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Jian-min Shen

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Zheng-mao Sheng

INTERNATIONAL ATOMIC ENERGY AGENCY

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Jian–min Shen

International Centre for Theoretical Physics, Trieste, Italy **

and

Zhejiang Institute of Modern Physics, Zhejiang University, Hanzhou, People's Republic of China †

and

Zheng–mao Sheng

Zhejiang Institute of Modern Physics, Zhejiang University, Hanzhou, People's Republic of China ††

and

Department of Physics, Hanzhou University, Hanzhou, People's Republic of China.

ABSTRACT

We consider in this paper the classical Liouville field theory on the Riemann sphere with n punctures. In terms of the uniformization theorem of Riemann surface, we show explicitly the classical exchange algebra (CEA) for the chiral components of the Liouville field. We find that the matrice which dominate the CEA is related to the symmetry of the Lie group $SL(n)$ in a nontrivial manner with $n > 3$.

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** Present address until 21 March, 1992.
† Permanent address.
†† Mailing address.

1. Introduction

To study the exchange algebraic structure in both classical and quantum theory is deeply related to integrable theory. Since the Liouville theory plays so important a role in string theory (ST), 2-d quantum gravity (QG), and conformal field theory (CFT), much attention has been focused in this area[1]. The structure of the Poisson bracket algebras, i.e. CEA, of Liouville theory on cylinder has already been explored by Gervais and Neveu[2] and etc.. In order to match the study of ST and CFT, it is necessary to find out the properties of the exchange algebraic structure for the classical Liouville theory on Riemann surfaces. In this paper, we will consider the case of the Riemann sphere with $n \geq 3$ punctures.

It is already known[3] that there are some relations between the Liouville equation and the Schröding equation with zero eigenvalue. This can be shown as follows. If ϕ is a real valued function and γ a real constant, then the following equations are equivalent:

$$\partial_z \partial_{\bar z} \phi = \gamma e^{\phi} \tag{1}$$

$$\left(\frac{\partial^2}{\partial z^2} + \frac{1}{2} h(z)\right) e^{-\frac{1}{2}\phi} = 0 \tag{2}$$

where $h(z)$ is a holomorphic function. Now let η_1 and η_2 be a pair of linearly independent solutions of equation (2) which are analytic, then we have

$$e^{-\frac{1}{2}\phi} = \frac{Im(\eta_1 \bar{\eta}_2)}{|\eta_1' \eta_2 - \eta_1 \eta_2'|} \tag{3}$$

On the other hand, it is easy to find that $h(z)$ is a Schwarz derivative by changing ϕ into a proper variable.

In the Liouville theory, starting from the Poisson bracket between the Liouville field ϕ and its conjugate momentum π, i.e. $\{\phi(\sigma), \pi(\sigma')\} = \delta(\sigma - \sigma')$, we can get

$$\{\eta_i(\sigma), \eta_j(\sigma')\} = S_{ij}^{kl} \eta_k(\sigma) \eta_l(\sigma') \tag{4}$$

where $i, j, k, l = 1, 2$ and

$$S_{ij}^{kl} = -\frac{1}{4}[r_+ \theta(\sigma - \sigma') + r_- \theta(\sigma' - \sigma)]_{ij}^{kl} \tag{5}$$

and $\theta(\sigma)$ is the step function. The 4×4 matrices $r_\pm$ are the solutions of the classical Yang-Baxter equation:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

and can be expressed by the generators of the Lie algebra $sl(2, r)$:

$$\begin{aligned} r_\pm &= \pm H \otimes H \pm 4E_\pm \otimes E_\mp \\ [H, E_\pm] &= \pm 2E_\pm, \quad [E_+, E_-] = H \end{aligned} \tag{6}$$

Furthermore we may express $exp(-k\phi/2)$ as follows

$$exp(-k\phi/2) = \sum_{\lambda=-\nu}^{\nu} F_\lambda^\nu(z)\bar{F}^{\nu,\lambda}(\bar{z}), \quad \nu = \frac{k}{2} \in \frac{1}{2}N, \quad \lambda + \nu \in N \tag{7}$$

