-L+\Delta+t)!!}$$

$$\times \frac{(\lambda+L+\Delta+t+1)!!}{(\lambda+L+\delta-J+t+2)!!} . \qquad (3.18)$$

In (3.16) and (3.18) the dashes in the summation symbols mean that the running indices run in step of 2.

Replacing (3.16)-(3.18) and the polynomial-type expression for the P's given in (I) into eq. (3.14) one obtains a polynomial-type expression for the matrix elements of the inversed overlap matrix D.

Replacing in the RHS of (3.13) the polymomial-type expression so obtained for the D's as well as the ones for the isoscalar factors and the SU(2) 6-j symbol, one obtains a polynomial-type expression for the U3IB. This polynomial-type expression was used to produce a computer code to evaluate the U3IB exactly as $(n_1/n_2)(n_3/n_4)^{1/2}$ with the n_i' s being integers. As example we list in Table I a few values of U3IB.

Now we exploit the expression of the U3IB in terms of particular U(3) 9-j symbols and isoscalar factors to obtain a second alternative way of computing it.

The U(3) isoscalar factors in Elliott's basis are related to the ones in the canonical one by [16,26])

$$C^{[E'][E'']\beta[E]}_{K'L'\,K''L''\,K\,L} = [A^{(\lambda\mu)\,K}_{KL\,(00)}]^{-1} \sum_{\substack{r's'M'\\ r''s''t''}} A^{(\lambda'\mu')\,M'}_{K'L'\,(p'q')}\, A^{(\lambda''\mu'')\,M''}_{K''L''\,(p''q'')} \times$$

$$C^{L'\,L''\,L}_{M'\,M''\,K}\, C^{(r'-s')/2\ (r''-s'')/2\ (h_2-h_3)/2}_{M'/2\quad M''/2\quad K/2}\, C^{[E']\ [E'']\ \beta[E]}_{[r's']\,[r''s'']\,[h_2h_3]} =$$

$$[A^{(\lambda\mu)\,K}_{KL\,(00)}]^{-1} \sum_{\substack{r's'M'\\ r''s''M''}} \mathcal{R}^{[E'][E''][E]}_{K'L'K''L''KL,r's'M'r''s''M''} \times$$

$$C^{[E']\ [E'']\ \beta[E]}_{[r's']\,[r''s'']\,[h_2h_3]} \qquad (3.19)$$

where the A's are defined by eq. (3.4) $[h_2,h_3]$ is equal to $[E_2E_3]$ when $\lambda \geq \mu$ and $[E_1E_2]$ when $\lambda < \mu$ and $\mathcal{R}$ is a shorthand notation for all the quantities at the left of the isoscalar factor inside the sums.

Using (3.19) in (2.18) and noticing that the U(3) 9-j symbol do not depend on the internal labels rst of the U(3) irreps involved, one sees that its RHS is transformed into the RHS of (3.12), so one concludes that

$$\sum_\beta \begin{Bmatrix} [\bar{E}] & [\epsilon] & [E] \\ & & \beta \\ [\bar{E}] & [\epsilon'] & [E'] \\ [0] & [E_0] & [E_0] \end{Bmatrix} C^{[E_0]\ \ [E']\ \ \beta[E]}_{K_0\,L_0\,K'\,L'\,K\,L} = [A^{(\lambda\mu)\,K}_{KL\,(00)}]^{-1} \times$$

$$\sum_{\substack{r's'M'\\ r''s''M''}} \mathcal{R}^{[E_0][E'][E]}_{K_0L_0K'L'KL,r's'M'r''s''M''} \left[\sum_\beta \begin{Bmatrix} [\bar{E}] & [\epsilon] & [E] \\ & & \beta \\ [\bar{E}] & [\epsilon'] & [E'] \\ [0] & [E_0] & [E_0] \end{Bmatrix} \right. \times$$

$$= C^{[E_0] \quad [E'] \quad \beta[E]}_{[r_0 s_0] \; [r's]' \; [h_2 h_3]} \quad . \tag{3.20}$$

Replacing the RHS of equation (2.19) into the RHS of (3.20) one obtains a second expression for the U3IB. Using the polynomial-type expression for the U3IB in canonical basis resulting from (2.19) and those for the constituent part of $\mathfrak{R}$ found in (1) one obtains a second polynomial-type expression for the U3IB. As before, we made a computer code based in this resulting polynomial-type expression to compute the U3IB exactly. The outputs of this computer code were compared with those of the computer code based in eq. (3.13) for a lot of cases, including those of Table I, and all of them were equal, confirming the exactness of computer codes and the consistency of the formulas involved.

