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Twistor-Theoretic Approach to Topological Field Theories

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Abstract

The two-dimensional topological field theory which describes a four-dimensional self-dual space-time (gravitational instanton) as a target space, which we constructed before, is shown to be deeply connected with Penrose's "twistor theory". The relations are presented in detail. Thus our theory offers a "twistor theoretic" approach to topological field theories.

In a previous paper,[1] we constructed a two dimensional topological field theory which describes four-dimensional self-dual space-time (gravitational instanton) as a target space. We obtained the model[1] by twisting a two-dimensional N=4 self-interacting harmonic superfield model[5)6)] which followed from four-dimensional N=2 model,[2)-4)] through a dimensional reduction. Due to the presence of the harmonic superspace, the self-dual structure of the four-dimensional space-time is incorporated automatically in the theory.

In the revised version of the previuos paper,[1] we showed briefly that our theory is deeply connected with Penrose's twistor theory.[10)-14)] The purpose of this paper is to elaborate on this relationship and show how deeply our harmonic superspace approach to topological field theory is connected with the twistor theory.

Before we proceed, we review briefly the model which we constructed in the previous paper.[1] The action is given by;

$$S = \int d^2\sigma_A \, d\theta^+_{++} \, d\theta^+ d\bar{\theta}^+ d\bar{\theta}^+_{--} \, du \, \mathcal{L}^{(+4)}$$

$$\mathcal{L}^{(+4)} = \overset{*}{q}{}^+_{tw} D^{++} q^+_{tw} + V^{(+4)}(q^+_{tw}, \overset{*}{q}{}^+_{tw}) \qquad \cdots (1)$$

Here, superfield q^+_{tw} is a twisted one on the analytic superspace $\{\sigma_{A\pm\pm}, \theta^+, \theta^+_{++}, \bar{\theta}^+, \bar{\theta}^+_{--}, u^\pm_i\}$ and can be expanded in terms of the Grassmann coordinates as follows;

$$\begin{aligned} q^+_{tw} = {} & F^+ + \theta^+_{++}\psi_{--} - \theta^+\psi + \bar{\theta}^+\bar{\varphi} - \bar{\theta}^+_{--}\bar{\varphi}_{++} \\ & + 2\theta^+_{++}\theta^+ M^-_{--} + 2\bar{\theta}^+\bar{\theta}^+_{--} N^-_{++} - \theta^+_{++}\bar{\theta}^+ A^-_{--} - \theta^+\bar{\theta}^+_{--} A^-_{++} \\ & + \theta^+\bar{\theta}^+ B^- + \theta^+_{++}\bar{\theta}^+_{--} C^- + 2\bar{\theta}^+\bar{\theta}^+_{--}(\theta^+_{++}\xi^{--} - \theta^+\xi^{--}_{++}) \\ & + 2\theta^+_{++}\theta^+(\bar{\theta}^+\bar{\chi}^-_{--} - \bar{\theta}^+_{--}\bar{\chi}^{--}) + 4\theta^+_{++}\theta^+\bar{\theta}^+\bar{\theta}^+_{--} D^{---} \end{aligned} \qquad \cdots (2)$$

Each component is a function of the harmonic coordinates u^+_i, u^-_i. For instance,

$$F^{+}=\sum_{n=0}^{\infty} f^{(i_1\cdots i_{n+1}\, j_1\cdots j_n)} u^{+}_{i_1}\cdots u^{+}_{i_{n+1}} u^{-}_{j_1}\cdots u^{-}_{j_n} \qquad \cdots(3)$$

The twisted harmonic derivative D^{++} is given by,

$$D^{++}=\partial^{++}+2i\,(\theta^{+}_{++}\bar\theta^{+}\partial_{--}+\theta^{+}\bar\theta^{+}_{--}\partial_{++}), \qquad \partial^{++}=u^{+i}\frac{\partial}{\partial u^{-i}} \qquad \cdots(4)$$

The action has fermionic symmetries generated by four charges : Q^{+} , Q^{-} , $\bar{Q}^{+}$, $\bar{Q}^{-}$, which have spin zero.

