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ON NONLINEAR EQUATIONS ASSOCIATED WITH LIE ALGEBRAS OF DIFFEOMORPHISM GROUPS OF TWO-DIMENSIONAL MANIFOLDS

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Abstract

Kashaev R.M. et al. On Nonlinear Equations Associated with Lie Algebras of Diffeomorphism Groups of Two-Dimensional Manifolds: IHEP Preprint 90-1. - Protvino, 1990. - p. 14, refs.: 11.

We investigate here nonlinear equations associated through a zero curvature type representation with Lie algebras $S_0\mathrm{Diff}\ T^2$ and of infinitesimal diffeomorphisms of $(S^1)^2$, and also with a new infinite-dimensional Lie algebra. The latter is some symbiosis of the former two algebras. In particular, the general solution (in the sense of the Goursat problem) of the heavenly equation which describes self-dual Einstein spaces with one rotational Killing symmetry is discussed, as well as the solutions to a generalized equation. The paper is supplied with Appendix containing the definition of the continuum graded Lie algebras and the general construction of the nonlinear equations associated with them.

Аннотация

Кашаев Р.М.и др. О нелинейных уравнениях, связанных с алгебрами Ли групп диффеоморфизмов двумерных многообразий: Препринт ИФВЭ 90-1. - Серпухов, 1990. - 14 с., библиогр.: 11.

В работе исследуются нелинейные уравнения, ассоциируемые посредством представления типа нулевой кривизны с алгебрами Ли $S_0\mathrm{Diff}\ T^2$ и инфинитезимальных диффеоморфизмов $(S^1)^2$, а также новой бесконечномерной алгеброй Ли, представляющей собой некий их симбиоз. В частности, обсуждается общее решение (в смысле задачи Гурса) "божественного" уравнения, описывающего автодуальные пространства Эйнштейна с одной (вращательной) симметрией Киллинга, а также обобщающего его уравнения. Работа снабжена дополнением, содержащим определение континуальных градуированных алгебр Ли и общую конструкцию ассоциируемых с ними нелинейных уравнений.

1. The symmetries generated by infinite-dimensional Lie algebras of diffeomorphism groups of two-dimensional manifolds have been of increasing interest in theoretical physics in the last few years. Suffice it to mention their applications under investigation of the evolution equations for an incompessible fluid on a manifold; of extended objects (strings, membranes, etc.) in gauge field theories and in statistical physics; of extended conformal symmetries and higher spin fields in a continuous limit; of classical and quantum gravity; etc. It is interesting to note here that in many of these problems one and the same algebra figures, though in different aspects. This is the algebra $S_0 \mathrm{Diff}\, T^2$, the infinitesimal area-preserving diffeomorphisms of the two-dimensional torus T^2, which is isomorphic to the Poisson brackets algebra on T^2 as well as to the simplest continuous limit of the series A. Naturally, the realizations of these algebras can be completely different, while their identification as Z-graded algebras is ascertained, probably in the most simple way, in the framework of an axiomatics of the continuum Lie algebras[1,2].

In this paper, considering the Poisson brackets algebra as a continuum Lie algebra we construct a new infinite-dimensional Lie algebra which looks like a symbiosis of the algebra $S_0 \mathrm{Diff}\, T^2$ and of the algebra of infinitesimal diffeomorphisms of $(S^1)^2$. The Fourier components of the elements of all these algebras generate natural two-indexed generalizations of the Virasoro algebra, some of which were known previously (see, e.g.[3,5]). Let us stipulate from the very beginning that we will not consider here their central extensions.

In what follows we investigate the nonlinear equations associated with Lie algebras in question by means of a zero curvature type representation. For the present, the only known equation among them is, perhaps, the so-called "heavenly" equation (7) which, for the first time, appeared probably in Ref./6/ as an equation completely defining the self-dual (real Euclidean) Einstein spaces with one rotational Killing symmetry. This equation has intensely been discussed for almost a decade in the physical literature (see, e.g./7/) in connection with its role in the theory of relativity. However, only the simplest special solutions, like the Eguchi-Hanson gravitational instanton, to equation (7) have been known up to now. In the discussed context this equation is related, in fact, with the algebra $S_0\mathrm{Diff}\, T^2$. Note, that equation (7) independently appeared also in paper/8/ in view of integrable (in the sense of Liouville) Hamiltonian systems associated with the Poisson brackets algebra on T^2, and in paper/9/ as a direct continuous limit of the two-dimensional Toda lattice for the series A_n for $n\to\infty$. Finally, in paper/10/ a continuous analogue (9)of the two-dimensional Toda lattice was introduced and integrated (in the sense of a formal solution of the Goursat problem). The simplest special case of this analogue is just equation (7), for which we will obtain here an expression for the general solution simpler than those in/10/.