4. A THIRD ALTERNATIVE EXPRESSION FOR U3IC

The expression of the U3IB in terms of a particular U(3) 9-j symbol and isoscalar factors can be further exploited if one has some particular classification β of the outer multiplicity with easy to compute 6-j symbols and isoscalar factors.

Generalizing an ideia introduced in [9]) for the splitting of the outer multiplicity, Ališauskas [28]) introduced the classifications $\beta = (+I_2+)$ and $(-I_2-)$ for the canonical basis.

The classification $\beta = (+I_2+)$ is defined requering that when the U(2) irreducible labels of the first and third irreps assume their minimum values (right tyed values) the corresponding isoscalar factor is one if the U(2) irreducible labels of the second irrep satisfy $(r''-s'')/2 = 2I_2$ and 0 if not, namely,

$$C\begin{matrix}[E'_1E'_2E'_3] & [E''_1E''_2E''_3] & (+I_2+)[E_1E_2E_3] \\ [E'_2E'_3] & [r''s''] & [E_2E_3]\end{matrix} = \delta_{(r''+s''),(E_2+E_3-E'_2-E'_3)} \times \delta_{(r''-s''),2I_2} . \qquad (4.1)$$

The first δ is a consequence of the rules of the U(2) Kronecker product while the second defines the classification index I_2 .

As a consequence of (4.1), $2I_2$ can assume the values

$$I_2 = 0,\ 1/2,\ 1,\ \ldots\ldots ,\ (E''_1 - E''_3)/2 \qquad (4.2)$$

subject to the triangular conditions for the addition of angular momenta,

$$|(E_2-E_3)/2-(E'_1-E'_2)/2| \leq I_2 \leq (E_2-E_3)/2+((E'_1-E'_2)/2 . \qquad (4.3)$$

Likewise, the classification $\beta = (-I_2-)$ is defined by

$$C\begin{matrix}[E'_1E'_2E'_3] & [E''_1E''_2E''_3] & (-I_2-)[E_1E_2E_3] \\ [E'_1E'_2] & [r''s''] & [E_1E_2]\end{matrix} = \delta_{(r''+s''),(E_1+E_2-E'_1-E'_2)} \times$$

$$\times \ \delta_{(r''-s''),2I_2} \qquad (4.4)$$

with I_2 assuming the values (4.2) now subjected to

$$|(E_1-E_2)/2-(E'_1-E'_2)/2| \leq I_2 \leq (E_1-E_2)/2+((E'_1-E'_2)/2 . \qquad (4.5)$$

Taking (2.18) with $(r_0,s_0) = (E_{02},E_{03})$, $(r,s) = (E_2,E_3)$ and using (4.1) one sees that only one term in the sum of the RHS survives and one obtains for the particular U(3) 9-j symbol in this classification the expression

$$\left\{\begin{matrix}[\bar{E}] & [\varepsilon] & [E] \\ & & (+I_2+) \\ [\bar{E}] & [\varepsilon'] & [E'] \\ [0] & [E_0] & [E_0]\end{matrix}\right\} = F_1([\bar{E}][\varepsilon][\varepsilon'][E][E'][E_0];E_2E_3,r's',E_{02}E_{03}) \qquad (4.6)$$

with

$$r' = (E_2+E_3-E_{02}-E_{03})/2+I_2 \quad \text{and} \quad s' = (E_2+E_3-E_{02}-E_{03})/2-I_2 .$$

In this way using the polynomial-type expression based in (2.19) with the values of (r,s), (r_0,s_0), (r',s') above, one obtains a polynomial-type expression for the particular U(3) 9-j symbol in the classification $\beta = (+I_2+)$.

Analogously, for the classification $\beta = (-I_2-)$ one has

$$\left\{\begin{matrix} [\bar{E}] & [\epsilon] & [E] \\ & & (-I_2-) \\ [\bar{E}] & [\epsilon'] & [E'] \\ [0] & [E_0] & [E_0] \end{matrix}\right\} = F_1([\bar{E}][\epsilon][\epsilon'][E][E'][E_0];E_1E_2,r's',E_{01}E_{02}) \tag{4.7}$$

with

$$r' = (E_1+E_2-E_{01}-E_{02})/2+I_2 \quad \text{and} \quad s' = (E_1+E_2-E_{01}-E_{02})/2-I_2 ,$$

what allows one to obtain a polynomial-type expression for the U_3 9-j symbol in the classification $\beta = (-I_2-)$.