$F_\lambda^\nu(z)$ and $\bar{F}^{\nu,\lambda}(\bar{z})$ are just the representations of the Lie group $SL(2,R)$ of spin ν[1]

2. The global solution of Liouville equation on punctured sphere.

To work on the punctured Riemann sphere, we may consider a functional[5] at

$$S(\phi) = \int_\Omega (\partial\phi)^2 + e^\phi + regularization \quad terms \tag{8}$$

where $\Omega = S^2/\{P_1, \cdots, P_n\}$ with a metric of constant negative curvature -1. Since the universal covering of Ω is the upper half plane H[6], we may define a conformal map $J : H \to \Omega$ such that

$$e^{\phi(z)} = \frac{|(J^{-1})'(z)|^2}{(ImJ^{-1}(z))^2} \tag{9}$$

where $J^{-1}(z)$ is a locally univalent linearly polymorphic function on Ω. It is easy to check that the right hand side of eq.(9) satisfies the Liouville equation $\Delta\phi = \frac{1}{2}e^\phi$. On the other hand, it was proved[3,7] that the Liouville equation is uniquely solvable in the class of smooth real- valued functions on Ω which have asymptotic behavior near puncture P_λ

$$\begin{aligned} \phi(z) &= -2\ln|z - z_\lambda| - 2\ln|\ln|z - z_\lambda|| + const. \quad z \to z_\lambda \\ \phi(z) &= -2\ln|z| - 2\ln|\ln|z|| + const. \quad z \to \infty \end{aligned} \tag{10}$$

In fact the Schördinger equation (2) can also be globally defined on Ω[8]

$$\frac{d^2\eta(P)}{dP^2} + \frac{1}{2}h(P)\eta(P) = 0 \tag{11}$$

for all points $P \in \Omega$,if we view $\eta(P)$ as the minus half order differential and $h(P)$ the Schwarz connection. It is well known that the uniformization problem of the Riemann surface is related to equation (11) with $h(P) = \{J^{-1}(P), P\}$, i.e. the Fuchsian equation:

$$\frac{d^2\eta(P)}{dP^2} + \frac{1}{2}\{J^{-1}(P), P\}\eta(P) = 0 \tag{12}$$

where $J^{-1} = \frac{\eta_1}{\eta_2}$. The Wronskian $W(P)$ of η_1 and η_2 must be a constant because of equation (11). By normalizing $W(P) = 1$, we may globally express the solution of the Liouville equation as follows

$$\phi(P) = -2\ln Im(\eta_1(P)\bar{\eta}_2(P)) \tag{13}$$

where$\phi(z)$ transforms as a pseudoscalar under the coordinate transformation.

3. Monodromy matrices and exchange algebra.

In case of $m_\lambda = m_\lambda^*$, m_λ being the accessory parameters, which appears in the Schwarz equation:

$$\{J^{-1}(z), z\} = \sum_{i=1}^{n}\left(\frac{1}{2(z - z_\lambda)^2} + \frac{m_\lambda}{z - z_\lambda}\right) \tag{14}$$

The monodromy group here is $PSL(2,R)$[9]. The elements of this group are parabolic. We therefore find that $\phi(P)$ is periodic for a closed path Γ_λ around any puncture on Ω when the analytic continuation of a pair of solutions $\eta_1(P)$ and $\eta_2(P)$ along Γ_λ results in a pair of new solutions $\eta_1^\lambda(P)$ and $\eta_2^\lambda(P)$.

That $\phi(z)$ is single valued around puncture on Ω enables us to make the following assumptions: First,

$$\{\phi(P), \pi(P')\} = \{\phi(P + \Gamma_\lambda), \pi(P' + \Gamma_\lambda)\} = \Delta(P - P') \tag{15}$$

where $P, P' \in C_\tau$ and C_τ is the level curve, $\Delta(P - P')$ is the delta function on C_τ. We define C_τ by

$$C_\tau = \{Q | Re\int_{Q_0}^{Q} \omega = \tau.\} \tag{16}$$

where $e^{\phi(z)} = \omega(z)\bar{\omega}(\bar{z})$. From (10) we know that ω can be expressed as