$$Q^{+}=\frac{\partial}{\partial\theta^{-}}-2i\,\bar\theta^{+}_{--}\partial_{++}\,, \qquad Q^{-}=-\frac{\partial}{\partial\theta^{+}}$$

$$\bar{Q}^{+}=-\frac{\partial}{\partial\bar\theta^{-}}-2i\,\theta^{+}_{++}\partial_{--}, \qquad \bar{Q}^{-}=\frac{\partial}{\partial\bar\theta^{+}} \qquad \cdots(5)$$

$Q_L=Q^{+}+Q^{-}$, $Q_R=\bar{Q}^{+}+\bar{Q}^{-}$ obey a relation $Q_L^2=Q_R^2=\{Q_L,Q_R\}=0$. We regard $Q_B=Q_L+Q_R$ as the charge which generate the BRST transformation of our theory (BRST charge). This charge satisfies the nilpotency condition: $Q_B^2=0$. Acting the scalar charges on the twisted superfield q^{+}_{tw} ,we find the following transformation rules.

$$\begin{aligned}
\delta F^{+}&=\varepsilon^{+}\psi+\bar\varepsilon^{+}\bar\varphi\\
\delta\psi_{--}&=-2\varepsilon^{+}M^{-}_{--}+2i\bar\varepsilon^{-}\partial_{--}F^{+}-\bar\varepsilon^{+}A^{-}_{--}\\
\delta\bar\varphi&=\varepsilon^{+}B^{-}\\
\delta\psi&=-\bar\varepsilon^{+}B^{-}\\
\delta\bar\varphi_{++}&=\varepsilon^{+}A^{-}_{++}-2i\varepsilon^{-}\partial_{++}F^{+}+2\bar\varepsilon^{+}N^{-}_{++}\\
\delta M^{-}_{--}&=i\bar\varepsilon^{-}\partial_{--}\psi+\bar\varepsilon^{+}\bar\chi^{--}_{--}\\
\delta N^{-}_{++}&=i\varepsilon^{-}\partial_{++}\bar\varphi+\varepsilon^{+}\xi^{--}_{++}\\
\delta A^{-}_{--}&=-2\varepsilon^{+}\bar\chi^{--}_{--}+2i\bar\varepsilon^{-}\partial_{--}\bar\varphi\\
\delta A^{-}_{++}&=2i\varepsilon^{-}\partial_{++}\psi-2\bar\varepsilon^{+}\xi^{--}_{++}\\
\delta B^{-}&=0
\end{aligned}$$

$$\delta C^{-} = 2i\varepsilon^{-}\partial_{++}\psi_{--} - 2\varepsilon^{+}\bar{\chi}^{--} + 2i\bar{\varepsilon}^{-}\partial_{--}\bar{\varphi}_{++} - 2\bar{\varepsilon}^{+}\xi^{--}$$

$$\delta\xi^{--} = -i\varepsilon^{-}\partial_{++}A^{-}_{--} - 2\varepsilon^{+}D^{---} + 2i\bar{\varepsilon}^{-}\partial_{--}N^{-}_{++}$$

$$\delta\xi^{-}_{++} = -i\varepsilon^{-}\partial_{++}B^{-}$$

$$\delta\bar{\chi}^{--} = -2i\varepsilon^{-}\partial_{++}M^{-}_{--} + i\bar{\varepsilon}^{-}\partial_{--}A^{-}_{++} + 2\bar{\varepsilon}^{+}D^{---}$$

$$\delta\bar{\chi}^{--}_{--} = i\bar{\varepsilon}^{-}\partial_{--}B^{-}$$

$$\delta D^{---} = i\varepsilon^{-}\partial_{++}\bar{\chi}^{--}_{--} + i\bar{\varepsilon}^{-}\partial_{--}\xi^{--}_{++} \tag{6}$$

Now the BRST transformation rules are obtained by setting $\varepsilon^{+} = \varepsilon^{-} = \bar{\varepsilon}^{+} = \bar{\varepsilon}^{-} \equiv \varepsilon$, in the above formulae. The BRST transformation rules satisfy the nilpotency condition $Q_B^2 = 0$, as they should.