The paper is supplied with Appendix where the definitions of the continuum Lie algebras are given in a quite general form together with the construction of the nonlinear systems associated with these algebras.

2. Consider the Lie algebra of the functions of two variables (t_1 and t_2) with the product [,] defined by their Poisson bracket

$$[U_1,\ U_2] = -c_0\{U_1,\ U_2\} \equiv -c_0(U_{1,1}U_{2,2} - U_{1,2}U_{2,1}),$$

where $U_{i,j}\equiv\partial u_i/\partial t_j$. Let us describe some heuristic arguments clarifying a transition from this initial bracket to the bracket of form (1) or (2) which we are interested in. For this aim parametrize these elements as

$$U_i = u_i(t)\exp i\left(\frac{c_1}{c_0}t_2 - \frac{c_2}{c_0}t_1\right),\ c_0 \neq 0.$$

Then, define for the functions u_1 and u_2 a new bracket by the formula

$$[u_1, u_2] = -c_o\left(\frac{\partial u_1}{\partial t_1}\frac{\partial u_2}{\partial t_2} - \frac{\partial u_1}{\partial t_2}\frac{\partial u_2}{\partial t_1}\right) - ic_1\left(\frac{\partial u_1}{\partial t_1}u_2 - u_1\frac{\partial u_2}{\partial t_1}\right) +$$

$$+ic_2\left(\frac{\partial u_1}{\partial t_2}u_2 - u_1\frac{\partial u_2}{\partial t_2}\right) \equiv -c_o\{u_1, u_2\} - ic_1(u_{1,1}u_2 - u_1u_{2,1}) + ic_2(u_{1,2}u_2 -$$

$-u_1u_{2,2})$. Note, that the imaginary unit at c_o and c_1 in the brackets given above, was introduced to avoid its appearance in formula (1). From this formula we assume that $\hat{c}\equiv(c_o, \mathbf{c})\equiv$ $\equiv(c_o, c_1, c_2)$ is an arbitrary three-dimensional vector in $\mathbb{R}^3$, $\mathbb{Q}^3$, or $\mathbb{Z}^3$. Such a description of the Lie algebra is equivalent to its formulation in terms of the elements $X(u(\mathbf{t}))$, or, which is the same, via their Fourier components $X_m(\phi)$, for which

$$\left[X_{m_1}(\phi), X_{n_1}(\psi)\right] = X_{m_1+n_1}(-ic_o(m_1\phi\psi_{,2} - n_1\phi_{,2}\psi) +$$

$$+ c_1(m_1 - n_1)\phi\cdot\psi - ic_2(\phi_{,2}\psi - \phi\psi_{,2})). \qquad (1)$$

Here the functions ϕ and ψ depend on variable t_2. This form of writing down the algebra under discussion is nothing else but its formulation as the continuum $\mathbb{Z}$-graded Lie algebra $g = \bigoplus_{m_1\in\mathbb{Z}} g_{m_1}$. If one carries out the repeated Fourier expansion, now over the variable t_2 and with the identification $Y_{\mathbf{m}} = X_{m_1}(e^{im_2t_2})$, then the following relations come from (1):

$$[Y_{\mathbf{m}}, Y_{\mathbf{n}}] = (c_o\mathbf{m}\times\mathbf{n} + \mathbf{c}(\mathbf{m}-\mathbf{n}))Y_{\mathbf{m}+\mathbf{n}}. \qquad (2)$$

Here $\mathbf{m}=(m_1, m_2)$ and $\mathbf{n}=(n_1, n_2)$ are 2-dimensional integer vectors, $\mathbf{m}\times\mathbf{n} = m_1n_2 - m_2n_1$.