Now that we have polynomial-type expressions for the particular 9-j symbols we need the isoscalar factors in these classifications expressed in Elliott's basis. By (3.20) one sees that the isoscalar factors in canonical basis that are needed are only the ones with $(r,s) = (E_2,E_3)$ (right tyed labels) for $\lambda \geq \mu$ or $(r,s) = (E_1,E_2)$ (left tyed labels) for $\lambda < \mu$. So we need the classification $(+I_2+)$ for $\lambda \geq \mu$ and $(-I_2-)$ for $\lambda < \mu$.

In order to construct the isoscalar factor with the U_2 labels of the third irrep tyed to the right we first introduce the quantity

$$\left[\begin{matrix} (\lambda_1\mu_1) & (\lambda_2\mu_2) & (+I_2+)(\lambda\mu) \\ (z_1)i_1 & (z_2)i_2 & (\mu/2)\ \mu/2 \end{matrix}\right] = (-)^{(\mu-\mu_1)/2+I_2+2i_1} \times$$

$$\left[\frac{(2I_2+1)(2i_1+1)(2i_2+1)(\lambda_1+z_1-i_1)!(\lambda_1+z_1+i_1+1)!\Gamma(\lambda_2\mu_2 i_2 z_2)}{\lambda_1!(\lambda_1+\mu_1+1)!\nabla((\mu_1-2z_1)/2,I_2,i_2)\ \Gamma(\lambda_2\mu_2 I_2\ z_2)}\right]^{1/2} \times$$

$$\times \left\{\begin{matrix} i_1 & i_2 & \mu/2 \\ I_2 & \mu_1/2 & (\mu_1-2z_1)/2 \end{matrix}\right\} . \tag{4.8}$$

with

$$\nabla(abc) = \frac{(a-b+c)!(a+b-c)!(a+b+c+1)!}{(b+c-a)!} ,$$

$$\Gamma(abcc) = \frac{(c+d)!(a+d-c)!(a+d+c+1)!}{(c-d)!(b-d-c)!(b-d+c+1)!} , \qquad (4.9)$$

$$\underline{z}_2 = (\mu-\mu_1)/2-(\lambda_1-\mu_1+\lambda_2-\mu_2-\lambda+\mu)/3$$

and for each of the three irreps involved,

$$\lambda = E_1-E_2, \quad \mu = E_2-E_3, \quad z = (2E_2-r-s)/2, \quad i = (r-s)/2 \ . \qquad (4.10)$$

The values allowed for I_2 in (4.8) are

$$| \mu_1/2-\mu/2 | \le I_2 \le \mu_1/2+\mu/2 \qquad (4.11)$$

implyed by the presence in (4.8) of an SU(2) 6-j symbol.

When the irreducible labels of the U(3) irreps satisfy

$$E'_1+E''_3+E'_2 \ge 0 \quad \text{or} \quad -E'_2-E''_1+E_1 \ge 0 \qquad (4.12)$$

one has

$$C^{[E'_1E'_2E'_3]\ [E''_1E''_2E''_3]\ (+I_2+)[E_1E_2E_3]}_{[r's']\quad [r''s'']\qquad\qquad [E_2\ E_3]} = \begin{bmatrix} (\lambda'\mu') & (\lambda''\mu'') & (+I_2+)(\lambda\mu) \\ (z')i' & (z'')i'' & (\mu/2)\ \mu/2 \end{bmatrix}$$

(4.13)

Using $(E'_1+E'_2+E'_3) + (E''_1+E''_2+E''_3) = (E_1+E_2+E_3)$ and the relations (4.10) the quantity $\underline{z}_2$ that will appear in the RHS of (4.13) can be expressed as

$$\underline{z}_2 = E''_2 + (E'_2+E'_3-E_2-E_3)/2 \ . \qquad (4.14)$$

When none of conditions (4.12) is satisfied one has

$$C^{[E'_1E'_2E'_3]\ [E''_1E''_2E''_3]\ (+I_2+)[E_1E_2E_3]}_{[r's']\quad [r''s'']\qquad\qquad [E_2\ E_3]} = \begin{bmatrix} (\lambda'\mu') & (\lambda''\mu'') & (+I_2+)(\lambda\mu) \\ (z')i' & (z'')i'' & (\mu/2)\ \mu/2 \end{bmatrix} -$$

$$
- \sum_{J_2 \neq I_2} \left[\begin{matrix} (\lambda'\mu') & (\lambda''\mu'') & (+J_2+) & (\lambda\mu) \\ (z')1' & (z'')1'' & (\mu/2) & \mu/2 \end{matrix} \right] \frac{(-)^{I_2-I_0+A} \quad (I_2+I_0+A)!}{(I_2-J_2)(I_2-J_2+1)(I_2-I_0-A)!} \times
$$