Component fields can be truncated consistently by imposing the following conditions:

$$\psi = \bar{\varphi} \qquad \xi^{--} = -\bar{\chi}^{--}$$

$$\partial_{++}M^{-}_{--} - \partial_{--}N^{-}_{++} = 0$$

$$B^{-} = 0 \qquad C^{-} = 0$$

$$\partial_{++}A^{-}_{--} - \partial_{--}A^{-}_{++} = 0$$

$$\partial_{++}\psi_{--} + \partial_{--}\bar{\varphi}_{++} = 0$$

$$\partial_{++}\bar{\chi}^{--}_{--} - \partial_{--}\xi^{--}_{++} = 0 \qquad \cdots (7)$$

After the truncation, the BRST transformation rules become,

$$\delta F^{+} = 2\varepsilon^{+}\psi \qquad \delta\psi = 0$$

$$\delta\psi_{--} = -2\varepsilon^{+}H^{-}_{--} + 2i\varepsilon^{-}\partial_{--}F^{+}$$

$$\delta H^{-}_{--} = 2i\varepsilon^{-}\partial_{--}\psi$$

$$\delta G^{-}_{--} = 2\varepsilon^{+}\bar{\chi}^{--}_{--} \qquad \delta\bar{\chi}^{--}_{--} = 0$$

$$\delta\bar{\chi}_{--}^{--} = -2i\varepsilon^{-}\partial_{++} G_{--}^{-} + 2\varepsilon^{+} D^{---}$$
$$\delta D^{---} = 2i\varepsilon^{-}\partial_{++}\bar{\chi}_{--}^{--}$$

where:

$$H_{--}^{-} = M_{--}^{-} + \frac{1}{2} A_{--}^{-}$$
$$G_{--}^{-} = M_{--}^{-} - \frac{1}{2} A_{--}^{-} \qquad \cdots (8)$$

The model defined by the above action is topological invariant, which is proved as follows. The action is manifestly BRST invariant and it is manifestly a BRST commutator. Actually, it is written as:

$$S = -\left\{ Q_B, \int d^2\sigma_A du d\bar{\theta}^{+} d\bar{\theta}_{--}^{+} d\theta_{++}^{+} \mathcal{L}^{(+4)} + \int d^2\sigma_A du d\theta_{++}^{+} d\theta^{+} d\bar{\theta}_{--}^{+} \mathcal{L}^{(+4)} \right\} \qquad \cdots (9)$$

The energy momentum tensor is the variation of the action with repect to the metric of two dimensions $h_{\alpha\beta}$;[7)]

$$\delta S = \frac{1}{2} \int_{\Sigma} \sqrt{h}\, \delta h^{\alpha\beta} T_{\alpha\beta} \qquad \cdots (10)$$

Since the action is a BRST commutator, it follows that the energy momentum tensor is also a BRST commutator. On the other hand, a criterion of being a topological field theory is that the energy momentum tensor be a BRST commutator.[7),8)] Hence the model defined by the above action is indeed a topological field theory.

Our theory is obtained from self-interacting N=2,d=4 harmonic superfield models(refs. 2−4), by two steps. First, one reduces the dimension from d=4 to d=2, and then twist it.

We can show that the topological field theory which we have constructed describes the "topological phase" of four dimensional self-dual gravity, using the same method as that used in the case of the extended supersymmetry theory.[2)-4)] For example, consider the case where the potential $V^{(+4)}$ is given by

$$V^{(+4)} = \frac{\lambda}{2}(q^+ \overset{*}{q}{}^+)^2 \qquad \cdots (11)$$

Putting the theory on the mass shell, and eliminating auxiliary fields, by use of equations of motion, the effective target space in the $\mathcal{R}^4$ - space$\{f^i, \bar{f}_i\}$ (i=1,2) emerges[2)-4)] whose metric is:

$$ds^2 = g_{ij}\, df^i df^j + \bar{g}^{ij}\, d\bar{f}_i d\bar{f}_j + 2h^i{}_j\, df^j d\bar{f}_i$$