The introduced algebra can be reduced to several known particular cases by choosing some components of the constant 3-vector $\hat{c}$ equal to zero. For example, if $\mathbf{c}=0$ we come to the algebra $S_o\mathrm{Diff}\,T^2$, if $c_o=0$ and one of the components of $\mathbf{c}$ equals zero we obtain the Witt algebra (i.e., centre-free Virasoro algebra), if $c_o=0$ we have to do with the Ramos-Shrock algebra /5/. In other words, the mentioned (linear) manner of "mixing" of the algebras

$S_o \mathrm{Diff}\, T^2$ and $\mathrm{Diff}(S^1)^2$ gives again Lie algebra (2) which in what follows will be denoted as $W^{(c_o, c_1, c_2)}$.

It is evident that the case with $c_o \neq 0$ is reduced to the case with $c_o = 1$ through the trivial substitution. Finally, there takes place the following proposition.

Proposition 1.

If $c_o = 0$ and c_1/c_2 is an irrational number then the algebra $W^{(0, c_1, c_2)}$ for this case is not reduced to the Witt algebra being at the same time one-dimensional algebra (in the sense that the corresponding vector fields are taken along a one-dimensional foliation). If $c_o = 0$ and c_2/c_2 is a rational number then such an algebra $W^{(0, c_1, c_2)}$ is reduced to the Witt algebra.

Now consider the nonlinear equations which are generated, in accordance with a group-algebraic approach[11], by means of the zero curvature type representation

$$[\partial/\partial z_+ + A_+,\ \partial/\partial z_- + A_-] = 0 \tag{3}$$

with the functions $A_\pm(z_+, z_-)$ taking values in subspaces $\bigoplus_{m_1 \geqslant 0} g_{\pm m_1}$ of the algebra $g = W^{(\hat{c})}$. Here we will confine ourselves by consideration of the local part $\hat{g} = g_{-1} \oplus g_o \oplus g_{+1}$ of $W^{(\hat{c})}$, i.e., we choose

$$A_\pm = X_o(\Phi_o^\pm) + X_{\pm 1}(\Phi_1^\pm). \tag{4}$$

Substituting expansion (4) in representation (3) we obtain the following system of three equations:

$$\Phi^-_{0,z_+} - \Phi^+_{0,z_-} + c_2\Big[\{\Phi_o^+, \Phi_o^-\} + \{\Phi_1^+, \Phi_1^-\}\Big] +$$

$$+ (c_o \partial/\partial t + 2c_1)\Phi_1^+\Phi_1^- = 0, \tag{5_1}$$

$$\Phi^\pm_{1,z_+} + \Phi_1^\pm(c_o\partial/\partial t + c_1)\Phi_o^+ + c_2\{\Phi_o^+, \Phi_1^\pm\} = 0, \tag{5_2}$$

where $\{\Phi^+, \Phi^-\} = \Phi^+_{,t}\Phi^- - \Phi^+\Phi^-_{,t}$. Equations ($5_2$) serve for determination of the functions $\Phi_o^\pm$ which, being substituted in (5_1), lead to the unknown nonlinear equation. It will be convenient to consider the cases $c_2 = 0$ and $c_2 \neq 0$ separately.

3a. **Case of algebras $W^{(c_0,c_1,0)}$**. Here equations (5_2) are rewritten as

$$(\ln \Phi_1^{\pm})_{,z_+} = \pm(c_0 \partial/\partial t + c_1)\Phi_0^{+}.$$

Owing to this formula the equation we are interested in, results from (5_1) and can be presented in terms of the function $\rho \equiv \ln \Phi_1^{+}\Phi_1^{-}$ as

$$\Delta\rho + (2c_1^2 + 3c_0c_1 \partial/\partial t + c_0^2 \partial^2/\partial t^2)\exp\rho = 0, \tag{6}$$

where $\Delta \equiv \partial^2/\partial z_+\partial z_-$ is two-dimensional Laplacian. This equation has two important subcases. i) If $c_1=0$, that is for the algebra $S_0\mathrm{Diff}\, T^2 \simeq W^{(c_0,0,0)}$, then it coincides, after a trivial changing of the variables, with the heavenly equation[6]

