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COEXISTENCE OF UNIQUELY ERGODIC SUBSYSTEMS OF INTERVAL MAPPING

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ABSTRACT

We prove that if (I,f) has a uniquely ergodic subsystem $(X, f|_X, \alpha)$, then for every $s \lhd f_\alpha$ [see Section 1] (I,f) has a uniquely ergodic subsystem $(Y, f|_Y, \beta)$ such that $f_\beta = s$, where $I = [0,1], f \in C(I,I)$.

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1 Preliminaries

A theorem due to A.N.Sharkovskii gives a surprising answer to the following question: if f has a periodic orbit of period k, must f also have periodic orbits of other periods?

Theorem 1.1 [Sh]*(Sharkovskii's theorem) Order the positive integers as follows:*

$$3 \rhd 5 \rhd 7 \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$$

If a continuous function $f: \mathbb{R} \to \mathbb{R}$ has a periodic orbit of period k, then f has periodic orbits of all periods which follow k in the above Sharkovskii's order.

Sharkovskii's theorem is attractive to mathematicians because of its simple setting and surprising answer. For the original proof, some simpler proof see [Sh],[St] for references. In [K] the author generalized Sharkovskii's theorem to special types of functions in higher-dimensional Euclidean space, in [M] T.Matsuoka gave a 2-dimensional analogue of Sharkovskii's ordering. In spite of this, Sharkovskii's theorem is basically a one-dimensional phenomenon. In other words Sharkovskii's theorem holds for all connected ordered spaces on the order topology [Sc]. The reader can also find a refined Sharkovskii's theorem about orbits of periodic points and an analogue of Sharkovskii's theorem for the space $Y = \{z \in \mathcal{C} : z^3 \le 1\}$ in [Ba1], [Be], [ALM] and [Ba2].

The other surprising thing is that more complicate orbits of interval mapping, minimal sets, also coexist in a way described by some partial order [Y1]. The purpose of this paper is to show that uniquely ergodic subsystems of interval mapping also coexist in the way same as minimal sets do.

To do this we in section 2 give some notations. In section 3 we define D-function of a uniquely ergodic system and show its basic properties. We prove the coexistence of uniquely ergodic subsystems of interval mapping in section 4. At last we give the examples of uniquely ergodic systems with given D-functions in section 5.

In the following we state the main results of this paper.

Let $\mathcal{Z}$ be the set of functions s from N to N satisfying

(a) $s(k)|k$ for every $k \in$ N,

(b) for every $l, k \in$ N, if $l|k$ then $s(l) = (l, s(k))$, and $E = \{s \in \mathcal{Z} : s(k) = (n,k)$ for some $n \in \mathrm{N}$ and all $k \in \mathrm{N}\}$ be a subset of $\mathcal{Z}$. We shall identify E with N, namely if $n \in \mathrm{N}$ then the function s defined by $s(k) = (n,k)$ for $k \in \mathrm{N}$ will be identified with n. Denote $\mathcal{Y} = \mathcal{Z} \cup \mathrm{N}'$, where $\mathrm{N}' = \{n' : n \in \mathrm{N}\}$.

Let X be a compact metric space, $T \in C(X,X)$ and (X,T,μ) be uniquely ergodic. A $D-function$ of (X,T,μ), denoted by T_μ, is $n \in \mathrm{N}$ if $supp(\mu)$ is a periodic orbit of T with period n; is $n' \in \mathrm{N}'$ if $supp(\mu)$ is not a periodic orbit of T but the number of the ergodic components of T^k is (n,k) for all $k \in \mathrm{N}$; is a function from N to N such that for every $k \in \mathrm{N}$, $T_\mu(k)$ is the number of ergodic components of T^k and T_μ is not a bounded function.

Our first main result is:

Theorem 3.5 Let X be a compact metric space, $T \in C^0(X,X)$ and (X,T,μ) be a uniquely ergodic system. Then $T_\mu \in \mathcal{Y}$.