where:
$$g_{ij} = \frac{\lambda(2+\lambda f\bar{f})}{2(1+\lambda f\bar{f})}\bar{f}_i\bar{f}_j$$
$$\bar{g}^{ij} = \frac{\lambda(2+\lambda f\bar{f})}{2(1+\lambda f\bar{f})}f^i f^j$$
$$h^i{}_j = \delta^i{}_j(1+\lambda f\bar{f}) - \frac{\lambda(2+\lambda f\bar{f})}{2(1+\lambda f\bar{f})}f^i\bar{f}_j \qquad \cdots (12)$$

where f^i , $\bar{f}_i$ are related to the scalar component of the twisted superfield F^+ by:[2)-4)]

$$F^+ = f^i u_i^+ e^{\lambda f^{(j}\bar{f}^{k)}u_j^+ u_k^-} \qquad \cdots (13)$$

In terms of "spherical coordinates" (ρ , θ , φ , ψ):

$$f^1 = \rho\cos\frac{\theta}{2}\, e^{\frac{i}{2}(\psi+\varphi)}, \quad f^2 = \rho\sin\frac{\theta}{2}\, e^{\frac{i}{2}(\psi-\varphi)}, \quad \rho^2 = 2(r-m)m$$

the metric is rewritten as:[2)-4)]

$$m = \frac{1}{2\sqrt{\lambda}} \qquad \cdots (14)$$

$$ds^2 = 2\left\{\frac{1}{4}\frac{r+m}{r-m}dr^2 + \frac{1}{4}(r^2-m^2)(d\theta^2+\sin^2\theta\, d\varphi^2) + m^2\frac{r-m}{r+m}(d\psi+\cos\theta\, d\varphi)^2\right\} \cdots (15)$$

This is the Taub-NUT metric,[3),9)] which is a four dimensional self-dual metric. Changing the potential term to other ones, we can construct a topological field theory of more general self-dual metric in four dimensions. The correspondence between the form of the potential and the target space species, is the same as in the case of extended supersymmetry.[2)-4)]

We have seen that if we put our model on the mass-shell, and eliminate an infinite number of auxiliary fields, it reduces to a topological sigma model with a self-dual metric. Then what is the virtue to start from self-interacting N=4 harmonic superfield model and twist it?

The most remarkable virtue of the harmonic superspace approach to topological field theories, is its direct and deep connection to Penrose's "twistor theory".[10)-14)] In the revised version of ref. 1), we have presented the connection briefly. In this paper, we give a more detail about it. (A relation between harmonic superspace and the twistor, in the context of N=2 string theory, was suggested in ref.5 .

One of the virtues of the twistor theory is that the topological structure of the manifold is represented quite explicitly.[10)-14)] In fact, information of topology of the manifold is encoded in "transition function" in twistor theory,[10)-14)] which, in more tradditional formulation, emerges only after the integration of local quantities over the whole manifold. Another virtue is that, in twistor theory, one can construct a set of classical solutions of gauge or gravity theory systematically and classify them.[10)-14)] Even, new solutions have been found which had not been reached in a tradditional approach.[10)-14)]

On the other hand, in topological field theories, a classification of classical solutions of the theory and their moduli space are very important rather than the metric of a manifold.[7),8)]

In view of these facts, twistor theoretic approach is particularly well suited to topological field theories. Now we will see below that if we twist N=4 harmonic superfield model, the "twistor geometry" is incorporated automatically into the theory. Therefore, N=4 harmonic superspace approach is expected to be a good approach which inherits virtues of the twistor theory.

Now we explain the relation between our theory and the twistor theory. (A relation between harmonic superspace and the twistor, in the context of N=2 string theory, was suggested in ref. 5). In the twistor theory, the space-time manifold, in which physical phenomena take place, is replaced by the "twistor space" which is parametrized by two sets of two-component spinors. $Z^{\alpha} = (\omega^A, \pi_{A'})$ $(A = 0, 1, A' = 0, 1)$. A topological field theory obtained by twisting N=4 harmonic superfield with self-interaction is connected to a curved twistor space [11)-14)]. A curved twistor space is a fiber bundle whose base and the fiber are $\{\pi_{A'}\}$ and $(\omega^A, \pi_{A'})$, respectively, and is defined by "transition functions" f^A which relates twistor coordinates of two patches $\mathcal{U}(\omega^A, \pi_{A'})$ and $\hat{\mathcal{U}}(\omega^A, \pi_{A'})$, through,

$$\hat{\pi}_{A'} = \pi_{A'}, \qquad \hat{\omega}^A = \omega^A + f^A(\omega^B, \pi_{B'}) \qquad \cdots (16)$$