$$\Delta\rho = (\exp\rho_h)_{,tt}. \tag{7}$$

ii) If $c_0=0$, that is for the Witt algebra $W \simeq W^{(0,c_1,0)} \simeq W^{(0,0,c_1)}$, then equation (6) reduces to the Liouville equation

$$\Delta\rho_L = 2\exp\rho_L, \tag{8}$$

whose general solution was constructed about 150 years ago. It can be expressed as

$$\exp\left(-\frac{\rho_L}{2}\right) = \exp\left(-\frac{\rho_0}{2}\right)\left[1 - \int^{z_+}\int^{z_-} dz'_+ dz'_- \exp\rho_0(z'_+, z'_-)\right]$$

in terms of the solution $\rho_0(z_+,z_-)$ of the Laplace equation $\Delta\rho_0=0$.

Note that equation (6) is a particular case of a continuous analogue

$$\Delta\rho - K\exp\rho = 0 \tag{9}$$

of the two-dimensional Toda lattice. This analogue was proposed and integrated in a formal series in Ref.[10] for invertible operator K. It is remarkable that for equation (6) the corresponding K can not be reduced (for any value of the parameters c_0 and c_1) to the differential operator of the first order. In this connection let us also remind that the case with the Cartan operator K proportional to $\partial/\partial t$ corresponds to the continuum Lie algebra of temperate but not constant (as for $K \sim \partial^2/\partial t^2$) growth[1].

Before going to the description of the general solution for equation (7) let us say a few words on it. Firstly, this equation can be rewritten in the following two equivalent forms

$$\Delta\vartheta = \exp \vartheta_{,tt}, \quad \Delta\Phi = \Phi_{,tt}\exp \Phi_{,t}. \tag{10}$$

Here ϑ and Φ, as well as $\rho_h=\vartheta_{,tt}=\Phi_{,t}$, are functions of three spatial variables $x_1\equiv z_++z_-$, $x_2\equiv -i(z_+-z_-)$ and $x_3\equiv it$. Secondly, the Lagrangian density for equation (7) has the form

$$L = \int dt\left[-\tfrac{1}{2}\vartheta_{,z_+t}\vartheta_{,z_-t} + \exp \vartheta_{,tt}\right].$$

The system under consideration possesses an improved energy-momentum tensor

$$W_2^{\pm\pm} = c\int dt\left\{\partial^2\vartheta(t)/\partial z_\pm^2 - \tfrac{1}{2}\partial\vartheta(t)/\partial z_\pm\,\partial^2/\partial t^2\,\partial\vartheta(t)/\partial z_\pm\right\}$$

with a vanishing trace, $W_2^{+-}=0$, on shell. This tensor is nothing but the integral of the second order[10] of the characteristic equation which corresponds to equation (7).

And, finally, the Riemannian metric corresponding to such a system is given by the formula[6,7]

$$ds^2 = \partial\rho_h/\partial x_3\left[(dx_1^2 + dx_2^2)\exp \rho_h + dx_3^2\right] +$$
$$+ (\partial\rho_h/\partial x_3)^{-1}\left[\pm\left\{(-\partial\rho_h/\partial x_2)dx_1 + (\partial\rho_h/\partial x_1)dx_2\right\} + dx_4\right]^2.$$

Clearly, equation (7) admits a special solution of the form

$$\exp \rho_h(z_+, z_-; t) = (d_0 + d_1t + t^2/2)\exp \rho_L(z_+, z_-),$$

which, in particular, describes the Eguchi-Hanson gravitational instanton

$$\exp \rho_h = \tfrac{1}{2}(x_3^2 - a^2)(1 + x_1^2 + x_2^2)^{-1} \quad \text{with} \quad x_3^2 \geqslant a^2.$$

A formal solution to this equation which is obtained from those constructed in[10] if $K=\partial^2/\partial t^2$ contained the corresponding number of integrations over the variables of type t at each term of the infinite series. It is clear that these integrations can be performed explicitly (in fact, as for the case of an arbitrary operator K with a support on the diagonal).