Now let X be a compact, metric space and $T \in C(X,X)$. Denote $DF(X,T)$ to be the set of all D-functions of $(Y, T|_Y)$, where $Y \subset X$ is compact and $(Y, T|_Y)$ is uniquely ergodic. Then we have the following

Theorem 5.5 Let Σ_M be one-sided subshift of finite type with $k \times k$ matrix $M = (m_{ij})$ satisfying that $m_{01} = m_{12} = \cdots = m_{(k-1)0} = m_{00} = 1$ and σ the left shift. Then $DF(\Sigma_M, \sigma) \supset (\mathcal{Y} \backslash E)$.

At last we prove an extension of Sharkovskii's theorem

Theorem 4.5 (a) If $f \in C(I,I), I = [0,1]$ then $DF(I,f) = \mathcal{Y}(n), n \in \mathrm{N} \cup \{\infty\} \cup 2^\infty$,

(b) If $n \in \mathrm{N} \cup \{\infty\} \cup 2^\infty$, then there exists $f \in C(I,I)$, such that $DF(I,f) = \mathcal{Y}(n)$, where

$Y_i = \{s \in \mathcal{Z} \backslash E | s(2^l) = (2^l.2^i), \forall l \in \mathrm{N}\} \cup \{(2^i(2n-1))' : n \in \mathrm{N}\}, 0 \le i < \infty$,

$Y_\infty^2 = \{s \in \mathcal{Y} | s(2^l \cdot p) = 2^l$ for all $l \in \mathrm{N}$, all odd $p \in \mathrm{N}\}$,
$Y_\infty^1 = \{s \in \mathcal{Y} \backslash Y_\infty^2 | s(2^l) = 2^l$ for all $l \in \mathrm{N}\}$,
$\mathcal{Y}(2^\infty) = \{2^n : n = 0,1,2,\cdots\}, \mathcal{Y}(\infty) = Y_\infty^2 \cup \mathcal{Y}(2^\infty)$,

$$\mathcal{Y}(n) = \begin{cases} \{n\} \cup \{k : n \rhd k\} \cup (\cup_{i=l}^{\infty} Y_i) \cup Y_\infty^1 \cup Y_\infty^2, & n = 2^l p (p \ge 3, \text{ odd }); \\ \{2^l, 2^{l-1}, \cdots, 2, 1\}, & n = 2^l. \end{cases}$$

Remark 1.2 *Theorem 4.5 can be expressed in the way like Sharkovskii's Theorem as follows. Give $\mathcal{Y}$ a partial linear order :*

$3 \rhd 5 \rhd 7 \cdots \rhd Y_0 \rhd$
$2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \cdots \rhd Y_1 \rhd$
$2^2 \cdot 3 \rhd 2^2 \cdot 5 \rhd 2^2 \cdot 7 \cdots \rhd Y_2 \rhd \cdots$
$\rhd Y_\infty^1 \rhd Y_\infty^2 \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1$

If a continuous function $f : \mathrm{R} \longrightarrow \mathrm{R}$ has a uniquely ergodic subsystem $(X, f|_X, \alpha)$, then f has uniquely ergodic subsystems with D-functions which follow f_μ in the above partial order. Furthermore for every $i \in \{0\} \cup \mathrm{N}$ the existence of a uniquely ergodic subsystem $(X, f|_X, \alpha)$ with $f_\alpha \in Y_i$ (res. $f_\alpha \in Y_\infty^1$) implies the existence of periodic orbit with period $2^i \cdot q$ for some odd $q \ge 3$ (res. $2^l \cdot p$ for some $l \in \{0\} \cup \mathrm{N}$ and some odd $p \ge 3$).

2 Definitions and some lemmas

Let X be a compact, metric space, $T \in C(X,X)$. Inductively we define T^0 is the identity, $T^n = T^{n-1} \circ T, n \in \mathrm{N}$. $O(x,T) = \{x, T(x), T^2(x), \cdots\}$ is called the orbit of x under T. A point $x \in X$ is a periodic point of T of period $k \in \mathrm{N}$ if $T^k(x) = x$ but $T^i(x) \ne x$ for $0 < i < k$. If $k = 1$, x is called a fixed point. For periodic point x of period k we call $O(x,T)$ a periodic orbit of period k. The sets of periodic points and fixed points are denoted by $P(T), F(T)$ respectively. The ω-limit set of x under T, $\omega(x,f)$, is the set $\cap_{i=1}^{\infty} \overline{O(T^i(x), f)}$. Two dynamical systems $(X,T), (Y,S)$ are said to be topologically conjugate if there is a homeomorphism $h \in C(X,Y)$ such that $hT = hS$.