$$\omega(z) = \frac{1}{z - z_\lambda}\frac{1}{\ln|z - z_\lambda|} \times const, \quad z \to \quad z_\lambda \tag{17}$$

let $z - z_\lambda = r_\lambda e^{i\varphi}$, we find the integral $Re\int_{Q_0}^{Q}\omega \simeq \ln\ln r_\lambda(Q)$ is independent of φ in the neighbourhood of the puncture P_λ. This means that C_τ is a circuit around P_λ as z approaches z_λ. It is obvious that the level line will be connected or split at the zero points of ω. There are at least two loops, with same level, to move around the punctures as $\tau \to \infty$. We thus have the second assumptions:

$$\{\phi(P), \pi(P')\} = \{\phi(P + \Gamma_\lambda + \Gamma_\rho), \pi(P' + \Gamma_\lambda + \Gamma_\rho)\} = \Delta(P - P') \tag{18}$$

if C_τ will encircle the punctures P_λ and P_ρ seperately.

In terms of (15) and (18),we generalize (4) to

$$\{\eta_i^\lambda(P), \otimes\eta_j^\lambda(P')\} = S_{ij}^{kl}\eta_k^\lambda(P)\eta_l^\lambda(P') \tag{19}$$

and

$$\{\eta_i^{\lambda\rho}(P), \otimes\eta_j^{\lambda\rho}(P')\} = S_{ij}^{kl}\eta_k^{\lambda\rho}(P)\eta_l^{\lambda\rho}(P') \tag{20}$$

where $P, P' \in C_\tau, i, j, k, l = 1, 2$.

$$\begin{pmatrix} \eta_1^\lambda \\ \eta_2^\lambda \end{pmatrix} = \begin{pmatrix} M_\lambda \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \tag{21}$$

$$\begin{pmatrix} \eta_1^{\lambda\rho} \\ \eta_2^{\lambda\rho} \end{pmatrix} = \left(M_\lambda\right)\left(M_\rho\right)\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \tag{22}$$

M_λ, M_ρ are monodromy matrices. In the monodromy group we choose a standard system of parabolic generators $M_1, \cdots, M_n$ satisfying the single relation $M_1 \cdots M_n = 1$. Since $TrM_\lambda = 2$ and M_λ has one fixed point z_λ, we can write

$$M_\lambda = \begin{pmatrix} 1+\alpha_\lambda z_\lambda & -\alpha_\lambda z_\lambda^2 \\ \alpha_\lambda & 1-\alpha_\lambda z_\lambda \end{pmatrix} \tag{23}$$

and

$$M_n = \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix} \tag{24}$$

where we take $z_n = \infty$

From (19) and (20), the following Poisson brackets can be uniqualy determined

$$\begin{aligned} &\{\eta_1, \alpha_\lambda\} = -\tfrac{1}{4}\alpha_\lambda z_\lambda \eta_2, \quad &&\{\eta_1, z_\lambda\} = -\tfrac{1}{8} z_\lambda^2 \eta_2, \\ &\{\eta_2, \alpha_\lambda\} = 0, &&\{\eta_2, z_\lambda\} = -\tfrac{1}{8}\eta_1 \\ &\{\alpha_\lambda, \alpha_\rho\} = 0, &&\{z_\lambda, z_\rho\} = \tfrac{1}{8}(z_\rho^2 - z_\lambda^2) \\ &\{\alpha_\lambda, z_\rho\} = \tfrac{1}{4}\alpha_\lambda z_\rho \end{aligned} \tag{25}$$

where $\lambda, \rho = 1,\ 2,\ \cdots,\ k$. $k \geq 2$ is the number of punctures. surrounded by the level circles. Thus the elements of the monodromy matrices are dynamical variables in our case.

We analytically continue an arbitrary but fixed pair $\eta_i(P), i = 1, 2$ of independent solutions of Fuchsian equation (12) along a closed path Γ which surrounds k punctures on Ω. On returning to the starting point on Γ, we will have new solutions $\eta_i^{\Gamma}(P)$, which are related to the original solution by equation (21). By the non-trival properties shown in (25) of the elements of the monodromy matrices, we may consider all of $2k$ solutions $\eta_i^\lambda(P)$, $i = 1,\ 2,\ \lambda = 1,\ \cdots,\ k$ are independent.