Let us introduce the following notations:

$$\Phi_{m,\omega^{-1}(m)} = \exp\left[\rho_o^+(z_m^+;\ t) + \rho_o^-(z^-_{\omega^{-1}(m)};\ t)\right],$$

where $\rho_o \equiv \rho_o^+(z_+;\ t) + \rho_o^-(z_-;\ t)$ is the solution of the Laplace equation $\Delta\rho_o = 0$; $z_o^{\pm} \equiv z_{\pm}$, ω is any permutation of the indices from 1 to n-1; $\vartheta(z)$ is the Heaviside function;

$$D^{l_m}_{n,m} = \begin{cases} \varepsilon_m(\omega) - \partial^2/\partial t^2, & l_m = 0, \\ -\vartheta[\omega^{-1}(m) - \omega^{-1}(l_m)]\partial^2/\partial t^2, & l_m \neq 0; \end{cases}$$

$$\varepsilon_m(\omega) = \begin{cases} 2 \text{ if for all} \\ 1 \quad \text{not for all} \end{cases} 1 \leqslant l \leqslant m-1 \text{ the inequality } \omega^{-1}(m) < \omega^{-1}(l_m)$$

takes place.

Then we come to the :

Proposition 2.

The solution of the Goursat (boundary value) problem for equation (7) has the form

$$\rho(z_+, z_-;\ t) = \rho_o(z_+, z_-;\ t) - \partial^2/\partial t^2 \ln\left[1 + \sum_{n \geqslant 1} (-1)^n Q_n\right], \qquad (11)$$

$$Q_n = \int\ldots\int \prod_{m=1}^{n-1} dz_m^{\pm}\vartheta(z^{\pm}_{m-1} - z^{\pm}_m)\Phi_{oo} \sum_{\omega} \sum_{l_1=0}^{0} \sum_{l_2=0}^{1} \cdots$$

$$\cdots \sum_{l_{n-1}=0}^{n-2} \left(D^{l_1}_{n,1}\Phi_{1,\omega^{-1}(1)}\right)\left(D^{l_2}_{n,2}\Phi_{2,\omega^{-1}(2)}\right)\cdots\left(D^{l_{n-1}}_{n,n-1}\Phi_{n-1,\omega^{-1}(n-1)}\right).$$

In this, the operator $D^{l_m}_{n,m}\Phi_{m,\omega^{-1}(m)}$ acts on all operator factors $\left(D^{l_s}_{n,s}\Phi_{s,\omega^{-1}(s)}\right)$ with s>m, for which $l_s = m$.

The proof of the given proposition is based on the following formula from[10]:

$$Q_n = \int\ldots\int dz_n^{\pm} \prod_{m=1}^{n-1} dt_m dz_m^{\pm}\vartheta(z^{\pm}_{m-1} - z^{\pm}_m)\cdot\sum_{\omega}\Phi_{m,\omega^{-1}(m)}\times$$

$$\times\left\{\varepsilon_m(\omega)\delta(t - t_m) - K(t, t_m) - \sum_{l=1}^{m-1} K(t_l, t_m)\vartheta\left[\omega^{-1}(m) - \omega^{-1}(l)\right]\right\}.$$

Then the integration over the variables τ_m, $1 \leq m \leq n-1$, can be performed if the kernel $K(t, t')$ of the operator K is of the δ-type, i.e., $K(t, t') = \sum_{m \geq 0} c_m (\partial/\partial t)^m \delta(t-t')$. To clarify the structure of the terms in series (11) let us write the integrands $\hat{Q}_n$, for example, for the first three functions Q_n:

$$\hat{Q}_1 = \Phi_{oo}, \quad \hat{Q}_2 = \Phi_{oo}(2 - \partial^2/\partial t^2)\Phi_{11},$$

$$\hat{Q}_3 = \Phi_{oo}\Big\{\Big[(2 - \partial^2/\partial t^2)\Phi_{22}\Big]\Big[(2 - \partial^2/\partial t^2)\Phi_{11}\Big] -$$

$$- (2 - \partial^2/\partial t^2)(\Phi_{11}\partial^2/\partial t^2\Phi_{22}) + \Big[(2 - \partial^2/\partial t^2)\Phi_{21}\Big]\Big[(2-\partial^2/\partial t^2)\Phi_{12}\Big]\Big\}.$$