$$
\frac{(I_2-I_0+A)!(J_2+I_0-A)!}{(I_2+I_0-A)!(J_2+I_0+A)!(J_2-I_0+A)!(I_0+A-J_2-1)!} \left[(2I_2+1)(2J_2+1) \times \right.
$$

$$
\times \frac{(I_2+\underline{z}_2)!(J_2-\underline{z}_2)!(\mu''-\underline{z}_2-J_2)!(\mu''-\underline{z}_2+J_2+1)!(\lambda''+\underline{z}_2-J_2)!(\lambda''+\underline{z}_2+J_2+1)!}{(J_2+\underline{z}_2)!(I_2-\underline{z}_2)!(\mu''-\underline{z}_2-I_2)!(\mu''-\underline{z}_2+I_2+1)!(\lambda''+\underline{z}_2-I_2)!(\lambda''+\underline{z}_2+I_2+1)!}
$$

$$
\left. \times \frac{\nabla(\mu'/2,\mu/2,J_2)}{\nabla(\mu'/2,\mu/2,I_2)} \right]^{1/2} , \qquad (4.15)
$$

with

$$
I_0 = (\mu''-\lambda'+s+|s|)/2, \quad A = (\lambda''-\lambda-s)/2; \quad s = (\lambda'-\mu'+\lambda''-\mu''-\lambda+\mu)/3 . \qquad (4.16)
$$

The summation index J_2 has the same range as I_2 in (4.2), excluded those values that produce factorials with negative arguments.

Now using (3.19) with (4.13) or (4.15), depending whether the irreducible labels satisfy or not eqs. (4.12), in place of the canonical isoscalar factor with the last irrep tyed, one obtains the general U(3) isoscalar factor in the Elliott's basis and the classification $\beta = (+I_2 +)$. Multiplying the resulting expression by the particular U(3) 9-j symbol (4.6) in that classification and performing the sum in I_2 one obtains a new expression for the U3IB.

Again the final expression is of a polynomial-type kind and can be computed exactly. A computer code was also produced for its evaluation.

For the case $\lambda < \mu$ one uses the symmetry relation [28])

$$C^{[E_1'E_2'E_3'][E_1''E_2''E_3''](-I_2-)[E_1E_2E_3]}_{[r's'][r''s''][E_1E_2]} = (-)^{(r'-s'+r''-s''-E_1'+E_2')/2-I_2} \times$$

$$C^{[-E_3'-E_2'-E_1'][-E_3''-E_2''-E_1''](+I_2+)[-E_3-E_2-E_1]}_{[-s'-r'][-s''-r''][-E_2-E_1]} \qquad (4.17)$$

and applies (4.13) or (4.15) to the isoscalar factor in the RHS of (4.17) which has $\bar{\lambda}=-E_3+E_2=\mu$; $\bar{\mu}=-E_2+E_1=\lambda$ with $\bar{\lambda} > \bar{\mu}$.

A computer code was also developed to compute the U3IB using this third alternative for both $\lambda \geq \mu$ and $\lambda < \mu$. It was run for a lot of cases giving for all of them the same results found in alternatives 1 and 2.

5. Concluding remarks

We developed three alternative ways of computing U3IB, the isoscalar part of the U(3) content of the matrix elements of one- and two-body operators in Elliott's basis. All of them lead to polynomial-type expressions that can be evaluated exactly in terms of integers of arbitrary size. The first of them, based in eq. (3.13) uses the U(3) isoscalar factors calculated in (I). The second computes the U3IB in canonical basis using (2.19) and transforms it to the Elliott's basis by means of (3.20). The third one uses (3.12), the expression of the U3IB in terms of particular U(3) 9-j symbols and general isoscalar factors in Elliott's basis, chooses a particular classification β, computes separately the 9-j symbols and the iscscalar iactors to obtain the U3IB in canonical basis and then transforms it to the Elliott's basis.

Looking closely at eq. (3.13) and eq. (2.19) transformed to the Elliott's basis one sees that the computer code based in the later is much more efficient than the one based in the former since the transformation procedure to the Elliott's basis is done only once in the second and three times, one for each isoscalar factor, in the first.