A finite deformation arises from an infinitesimal deformation,

$$\hat{\omega}^A = \omega^A + \varepsilon \frac{\partial f}{\partial \omega_A} \qquad \cdots (17)$$

as the integral curves of the vector field: $\dfrac{\partial f}{\partial \omega_A} \dfrac{\partial}{\partial \omega^A}$,

i. e. by solving the differential equation,[11)-14)]

$$\frac{d\nu^A}{d\xi} = \varepsilon^{AB} \frac{\partial f(\nu^C, \pi_{C'})}{\partial \nu^B} \qquad \cdots (18)$$

The twistor curve [11)-14)] is a solution of the above differential equations, with $\omega^A = \nu^A(0)$ and $\hat{\omega}^A = \nu^A(\xi)$ for some $\xi \neq 0$

On the other hand, $\theta = 0$ components of the equations of motion of our harmonic superfield topological field theory are:

$$\partial^{++} F^{++} + \lambda (F^{+} \overset{*}{F}{}^{+}) F^{+} = 0$$
$$\partial^{++} \overset{*}{F}{}^{++} - \lambda (F^{+} \overset{*}{F}{}^{+}) \overset{*}{F}{}^{+} = 0 \qquad \partial^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} \qquad \cdots (19)$$

which take the same form as those of susy theories[2)-4)]. The potential is taken to be: ($V^{(+4)} = \frac{\lambda}{2} (q^{+} \overset{*}{q}{}^{+})^2$)

There is a remarkable correspondence between harmonic

superspace coordinates and the twistor coordinates, as follows;
$$\mathcal{U}_1^+ = \pi_{0'}, \quad \mathcal{U}_2^+ = \pi_{1'}, \qquad F^+ = \omega^0, \quad \overset{*}{F}{}^+ = -\omega^1$$
$$\mathcal{U}^{+1} = \pi_{1'} \quad \mathcal{U}^{+2} = -\pi_{0'} \qquad \cdots (20)$$

Rewriting the equations of motion, using these correspondences, we find:

$$\frac{\partial}{\partial\xi}\omega^0 = \lambda\frac{\omega^0\omega^1}{\pi_{0'}\pi_{1'}}\omega^0, \quad \frac{\partial}{\partial\xi}\omega^1 = -\lambda\frac{\omega^0\omega^1}{\pi_{0'}\pi_{1'}}\omega^1, \quad \frac{\partial}{\partial\xi} = \frac{1}{\pi_{1'}}\frac{\partial}{\partial\bar{\pi}_{1'}} - \frac{1}{\pi_{0'}}\frac{\partial}{\partial\bar{\pi}_{0'}}$$

Rewritten in the following form, $\cdots (21)$

$$\frac{\partial}{\partial\xi}\omega^A = \varepsilon^{AB}\frac{\partial f}{\partial\omega^B}, \qquad f = \frac{\lambda}{2}\frac{(\omega^0\omega^1)^2}{\pi_{0'}\pi_{1'}} \qquad \cdots (22)$$

it turns out that these equations are nothing but differential equations satisfied by twistor curves corresponding to Taub-NUT metric.[1)-14)] Hence our harmonic superfield approach to topological field theories is directly connected to the twistor theory.