Let us make the following remark in connection with the convergence problem for the series in (11). This formula results from the representation[10]

$$\exp\Big[-\vartheta(z, t)\Big] = \exp\Big[-\vartheta_o(z; t)\Big]\Big\{1 + \sum_{n \geq 1} (-1)^n \int \ldots \int \prod_{m=1}^{n} dz_m^{\pm} \times$$

$$\times \vartheta(z_{m-1}^{\pm} - z_m^{\pm}) \langle t \mid X_+^{(1)} \cdots X_+^{(n)} X_-^{(n)} \cdots X_-^{(1)} \mid t \rangle\Big\} \qquad (12)$$

for the general solution of the Goursat problem to equation (9). Here

$$X_{\pm}^{(m)} \equiv \int dt\, X_{\pm}(t) \exp \rho_o^{\pm}(z_m^{\pm}; t), \quad \rho_o^{\pm} \equiv K\vartheta_o^{\pm};$$

vectors $|t\rangle$ satisfy the relations

$$X_o(\phi)|t\rangle = \phi(t)|t\rangle, \quad X_+(\phi)|t\rangle = 0,$$

and play the role of the highest weight vectors in the corresponding modules space. At the same time in the case under consideration, i.e., for $K = \partial^2/\partial t^2$, expression (12) is directly related with the analogous formula[11] for the solution of the Toda lattice for the series A_n. Here, in fact, if the functions $\vartheta(t)$ and $X_{\pm}(t)$ have a support at the points $t = 1, 2, \ldots, n$, then (12) leads to the solution of the discrete case A_n in the form of a finite polynomial. In other words, the solutions for the series A_n are the partial sums of their continuous limit.

An interesting problem is to construct the solutions like the gravitational instantons with the topological charge N which are special (parametric) solutions. They correspond to a definite choice of the arbitrary functions $\rho_o^{\pm}$.

Note, that solution (11) of the boundary value problem for equation (7) is determined via the solution of the Laplace equation which corresponds to the free Lagrangian, in other words, to the asymptotic values of noninteracting (free) fields $\rho_0^{\pm}$. Moreover, the fact that the general solution to this equation (as well as, in accordance with[10], for equation (9) with an arbitrary invertible K) depends on two arbitrary functions $\rho_0^{\pm}(z_{\pm};\ t)$ of two variables is conformed to that for the heavens of type III (see J.D.Finley, J.F.Plebanski. J. Math. Phys. 20(1979) 1938).

3b. Case of algebras $W^{(c_0,c_1,c_2)}$ with $c_2 \neq 0$. For this case it will be convenient to pass to the gauge with $\phi_0^+=0$, in which ϕ_1^- can be equated, for example, to 1. Here, of course, one sacrifices the symmetry of writing equations (5) over z_+ and z_-. Note, that the way back is always possible due to the form-invariance of representation (3) with respect to the gauge transformations $A_{\pm}\to G^{-1}(A_{\pm}+\partial/\partial z_{\pm})G$ which do not violate the gradation spectrum of the functions $A_{\pm}$ iff $G(z_+, z_-)$ are generated by subalgebra g_0. Moreover, it will be also convenient for us to consider the examples with $c_0=c_2$ and $c_0\neq c_2$ separately.

i) $W^{(c_0,c_1,c_0)}$ with $c_0\neq 0$.

It follows from equation (5_2) that $\phi_0^- = c_0^{-1}f_{,z_-}/f_{,t}$, where $f\equiv\phi_1^+\exp\left(\frac{c_1}{c_0}t\right)$. Substituting this expression into (5_1) we obtain the sought equation

$$(f_{,z_-}/f_{,t})_{,z_+} + 2c_0^2 f_{,t}\exp\left(-\frac{c_1}{c_0}t\right) = 0. \qquad (13)$$

ii) $W^{(c_0,c_1,c_2)}$ with $c_0\neq c_2$, $c_2\neq 0$.

Let us put $c(c_0-c_2)^{-1}\equiv d$ and denote

$\exp\sigma \equiv (\phi_1^+)^{d_2}\exp(d_1 t)$.