The third alternative also uses only once the procedure to transform to the Elliott's basis like alternative two, but if one counts the number of sums envolved in both alternatives one sees that the third is far superior. Another argument in favor of the third alternative is the following. In applications to a given nucleus one fixes the U(3) labels [E] and [E'] of the initial and final state and needs the U3IB for all allowed values of $[\bar{E}]$, $[\varepsilon]$, $[\varepsilon']$, $[E_0]$, K_0L_0, $K'L'$, KL. Alternative three based in (3.12) allows one to compute all the particular U(3) 9-j symbols that are independent of the K 's and L 's only once, store them and use them a lot of times combined with the isoscalar factors for all possibilities of the K 's and L 's. Also the isoscalar factors, being independent on $[\bar{E}]$, $[\varepsilon]$ and $[\varepsilon']$ can be computed only once, stored and used a lot of times combined with the U(3) 9-j symbols for all the possibilities of $[\bar{E}]$, $[\varepsilon]$ and $[\varepsilon']$. Since the sum in I_2 runs only a few values the combining procedure is not time consuming. Putting all together, the procedure of saving quantities to avoid computing them many times amounts in a very substantial economy of computing time.

We have applied the formulas for the U3IB for alpha nuclei [37]) and the mirror nuclei with $A \leq 40$ using the strictly restricted dynamics model [38]). In this model the quantities A's appearing in (2.8) and (2.9) have very simple expressions and we were able, using the polynomial-type expressions of the U3IB here exposed, to express the complete space part of two-body operators as linear combinations of Talmi integrals with polynomial-type coefficients.

APPENDIX: SOME ASPECTS OF THE USE OF DUAL BASIS IN COMPACT LIE GROUPS

Although in this paper we need dual basis only for the U(3) group the results that we are going to present do not depend on any special peculiarity of this group so we will make definitions and derivations valid for any compact Lie group.

Let

$$\{ \, | \lambda\mu \rangle ; \; \mu = 1,2,\ldots \, \dim[\lambda] \, \} \qquad (A.1)$$

be a basis for an irrep λ of a compact Lie group G labeled by internal labels μ. The non-orthogonality (in the internal labels) of the basis is characterized by a non-unity overlap matrix $B^{(\lambda)}$ with matrix elements

$$B^{(\lambda)}_{\mu\,\mu'} \equiv \langle \lambda\, \mu \,|\, \lambda\, \mu' \rangle \qquad (A.2)$$

that, by an appropriated choice of phase of the basis states (A.1) can be made real and therefore symmetric.

Using its inverse $D^{(\lambda)}$ one introduces a dual basis [34]) by the relation

$$| \lambda\, \mu^{D} \rangle = \sum_{\mu'} D^{(\lambda)}_{\mu\,\mu'} \, | \lambda\, \mu' \rangle \qquad (A.3)$$

with the useful property

$$\langle \lambda\ \mu^{D} \mid \lambda'\ \mu' \rangle = \delta_{\lambda\lambda'}\,\delta_{\mu\mu'}. \qquad (A.4)$$

One defines the Clebsch-Gordan coefficients for a general compact Lie group as the coefficients that .couples two <u>dual</u> bases to a resultant <u>usual</u> basis, namely

$$|(\lambda_1,\lambda_2)\beta\lambda\mu\rangle = \sum_{\mu_1\mu_2} C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1\ \mu_2\ \mu}\, |\lambda_1\ \mu_1^{D}\rangle\, |\lambda_2\ \mu_2^{D}\rangle =$$

$$\overline{\sum_{\mu_1\mu_2\mu_1'\mu_2'}}\, D^{(\lambda_1)}_{\mu_1\mu_1'}\, D^{(\lambda_2)}_{\mu_2\mu_2'}\, C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1'\ \mu_2'\ \mu}\, |\lambda_1\mu_1'\rangle\, |\lambda_2\mu_2'\rangle \qquad (A.5)$$

where β labels the possible multiple appearance of the irrep λ in the reduction of the Kronecker product $\lambda_1 \otimes \lambda_2$. Besides, one chooses orthogonal classification scheme for the outer multiplicity, real Clebsch-Gordan coefficients and requires that the coupled basis states (A.5) also have the same overlap (A.2) as the uncoupled states, i.e.,

$$\langle(\lambda_1,\lambda_2)\beta'\lambda'\mu' \mid (\lambda_1,\lambda_2)\beta\lambda\mu\rangle = \delta_{\beta\beta'}\,\delta_{\lambda\lambda'}\,B^{(\lambda)}_{\mu\mu'}. \qquad (A.6)$$

Condition (A.6) assure that the coupled basis states are also dual-orthogonal and gives for the Clebsch-Gordan coefficients the relation

$$\overline{\sum_{\mu_1\mu_2}}\, C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1^{D}\ \mu_2^{D}\ \mu^{D}}\, C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1\ \mu_2\ \mu'} = \delta_{(\beta\lambda\mu)}\ \delta_{(\beta'\lambda'\mu')} \qquad (A.7)$$

where we extended the notation (A.3) for any index μ of any quantity F^{λ}_{μ}

$$F^{\lambda}_{\mu^D} = \sum_{\mu'} D^{(\lambda)}_{\mu\mu'} F^{\lambda}_{\mu'} . \tag{A.8}$$