After some calculations, the Poisson bracket for $\eta_i^\lambda(P)$ is found to be

$$\{\eta_i^\lambda(P), \eta_i^\rho(P')\} = -\frac{1}{16}\epsilon(P-P')[2\eta_i^\rho(P)\eta_i^\lambda(P') - \eta_i^\lambda(P)\eta_i^\rho(P')] \tag{26}$$

where $i = 1, 2$, and

$$\begin{aligned} &\{\eta_1^\lambda(P), \eta_2^\rho(P')\} \\ &= \tfrac{1}{16}\epsilon(P-P')\eta_1^\lambda(P)\eta_2^\rho(P') - \tfrac{1}{8}\epsilon(P-P')\eta_2^\rho(P)\eta_1^\lambda(P') - \tfrac{1}{8}\eta_2^\lambda(P)\eta_1^\rho(P') \end{aligned} \tag{27}$$

$$\begin{aligned} &\{\eta_2^\lambda(P), \eta_1^\rho(P')\} \\ &= \tfrac{1}{16}\epsilon(P-P')\eta_2^\lambda(P)\eta_1^\rho(P') - \tfrac{1}{8}\epsilon(P-P')\eta_1^\rho(P)\eta_2^\lambda(P') + \tfrac{1}{8}\eta_1^\lambda(P)\eta_2^\rho(P') \end{aligned} \tag{28}$$

One can check that eqs. (26)-(28) coincide with eq. (19) if $\lambda = \rho$. We arrange $\Psi_i^\lambda, i = 1, 2, \lambda = 1, \cdots, k$ into a vector

$$\Psi(P) = (\eta_1^1(P), \eta_2^1(P), \eta_1^2(P), \eta_2^2(P), \cdots, \eta_1^k(P), \eta_2^k(P)),$$

the Poisson brackets (26)-(29) can be combined into the standard form

$$\{\Psi(P), \otimes\Psi(P')\} = -\frac{1}{4}[R_+\Theta(P-P') + R_-\Theta(P'-P)]\Psi(P)\otimes\Psi(P') \tag{29}$$

For convenience we also give the component form of (29) as follows

$$\{\Psi_l(P), \otimes\Psi_m(P')\} = -\frac{1}{4}\sum_{l'm'}[R_{+lm}^{\ l'm'}\Theta(P-P') + R_{-lm}^{\ l'm'}\Theta(P'-P)]\Psi_{l'm'}(P)\otimes\Psi(P') \tag{30}$$

(i): when l, m are both odd or even

$$\begin{aligned} R_{+lm}^{\ l'm'} &= -\tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} + \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} \\ R_{-lm}^{\ l'm'} &= \tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} - \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} \end{aligned} \tag{31}$$

(ii): when l is odd and m is even

$$\begin{aligned} R_{+lm}^{\ l'm'} &= -\tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} + \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} + \tfrac{1}{2}\delta_{l'}^{l+1}\delta_{m-1}^{m'} \\ R_{-lm}^{\ l'm'} &= \tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} - \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} + \tfrac{1}{2}\delta_{l'}^{l+1}\delta_{m-1}^{m'} \end{aligned} \tag{32}$$

(iii): when l is even and m is odd

$$\begin{aligned} R_{+lm}^{\ l'm'} &= -\tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} + \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} - \tfrac{1}{2}\delta_{l'}^{l-1}\delta_{m+1}^{m'} \\ R_{-lm}^{\ l'm'} &= \tfrac{1}{4}\delta_l^{l'}\delta_m^{m'} - \tfrac{1}{2}\delta_{l'}^{m}\delta_l^{m'} - \tfrac{1}{2}\delta_{l'}^{l-1}\delta_{m+1}^{m'} \end{aligned} \tag{33}$$

As an example, we present here a matrix R_+ with $l,\ m,\ l',\ m' = 1,\ 2,\ 3,\ 4$, this corresponds to the case of $i,\ j,\ \lambda',\ \rho = 1,\ 2$ for Poisson brackets (26)–(28). R_+ is a 16×16 matrix (Please see the appendix).