Then equation (5_2) in this case takes the form

$(\exp\sigma)_{,z_-} = c_2(\phi_0^-\exp\sigma)_{,t}$,

that is

$c_2^{-1}\exp\sigma = \Phi_{,t}$, $\quad \phi_0^-\exp\sigma = \Phi_{,z_-}$.

whereof

$$\Phi_o^- = c_2^{-1}\Phi_{,z_-}/\Phi_{,t}, \quad (\Phi_1^+)^{d_2} = c_2\Phi_{,t}\exp(-d_1 t).$$

Substituting these formulas into eq. (5_1) with $\Phi_o^+=0$ and $\Phi_1^-=1$, we come to the equation

$$(\Phi_{,z_-}/\Phi_{,t})_{,z_+}+c_2^{1+d_2^{-1}}\Big[(c_o+c_2)\partial/\partial t + 2c_1\Big]\Big(\Phi_{,t}\exp(-d_1 t)\Big)^{1/d_2}=0. \quad (14)$$

For the case of the Ramos-Shrock algebra, i.e., $W^{(0,c_1,c_2)}$, eq. (14) essentially becomes simpler. Here $d_2=-1$, and using the variables τ, $\partial/\partial\tau = \exp\Big(-\frac{c_1}{c_2}t\Big)\partial/\partial t$, we have

$$(\Phi_{,z_-}/\Phi_{,\tau})_{,z_+} + c_2(1/\Phi_{,\tau})_{,\tau} = 0. \qquad (15_1)$$

Hence, solving this equation as

$$\Phi_{,z_-}/\Phi_{,\tau} = c_2\Psi_{,\tau}, \quad 1/\Phi_{,\tau} = -\Psi_{,z_+},$$

it can be rewritten in an equivalent form

$$\Delta\Psi = c_2\Psi^2_{,\tau}(\Psi_{,z_+}/\Psi_{,\tau})_{,\tau}. \qquad (15_2)$$

Up to now we managed to find only special solutions to equation (15), namely

$$\Phi = c_1^{-1}\exp\Big[-\frac{c_1}{c_2}t + \rho_L(z_+, z_-)\Big], \qquad (16)$$

and

$$\Phi = \lambda\exp\Big(\frac{c_1}{c_2}t\Big) + \rho_o(z_+, z_-), \quad \lambda = \text{const}. \qquad (17)$$

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Appendix

The notion of the continuum Lie algebras has not jet enter into usual vocabulary of theoreticians. At the same time an inquisitive reader would be naturally interested in realizing Lie algebras of diffeomorphism groups of two-dimensional manifolds and nonalgebras equations assoiated with them (and discussed in this paper)as a part of the general construction of the continuum Lie algebras and equations generated by these algebras. Therefore we will briefly remind the reader in Appendix of the definition and the main relations of the continuum Lie algebras[1,2], however in a more general formulation.

Let E be an arbitrary associative algebra over the field Φ; $g^{(m_o)} \equiv \bigoplus_{|i| \leq m_o} g_i$ is a direct sum of one dimensional (in a functional sense) subspaces over E whose elements $X_i(\phi)$, $\phi \in E$, satisfy the relations

$$\left[X_i(\phi),\ X_j(\psi)\right] = X_{i+j}\left[K_{ij}(\phi,\ \psi)\right] \tag{A.1}$$

for all $|i|$, $|j|$, $|i+j| \leq m_o$ and for all ϕ, $\psi \in E$. Here K_{ij} are some bilinear mappings $E \times E \to E$. We call $\hat{g} = g^{(m_o)}$ a (modified) local Lie algebra if its elements satisfy the anticommutativity condition and the Jacobi identity, i.e., in terms of the operators K,

$$\begin{aligned} &K_{ij}(\phi,\ \psi) = -K_{ji}(\psi,\ \phi); \\ &K_{k,\,j+i}(\chi,\ K_{ji}(\psi,\ \phi)) + K_{j,\,i+k}(\psi,\ K_{ik}(\phi,\ \chi)) + \\ &+ K_{i,\,k+j}(\phi,\ K_{kj}(\chi,\ \psi)) = 0. \end{aligned} \tag{A.2}$$

Here all indices and their pair sums take values from $-m_o$ to $+m_o$; ϕ, ψ, $\chi \in E$.