From (A.7) and the finiteness of the basis (A.1) one obtains

$$\sum_{\beta\lambda\mu} C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1'\ \mu_2'\ \mu} \ C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1^D\ \mu_2^D\ \mu^D} = \delta_{(\mu_1\mu_2),(\mu_1'\mu_2')} . \tag{A.9}$$

Notation (A.8) allows us to write the coupled basis states in two alternative ways:

$$|(\lambda_1,\lambda_2)\beta\lambda\mu\rangle = \sum_{\mu_1\mu_2} C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1\ \mu_2\ \mu} |\lambda_1\ \mu^D_1\rangle\ |\lambda_2\ \mu^D_2\rangle =$$

$$\sum_{\mu_1\mu_2} C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1^D\ \mu_2^D\ \mu} |\lambda_1\ \mu_1\rangle\ |\lambda_2\ \mu_2\rangle . \tag{A.10}$$

The definition of an irreducible tensor $\{ T^{\lambda} \}$ of rank λ in non-orthogonal basis is given in analogy with the orthogonal one, by

$$[X_{\rho}, T^{\lambda}_{\mu}] = \sum_{\nu} \langle \lambda\ \nu^D |\ X_{\rho}\ | \lambda\ \mu\rangle\ T^{\lambda}_{\nu} , \tag{A.11}$$

where X_{ρ} are the infinitesimal generators of the group G.

Using the effect of an infinitesimal generator in the composite basis state (A.5) and (A.6) one obtains a relation between CLebsch-Gordan coefficients and matrix elements of infinitesimal generators that allows us to prove that the quantity

$$\sum_{\mu_1\mu_2} C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1\ \mu_2\ \mu}\, T^{\lambda_1}_{\mu_1^D}\, |\lambda_2\ \mu_2^D\rangle = |\ \Gamma(\lambda_1,\lambda_2)\beta\lambda\mu\rangle \qquad (A.12)$$

transforms as the basis state $|\ \lambda\ \mu\rangle$ of irrep λ. Using this fact and (A.4) one obtains

$$\langle\lambda\mu^D|\ T^{\lambda_1}_{\mu_1}\ |\lambda_2\mu_2\rangle = \sum_{\beta} \langle\lambda\mu^D|\ \Gamma(\lambda_1,\lambda_2)\beta\lambda\mu\rangle\ C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1\ \mu_2\ \mu^D}\ . \qquad (A.13)$$

Using normalized (in the sense of (A.4)) lowering operators to generate the whole basis starting from the state of maximum wight in order to define the internal label μ, one sees that the quantity $\langle\ldots\rangle$ in (A.13) does not depend on μ'. Therefore eq. (A.13) stablishes the Wigner-Eckart theorem for compact Lie groups with non-orthogonal bases:

$$\langle\lambda\mu^D|\ T^{\lambda_1}_{\mu_1}\ |\lambda_2\mu_2\rangle = \sum_{\beta} \langle\beta\lambda\|\ T^{\lambda_1}\ \|\ \lambda_2\rangle\ C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1\ \mu_2\ \mu^D}\ . \qquad (A.14)$$

Using the dual orthogonality condition (A.7) the reduced matrix element in (A.14) can be written as

$$\langle\beta\lambda\|\ T^{\lambda_1}\ \|\ \lambda_2\rangle = \sum_{\mu_1\mu_2} C^{\lambda_1\ \lambda_2\ \beta\lambda}_{\mu_1^D\ \mu_2^D\ \mu}\ \langle\lambda\mu^D|\ T^{\lambda_1}_{\mu_1}\ |\lambda_2\mu_2\rangle =$$

$$\langle\lambda\mu^D|\ \Gamma(\lambda_1,\lambda_2)\beta\lambda\mu\rangle\ . \qquad (A.15)$$

Let us consider a composite system with G-symmetry made up of two sub-systems also with G-symmetry. Using (A.14) the matrix elements of the component q of the G-irreducible tensor