The correspondence, actually, goes further, and turns out to be quite deep. There is a precise correspondence between the processes of solving the equations of motion of the topological field theory and solving the twistor differential equations. Following refs. 3), the equations of motion can be linearized by the change of variables from F to f, by:

$$F^+ = f^+ e^{-\frac{\lambda}{2}(f^+\overset{*}{f}{}^- + f^-\overset{*}{f}{}^+)}, \qquad \overset{*}{F}{}^+ = \overset{*}{f}{}^+ e^{\frac{\lambda}{2}(f^+\overset{*}{f}{}^- + f^-\overset{*}{f}{}^+)} \qquad \cdots (23)$$

Then the equations of motion reduce to the linear differential equations:[3)]

$$\partial^{++} f^+ = 0, \qquad \partial^{++}\overset{*}{f}{}^+ = 0 \qquad \cdots (24)$$

Therefore, the harmonic expansions of f^+ and $\overset{*}{f}{}^+$ involve only terms linear in $\mathcal{U}_i^+$.

$$f^+ = f^i\mathcal{U}_i^+ \qquad \overset{*}{f}{}^+ = \bar{f}_i\,\mathcal{U}^{+i} \qquad \cdots (25)$$

and F^{+} ($\overset{*}{F}{}^{+}$) can be written as:[3)]

$$\begin{pmatrix} F^{+} \\ -\overset{*}{F}{}^{+} \end{pmatrix} = \begin{pmatrix} e^{-\frac{\lambda}{2}(f^{+}\overset{*}{f}{}^{-}+f^{-}\overset{*}{f}{}^{+})} & 0 \\ 0 & e^{\frac{\lambda}{2}(f^{+}\overset{*}{f}{}^{-}+f^{-}\overset{*}{f}{}^{+})} \end{pmatrix} \begin{pmatrix} f^{1} & f^{2} \\ -\bar{f}_{2} & \bar{f}_{1} \end{pmatrix} \begin{pmatrix} u_{1}^{+} \\ u_{2}^{+} \end{pmatrix} \quad \cdots (26)$$

On the other hand, the differential equations satisfied by the twistor curves are rewritten in terms of $g(q, \pi_{A'})$, which is a function homogeneous of degree zero, as follows (following ref. 14).

$$\frac{d\omega^{A}}{d\xi} = -\phi^{AB}\omega_{B}\, g(q, \pi_{A'}) \qquad \phi^{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$g = -\lambda \frac{\omega^{0}\omega^{1}}{\pi_{0'}\pi_{1'}} \qquad q = \phi_{AB}\,\omega^{A}\omega^{B} \quad \cdots (27)$$

These differential equations (twistor equations) can be linearized by the following change of variables:[11)-14)]

$$\omega^{0} = \tilde{\omega}^{0} e^{\frac{\lambda}{2}(\tilde{\omega}^{0}\bar{\tilde{\omega}}^{0} - \tilde{\omega}^{1}\bar{\tilde{\omega}}^{1})}, \qquad \omega^{1} = \tilde{\omega}^{1} e^{-\frac{\lambda}{2}(\tilde{\omega}^{0}\bar{\tilde{\omega}}^{0} - \tilde{\omega}^{1}\bar{\tilde{\omega}}^{1})} \quad \cdots (28)$$

Then the twistor equations become:[11)-14)]

$$\frac{\partial}{\partial \xi}\tilde{\omega}^{0} = 0, \qquad \frac{\partial}{\partial \xi}\tilde{\omega}^{1} = 0 \quad \cdots (29)$$

We find the corespondences between the new twistor variables and the new harmonic superspace variables, as follows:

$$\tilde{\omega}^{0} = f^{+}, \quad \tilde{\omega}^{1} = -\overset{*}{f}{}^{+}, \quad \bar{\tilde{\omega}}^{0} = -\overset{*}{f}{}^{-}, \quad \bar{\tilde{\omega}}^{1} = -f^{-} \quad \cdots (30)$$

With this correspondence, the linearizations of the differential equations for the both theories turn out to be the same procedures. The twistor variables $\tilde{\omega}^{0}$, $\tilde{\omega}^{1}$ are global and homogeneous of degree one, hence they are linear in $\pi_{A'}$.[10)-14)]

$$\tilde{\omega}^A = x^{AB'}\pi_{B'} \qquad \cdots(31)$$

This corresponds to $f^+ = f^i u_i^+$, $\overset{*}{f}{}^+ = \bar{f}_i u^{+i}$, and from these correspondences, we find the following relations between $x^{AB'}$ and f^i .