Unlike r_+ appearing in (5), R_+ here is not the solution of classical Yang-Baxter equation. It, however, is interesting that R_+ can be expressed in terms of the generators of $SL(4, R)$ and

$$R_+ - R_- = C_{SL(4)\otimes SL(4)} - \frac{1}{4}I_{4\otimes 4} \tag{34}$$

where

$$C_{SL(4)\otimes SL(4)} = \sum H_i \otimes H_i + \sum_{\alpha\ positive} \frac{E_\alpha \otimes E_{-\alpha} + E_{-\alpha}\otimes E_\alpha}{(E_\alpha, E_{-\alpha})}$$

is the Casimir operator of $SL(4) \otimes SL(4)$ and $I_{4\otimes 4}$ is a $16 \otimes 16$ identity matrix.

4. Free field representation of Liouville field.

Finally, we will discuss the free field representation of the Liouville field in the neighbourhood of an arbitrary but fixed puncture. From equation (9) we know that the energy-momentum tensor for Liouville field can be expressed as

$$\partial^2\phi - \frac{1}{2}(\partial\phi)^2 = \{J^{-1}(z),\ z\} \tag{35}$$

Let ψ be a free field satisfying

$$\partial^2\psi - \frac{1}{2}(\partial\psi)^2 = \{J^{-1}(z),\ z\} \tag{36}$$

where ψ must be a pseudoscalar under conformal transformation such that $\partial^2\psi - \frac{1}{2}(\partial\psi)^2$ transforms as the Schwarzian connection. Let's assume $\psi = -2\ln\eta$, η is a $-\frac{1}{2}$ order differential, then we recover equation (12):

$$\eta'' + \frac{1}{2}\{J^{-1},\ z\}\eta = 0$$

This means that equation (32) has two linearly independent solutions $\psi_{1,2} = -2\ln\eta_{1,2}$. Denote $K = -\frac{1}{2}\partial\psi$, then $K_{1,2} = \frac{\partial\eta_{1,2}}{\eta_{1,2}}$. In terms of (4), we find

$$\{K(\sigma),\ K(\sigma')\} = \Delta'(\sigma - \sigma') \tag{37}$$

even though $K(\sigma)$ is not periodic here. The monodromy matrix in our case can not be diagonalized. We, however, may appropriately select the basis of the solutions η such that the monodromy matrix is changed into the upper (or down) triangular matrix

$$\tilde{M}_\lambda = S_\lambda M_\lambda S_\lambda^{-1} = \begin{pmatrix} 1 & x_\lambda \\ 0 & 1 \end{pmatrix}$$
$$\eta(\lambda) = S_\lambda\eta, \quad \eta^T = (\eta_1,\ \eta_2) \tag{38}$$

where

$$S_\lambda = \begin{pmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{pmatrix} \quad S_\lambda^{-1} = det^{-1}S_\lambda \begin{pmatrix} d_\lambda & -b_\lambda \\ -c_\lambda & a_\lambda \end{pmatrix}$$
$$S_\lambda, \quad S_\lambda^{-1} \in PSL(2,\ R) \tag{39}$$

with

$$a_\lambda = \frac{\alpha_\lambda(1-\alpha_\lambda z_\lambda)}{1-\alpha_\lambda z_\lambda+\alpha_\lambda z_\lambda^2} - f_\lambda, \quad b_\lambda = -\frac{(1-\alpha_\lambda z_\lambda)(1+\alpha_\lambda z_\lambda)}{1-\alpha_\lambda z_\lambda+\alpha_\lambda z_\lambda^2} + z_\lambda f_\lambda,$$
$$c_\lambda = \frac{1-\alpha_\lambda z_\lambda+\alpha_\lambda z_\lambda^2}{1-\alpha_\lambda z_\lambda}, \quad d_\lambda = -\frac{z_\lambda(1-\alpha_\lambda z_\lambda+\alpha_\lambda z_\lambda^2)}{1-\alpha_\lambda z_\lambda}$$
$$f_\lambda = -2 + 5\alpha_\lambda z_\lambda - 5\alpha_\lambda^2 z_\lambda^2 + \alpha_\lambda^3 z_\lambda^3 + 2\alpha_\lambda z_\lambda^2 - 4\alpha_\lambda^2 z_\lambda^3 + 2\alpha_\lambda^3 z_\lambda^4 \tag{40}$$
$$x_\lambda = \frac{-\alpha_\lambda(1+\alpha_\lambda z_\lambda)^2}{(1-\alpha_\lambda z_\lambda+\alpha_\lambda z_\lambda^2)^2}$$