Now, let $\hat{g}$ be the minimal (in accordance with m_o) local Lie algebra which freely generates a Lie algebra $g'(E;\ K)$, i.e., $\hat{g}$ is the local part of g'. Denote by J the largest homogeneous ideal which has a trivial intersection with g_o. Then, by analogy with the contragredient case $(m_o=1)$[1], it is natural to call an algebra $g'(E;\ K)/J \equiv g(E;\ K)$ a continuum Lie algebra, and relations (A.1) with condition (A.2) the defining relations. Clearly, it is a $\mathbb{Z}$-graded Lie algebra, $g = \bigoplus_{m \in \mathbb{Z}} g_m$. Further we will consider only such algebras $g(E;\ K)$, for which E is an associative commutative algebra over the field $\mathbb{R}$ and $\mathbb{C}$; the mappings K_{ij} will as a rule be realized by the linear operators $E \to E$. Moreover, we will confine ourselves to the equations generated (in the framework of the group-algebraic approach[11]) by the zero curvature type representation (3) with the functions

$$A_{\pm} = \sum_{0\leqslant i\leqslant m_o} X_{\pm i}(\phi_i^{\pm}), \tag{A.3}$$

taking values in the subspaces $\bigoplus_{0\leqslant i\leqslant m_o} g_{\pm i}$ of the local part $\hat{g}$ of Lie algebra g. Substituting expansion (A.3) into (3) with account of (A.1) and (A.2) we have

$$\phi^-_{o,z_+} - \phi^+_{o,z_-} + \sum_{0\leqslant i\leqslant m_o} K_{i,-i}(\phi_i^+, \phi_i^-) = 0, \tag{A.4$_1$}$$

$$\phi^{\pm}_{j,z_+} + \sum_{0\leqslant i\leqslant m_o-j} K_{\pm(i+j),\mp i}(\phi^{\pm}_{i+j}, \phi_i^{+}) = 0, \quad 1\leqslant j\leqslant m_o, \tag{A.4$_2$}$$

(cf. (5)). Here two equations in (A.4_2) serve for finding the functions $\phi_o^{\pm}$ (under appropriate conditions for the operators K_{ij}). The remaining $2(m_o-1)$ equations in (A.4_2) together with (A.4_1), in which $\phi_o^{\pm}$ are expressed via the functions $\phi_j^{\pm}$, $1\leqslant j\leqslant m_o$, and their derivatives, represent the sought nonlinear system associated with the algebra g(E; K).

Under the natural assumptions of the contragredient case, namely

$$K_{01}(\phi, \psi) = -K_{0-1}(\phi, \psi) = \psi K_{01}\phi, \quad K_{1-1}(\phi, \psi) = K_{1-1}(\phi\cdot\psi),$$
$$K_{oo}(\phi, \psi) = 0, \tag{A.5}$$

this system is reduced to the continuous analogue (9) of the Toda lattice. Here

$$Kf(t) \equiv K_{01}K_{1-1}f(t), \quad \exp\rho \equiv \phi_1^+\phi_1^-,$$

and all functions depend on the variables z_+, z_- and t. (In accordance with the terminology adopted in/1/, the operator K in this case is called the Cartan operator). Let us especially note once more that here we speak not about a continuous limit of the Toda lattice, for which K equals $\partial^2/\partial t^2$ if one consider the series A. We speak about an essentially more general situation when K is in arbitrary integro-differential operator. Nevertheless, here it is possible, as we have already mentioned, to obtain a formal solution of the boundary value problem for equation (9) when the operator K is invertible. This solution is represented as an infinite series whose convergence properties are related with the subsequent restrictions on the form of the operator K. Moreover, by the analogy with the discrete case, i.e., for example, with the equations associated with the affine Kac-Moody algebras, it is natural to assume[1,10] that the integrability (or the convergence of the corresponding series) of their continuous analogous is also related with the conditions on the growth of the corresponding algebra. Of course, here the growth of the algebra is understood in the functional sense[1].

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О нелинейных уравнениях, связанных с алгебрами Ли
групп диффеоморфизмов двумерных многообразий.

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