$$x^{00'} = f^1, \quad x^{01'} = f^2, \quad x^{10'} = -\bar{f}_2, \quad x^{11'} = \bar{f}_1 \qquad \cdots(32)$$

Therefore, the twistor variables ω^0 , ω^1 can be rewritten as:[11)-14)]

$$\begin{pmatrix} \omega^0 \\ \omega^1 \end{pmatrix} = \begin{pmatrix} e^{\frac{\lambda}{2}(\tilde{\omega}^0\bar{\tilde{\omega}}^0 - \tilde{\omega}^1\bar{\tilde{\omega}}^1)} & 0 \\ 0 & e^{-\frac{\lambda}{2}(\tilde{\omega}^0\bar{\tilde{\omega}}^0 - \tilde{\omega}^1\bar{\tilde{\omega}}^1)} \end{pmatrix} \begin{pmatrix} x^{00'} & x^{01'} \\ x^{10'} & x^{11'} \end{pmatrix} \begin{pmatrix} \pi_{0'} \\ \pi_{1'} \end{pmatrix} \qquad \cdots(33)$$

which is nothing but eq.(26), with the correspondence between twistor and harmonic variables. With these ω's , we can calculate the twistor function g which is homogeneous of degree zero.[11)-14)]

$$g = \lambda\left[x^{01'}x^{11'}\frac{\pi_{1'}}{\pi_{0'}} + (x^{00'}x^{11'} + x^{01'}x^{10'}) + x^{00'}x^{10'}\frac{\pi_{0'}}{\pi_{1'}}\right] \qquad \cdots(34)$$

"g" breaks up into two pieces,[11)-14)] h and $\hat{h}$, which are holomorphic in the two patches $\mathcal{U}$ and $\hat{\mathcal{U}}$,respectively: $\mathcal{U} = \{\pi_{A'} \mid \pi_{1'} \neq 0\}$, $\hat{\mathcal{U}} = \{\pi_{A'} \mid \pi_{0'} \neq 0\}$.

$$\begin{aligned} g &= h - \hat{h} \\ h &= \lambda\left[\frac{1}{2}(x^{00'}x^{11'} + x^{01'}x^{10'}) + x^{00'}x^{10'}\frac{\pi_{0'}}{\pi_{1'}}\right] \\ \hat{h} &= \lambda\left[-x^{01'}x^{11'}\frac{\pi_{1'}}{\pi_{0'}} - \frac{1}{2}(x^{00'}x^{11'} + x^{01'}x^{10'})\right] \end{aligned} \qquad \cdots(35)$$

This can be translated back to harmonic superfield topological field theory, in terms of the correspondences between th twistor and harmonic variables.

$$g = \lambda \left[f^2 \bar{f}_1 \frac{u_2^+}{u_1^+} + (f^1\bar{f}_1 - f^2\bar{f}_2) - f^1\bar{f}_2 \frac{u_1^+}{u_2^+} \right] \quad \cdots (36)$$

which breaks up into two pieces: $g = h - \hat{h}$ with

$$h = \lambda \left[\frac{1}{2}(f^1\bar{f}_1 - f^2\bar{f}_2) - f^1\bar{f}_2 \frac{u_1^+}{u_2^+} \right]$$

$$\hat{h} = \lambda \left[-f^2\bar{f}_1 \frac{u_2^+}{u_1^+} - \frac{1}{2}(f^1\bar{f}_1 - f^2\bar{f}_2) \right] \quad \cdots (37)$$

h and $\hat{h}$ are holomorphic in the patches $\mathcal{U} = \{ u_i^\pm \mid u_2^+ \neq 0 \}$ and $\hat{\mathcal{U}} = \{ u_i^\pm \mid u_1^+ \neq 0 \}$,respectively.