$\{ T^{\chi} \}$ made up of G-irreducible tensors of ranks χ_1 and χ_2 that act in the sub-systems 1 and 2 is given by

$$\langle(\lambda_1,\lambda_2)\alpha\lambda\mu^D \,|[U^{\chi_1} \times V^{\chi_2}]^{\beta_0\chi}_q\, |(\lambda_1',\lambda_2')\alpha'\lambda'\mu'\rangle =$$

$$\sum_{\beta_1\beta_2} \langle\beta_1\lambda_1\| U^{\chi_1} \|\lambda_1'\rangle \langle\beta_2\lambda_2\| V^{\chi_2} \|\lambda_2'\rangle \left[\sum_{\substack{\mu_1\mu_2\mu_1'\mu_2' \\ q_1\, q_2}} C^{\lambda_1\ \lambda_2\ \alpha\lambda}_{\mu_1\ \mu_2\ \mu^D} \times\right.$$

$$\left. C^{\chi_1\ \chi_2\ \beta_0\chi}_{q_1^D\ q_2^D\ q}\, C^{\lambda_1'\ \lambda_2'\ \alpha'\lambda'}_{\mu_1'^D\ \mu_2'^D\ \mu'}\, C^{\chi_1\ \lambda_1'\ \beta_1\chi_1}_{q_1\ \mu_1'\ \mu_1^D}\, C^{\chi_2\ \lambda_2'\ \beta_2\lambda_2}_{q_2\ \mu_2'\ \mu_2^D} \right] . \qquad (A.16)$$

On the other hand, applying (A.14) to T^{χ}_q ones sees that the dependence of its matrix elements in the internal labels μ, q, μ' must be contained only in a Clebsch-Gordan coefficient. Therefore

$$\langle(\lambda_1,\lambda_2)\alpha\ \lambda\ \mu^D \,|[U^{\chi_1} \times V^{\chi_2}]^{\beta_0\chi}_q\, |(\lambda_1',\lambda_2')\ \alpha'\ \lambda'\ \mu'\rangle =$$

$$\sum_{\beta_1\beta_2} \langle\beta_1\lambda_1\| U^{\chi_1} \|\lambda_1'\rangle \langle\beta_2\lambda_2\| V^{\chi_2} \|\lambda_2'\rangle \sum_{\beta} \left\{ \begin{matrix} \lambda_1 & \lambda_2 & \alpha\lambda \\ \beta_1 & \beta_2 & \beta \\ \lambda_1' & \lambda_2' & \alpha'\lambda' \\ \chi_1 & \chi_2 & \beta_0\chi \end{matrix} \right\} \times$$

$$C^{\chi\ \lambda'\ \beta\lambda}_{q\ \mu'\ \mu^D} . \qquad (A.17)$$

where {...} are, up to now, unknown expansion coefficients. Comparing (A.16) and (A.17) one obtains

$$\sum_{\substack{\mu_1\mu_2\mu_1'\mu_2' \\ q_1\, q_2}} C^{\lambda_1\ \lambda_2\ \alpha\lambda}_{\mu_1\ \mu_2\ \mu^D}\, C^{\chi_1\ \chi_2\ \beta_0\chi}_{q_1^D\ q_2^D\ q}\, C^{\lambda_1\ \lambda_2'\ \alpha'\lambda'}_{\mu_1'^D\ \mu_2'^D\ \mu'}\, C^{\chi_1\ \lambda_1'\ \beta_1\lambda_1}_{q_1\ \mu_1'\ \mu_1^D}\ \times$$

$$\times\ C^{\chi_2\ \lambda_2'\ \beta_2\lambda_2}_{q_2\ \mu_2'\ \mu_2^D} = \sum_{\beta} \begin{Bmatrix} \lambda_1 & \lambda_2 & \alpha\lambda \\ \beta_1 & \beta_2 & \beta \\ \lambda_1' & \lambda_2' & \alpha'\lambda' \\ \chi_1 & \chi_2 & \beta_0\chi \end{Bmatrix} C^{\chi\ \lambda'\ \beta\lambda}_{q\ \mu'\ \mu^D} \ . \qquad (A.18)$$

Using the dual-orthogonality relation (A.7) in (A.18) one obtains

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \alpha\lambda \\ \beta_1 & \beta_2 & \beta \\ \lambda_1' & \lambda_2' & \alpha'\lambda' \\ \chi_1 & \chi_2 & \beta_0\chi \end{Bmatrix} = \sum_{\substack{\mu_1\mu_2\mu_1'\mu_2' \\ q_1 q_2 q\ \mu'}} C^{\lambda_1\ \lambda_2\ \alpha\lambda}_{\mu_1\ \mu_2\ \mu^D}\, C^{\chi_1\ \chi_2\ \beta_0\chi}_{q_1^D\ q_2^D\ q}\, C^{\lambda_1'\ \lambda_2'\ \alpha'\lambda'}_{\mu_1'^D\ \mu_2'^D\ \mu'}$$