such that $\eta^\lambda_{2,(\lambda)} \equiv \eta_{2,(\lambda)}(\sigma + \Gamma_\lambda) = \eta_{2,(\lambda)}(\sigma)$ as well as $K^\lambda_{2,(\lambda)} \equiv K_{2,(\lambda)}(\sigma + \Gamma_\lambda) = K_{2,(\lambda)}(\sigma) = \frac{\partial\eta_{2,(\lambda)}}{\eta_{2,(\lambda)}}$. Compare the Liouville field ϕ with the free field ψ, we find that both of them can be expressed by η, i.e. $\phi = -2\ln Im(\eta_1\bar{\eta}_2)$ and $\psi = -2\ln\eta_{2,(\lambda)}$. However ϕ is periodic around any punctures on Ω and ψ is only periodic around an arbitrary but fixed puncture. Then $\eta_{2,(\lambda)} = c_\lambda\eta_1(\sigma) + d_\lambda\eta_2(\sigma)$ is periodic around puncture P_λ. By some algebraic calculation, we find

$$\{\eta_{2,(\lambda)}(\sigma), \eta_{2,(\lambda)}(\sigma')\} = -\frac{1}{16}\epsilon(\sigma - \sigma')\eta_{2,(\lambda)}(\sigma)\eta_{2,(\lambda)}(\sigma') \tag{41}$$

Thus one can check

$$\{K_{2,(\lambda)}(\sigma), K_{2,(\lambda)}(\sigma')\} = \Delta'(\sigma - \sigma') \tag{42}$$

From equation (12) we know $\frac{\eta_2''}{\eta_2} = \frac{\eta_1''}{\eta_1}$. Precisely, we may rewrite eqution (32) as

$$\partial^2\psi_{i(\lambda)} - \frac{1}{2}(\partial\psi_{i(\lambda)})^2 = \{J^{-1}(P), P\} \tag{43}$$

where $i = 1,\ 2$, $\lambda = 1, \cdots, n$, $\psi_{2,(\lambda)} = -2\ln\eta_{2,(\lambda)}$, $\psi_{1,(\lambda)} = -2\ln\eta_{1,(\lambda)}$. We will consider the relations among these free fields and their quantization elsewhere.[10]

5. Conclusion

In this paper, we discuss the relation between the solutions of the Fuchsian equation and the solution of the Liouville equation as well as the monodromy group on the Riemann surfaces with $n > 3$ punctures. We find the exchange algebra for the chiral components of Liouville field, and present the free representation of Liouville field near an arbitrary fixed puncture on punctured sphere.

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Appendix

$R_+ =$

	11	12	21	22	13	31	14	41	23	32	24	42	33	34	43	44
11	$\frac{1}{4}$	0	0	0												
12	0	$-\frac{1}{4}$	1	0												
21	0	0	$-\frac{1}{4}$	0												
22	0	0	0	$\frac{1}{4}$												
13					$-\frac{1}{4}$	$\frac{1}{2}$										
31					$\frac{1}{2}$	$-\frac{1}{4}$										
14							$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	0						
41							$\frac{1}{2}$	$-\frac{1}{4}$	0	$-\frac{1}{2}$						
23							$-\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{1}{2}$						
32							0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$						
24											$-\frac{1}{4}$	$\frac{1}{2}$				
42											$\frac{1}{2}$	$-\frac{1}{4}$				
11													$\frac{1}{4}$	0	0	0
34													0	$-\frac{1}{4}$	1	0
43													0	0	$-\frac{1}{4}$	0
44													0	0	0	$\frac{1}{4}$

(44)

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