Since g is a twistor function, which implies that it satisfies:[10)-14)]

$$\pi^{A'} \nabla_{AA'}\, g = 0 \quad \cdots (38)$$

and $g = h - \hat{h}$, it follows that: $\pi^{A'} \nabla_{AA'}\, h = \pi^{A'} \nabla_{AA'}\, \hat{h}$ $\cdots (39)$

This represents a pair of functions global in $\pi_{A'}$, homogeneous of degree one so they are linear in $\pi_{A'}$ (following refs. 12 - 14).

$$\pi^{A'} \nabla_{AA'}\, h = -A_{AA'} \pi^{A'} \quad \cdots (40)$$

In the case of $g = -\lambda \dfrac{\omega^0 \omega^1}{\pi_{0'}\pi_{1'}}$,

$$\begin{pmatrix} A_{00'} & A_{01'} \\ A_{10'} & A_{11'} \end{pmatrix} = \frac{\lambda}{2} \begin{pmatrix} -x^{11'} & x^{10'} \\ -x^{01'} & x^{00'} \end{pmatrix} \quad \cdots (41)$$

and the one-form $A_a dx^a$ is:

$$A_a dx^a = A_{AB'} dx^{AB'} = \frac{\lambda}{2}(-x^{11'}dx^{00'} + x^{10'}dx^{01'} - x^{01'}dx^{10'} + x^{00'}dx^{11'}) \quad \cdots (42)$$

Translated into the language of harmonic superfield topological field theory, the vector fields $A_{AA'}$ are defined by:

$$u^{+1}\frac{\partial h}{\partial f^1} + u^{+2}\frac{\partial h}{\partial f^2} = -A_{00'}u^{+1} - A_{01'}u^{+2} \quad \cdots (43)$$

and are calculated to become:

$$\begin{pmatrix} A_{00'} & A_{01'} \\ A_{10'} & A_{11'} \end{pmatrix} = -\frac{\lambda}{2}\begin{pmatrix} \bar{f}_1 & \bar{f}_2 \\ f^2 & -f^1 \end{pmatrix} \qquad \cdots (44)$$

by virtue od eq. (37).

Following refs. 12 − 14), in the twistor theory, one can calculate the metric of the manifold as follows:

$$g_{ab} = \theta\eta_{ab} - (\mathcal{P}_a A_b + A_a \mathcal{P}_b) + \frac{\mathcal{P}^2}{\theta} A_a A_b$$

$$\theta = 1 + \mathcal{P}^a A_a$$

where

$$\mathcal{P}_{AA'} = \phi_A{}^B \chi_{BA'} = -\phi_{AB}\, \chi^{BC'} \varepsilon_{C'A'} \qquad \cdots (45)$$

In terms of f^i , one-forms $\mathcal{P}_a dx^a$ and $A_a dx^a$ are

$$\mathcal{P}_a dx^a = \bar{f}_1 df^1 + \bar{f}_2 df^2 - f^1 d\bar{f}_1 - f^2 d\bar{f}_2$$

$$A_a dx^a = \frac{\lambda}{2}(-\bar{f}_1 df^1 - \bar{f}_2 df^2 + f^1 d\bar{f}_1 + f^2 d\bar{f}_2) \qquad \cdots (46)$$

Hence the metric is:

$$\begin{aligned} ds^2 = {} & 2(1+\lambda\rho^2)(df^1 d\bar{f}_1 + df^2 d\bar{f}_2) \\ & + \lambda(-f^1 d\bar{f}_1 + \bar{f}_1 df^1 - f^2 d\bar{f}_2 + \bar{f}_2 df^2)^2 \\ & - \frac{\lambda\rho^2}{2(1+\lambda\rho^2)}(-f^1 d\bar{f}_1 + \bar{f}_1 df^1 - f^2 d\bar{f}_2 + \bar{f}_2 df^2)^2 \\ & \rho^2 = f^i \bar{f}_i \end{aligned} \qquad \cdots (47)$$

This is nothing but the metric of the effective target space of harmonic superfield topological field theory,(eq.(12)), obtained by eliminating an infinite number of auxiliary fields, by use of equations of motion.

In conclusion, we have seen that the two-dimensional topological field theories, which describe four-dimensional sdelf-dual space-times(gravitational instantons) as target spaces, which we constructed in the previous paper, are deeply connected with Penrose's "twistor theory". Thus, our theory provides us with an alternative approach to topological field theory which incorporates automatically the "twistor geometry".

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