$$\times\ C^{\chi_1\ \lambda_1'\ \beta_1\chi_1}_{q_1\ \mu_1'\ \mu_1^D}\, C^{\chi_2\ \lambda_2'\ \beta_2\lambda_2}_{q_2\ \mu_2'\ \mu_2^D}\, C^{\chi\ \lambda'\ \beta\lambda}_{q^D\ \mu'^D\ \mu} =$$

$$\langle((\chi_1,\lambda_1')\beta_1\lambda_1,(\chi_2,\lambda_2')\beta_2\lambda_2)\alpha\lambda\mu^D \mid ((\chi_1,\chi_2)\beta_0\chi\ ,(\lambda_1',\lambda_2')\alpha'\lambda')\beta\lambda\mu\rangle\ .$$

(A.19)

from which one concludes that the quantities {...} in (A.17) are 9-j symbols of the group G.

When $\chi_1 = s$, the scalar representation of G, χ_2 is necessarily equal to χ, the Clebsch-Gordan coefficients envolving χ_1 are equal to 1 and one obtains

$$\langle(\lambda_1,\lambda_2)\alpha\ \lambda\ \mu^D \mid V^{\chi}_{q} \mid(\lambda_1',\lambda_2')\ \alpha'\ \lambda'\ \mu'\rangle = \delta_{\lambda_1\lambda_1'}\ \times$$

$$\sum_{\beta_2} \langle \beta_2 \lambda_2 \| V^{\chi} \| \lambda_2' \rangle \sum_{\beta} \left\{ \begin{matrix} \lambda_1 & \lambda_2 & \alpha\lambda \\ & \beta_2 & \beta \\ \lambda_1 & \lambda_2' & \alpha'\lambda' \\ s & \chi & \chi \end{matrix} \right\} C^{\chi \quad \lambda' \quad \beta\lambda}_{q \quad \mu' \quad \mu^D} \qquad (A.20)$$

and

$$\sum_{\beta} \left\{ \begin{matrix} \lambda_1 & \lambda_2 & \alpha\lambda \\ & \beta_2 & \beta \\ \lambda_1 & \lambda_2' & \alpha'\lambda' \\ s & \chi & \chi \end{matrix} \right\} C^{\chi \quad \lambda' \quad \beta\lambda}_{q \quad \mu' \quad \mu^D} =$$

$$= \sum_{\mu_1 \mu_2 \mu_2'} C^{\lambda_1 \quad \lambda_2 \quad \alpha\lambda}_{\mu_1 \quad \mu_2 \quad \mu^D} \; C^{\lambda_1 \quad \lambda_2' \quad \alpha'\lambda'}_{\mu_1^D \quad \mu_2'^D \quad \mu'} \; C^{\chi \quad \lambda_2' \quad \beta_2\lambda_2}_{q \quad \mu_2' \quad \mu_2^D} . \qquad (A.21)$$

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$[\bar{E}]$	ε	ε'	[E]KL	[E']K'L'	$[E_0]K_0L_0$	U3IB
[000]	0	0	[000]00	[000]00	[000]00	1
[000]	3	2	[300]01	[200]00	[100]01	$\sqrt{5/3}$
[100]	1	1	[200]02	[200]02	[210]11	$-3/(4\sqrt{2})$
[200]	1	1	[300]01	[210]11	[210]11	$(1/2)/\sqrt{5/6}$
[311]	3	6	[521]13	[731]24	[960]34	$(186/385)\sqrt{2/1337}$
[411]	3	6	[621]13	[741]13	[960]13	$-(16/105)\sqrt{13/451}$
[441]	2	4	[542]12	[643]13	[640]02	$-(2/15)\sqrt{2/77}$
[511]	3	6	[721]13	[751]24	[960]34	$-(951/2695)\sqrt{6/955}$
[521]	2	4	[541]12	[651]13	[640]22	$-1/(6/\sqrt{322})$
[521]	4	2	[831]13	[541]13	[620]02	$2/(7/\sqrt{43})$
[521]	6	3	[941]14	[542]13	[930]12	$(1487/4536)\sqrt{13/4895}$
[742]	3	6	[952]37	[973]25	[960]33	$-(118/165)\sqrt{1/83265}$
[851]	4	2	[882]04	[853]23	[620]02	$-(1/27)\sqrt{5/6}$
[851]	4	2	[882]04	[862]23	[620]02	$7/(27\sqrt{30})$
[950]	6	3	[1091]13	[971]22	[930]12	$-(383/3850)\sqrt{3/770}$

Table I: a few examples of U3IB.