A point $x \in X$ is said to be an almost periodic under T provided that for each neighbourhood U of x there corresponds a $m \in \mathrm{N}$ with the properties that in every set of m consecutive positive integers appears an integer n such that $T^n(x) \in U$. Denote the set of almost periodic points by $AP(T)$.

Let $\mathcal{B}(X)$ be the Borel σ-algebra generated by all open subsets of X, $M(X,T)$ be the set of T-invariant probability measures on $\mathcal{B}(X)$ and $E(X,T)$ be the ergodic elements of $M(X,T)$. For $\alpha \in M(X,T), T\alpha \in M(X,T)$ is defined by: $T\alpha(B) = \alpha(T^{-1}B), \forall B \in \mathcal{B}(X)$. For $\beta \in E(X,T)$,

$$G(\beta) = \{x \in X : \lim \frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) = \int_X f d\beta, \forall f \in C(X,X)\}.$$

We say that system (X,T) is *uniquely ergodic* if $\#M(X,T) = 1$. Uniquely ergodic system (X,T,μ) is *strictly ergodic* if X is a minimal set of T. That (X,T,μ) is an ergodic system means: $\forall B \in \mathcal{B}(X)$ if $T^{-1}B = B$ then $\mu(B) = 0$ or $\mu(B) = 1$.

For uniquely ergodic system (X,T,μ), let

$$supp(\mu) = \{x \in X : \mu(V_x) > 0 \text{ for all neighbourhood } V_x \text{ of } x\},$$

then $supp(\mu)$ is a minimal set of T [Man].

Measure-preserving transformations T_1 on $(X_1, \mathcal{B}_1, m_1)$ and T_2 on $(X_2, \mathcal{B}_2, m_2)$ are *spectrally isomorphic* if there is a linear operator $W : L^2(m_2) \to L^1(m_1)$ such that

(1) W is invertible;

(2) $(Wf, Wg) = (f,g), \forall f,g \in L^2(m_2)$;

(3) $U_{T_1}W = WU_{T_2}$.

T_1 and T_2 are said to be *isomorphic* if there exist $M_1, \in \mathcal{B}_1, M_2 \in \mathcal{B}_2$ with $m_1(M_1) = m_2(M_2) = 1$ such that

(1) $T_1M_1 \subset M_1, T_2m_2 \subset M_2$, and

(2) there is an invertible measure-preserving transformation

$$\phi : M_1 \longrightarrow M_2 \text{ with } \phi T_1(x) = T_2\phi(x), \forall x \in M_1.$$

Recall that if X is a compact metric space, $T \in C(X,X)$ and $A \subset X$ is minimal then the D-function of A, T_A, is n, if A is periodic orbit of T with period n, is n' if A is not a periodic orbit of T but the number of distinct minimal sets for T^k which contain in A is (n,k) for all $k \in$ N, is a function from N to N such that for every $k \in$ N $T_A(k)$ is the number of distinct minimal sets of T^k which contain in A and T_A is not a bounded function.

The following lemmas are necessary in our proofs.

Lemma 2.1 [W] *If T is a continuous transformation of compact metric space. Then*

(1) $E(X,T) \neq \emptyset$;

(2) $G(\alpha) \cap G(\beta) = \emptyset$, if $\alpha \neq \beta \in E(X,T)$.

Lemma 2.2 [Go] *If X is a metric space, $f \in C^0(X,X)$ and $x \in AP(f)$ then $\omega(x,f)$ is a minimal set under f; if X is a locally compact space, $f \in C^0(X,X)$ and A is a minimal set of f then $A \subset AP(f)$.*

Lemma 2.3 [Y1] *(1) $f_A \in \mathcal{Y}$;*

(2) $x \in \omega(f(x), f^k)$ for some $x \in A$ and some $k \in$ N if and only if $f_A(k) = 1$;

(3) $x \notin \omega(f^i(x), f^k)$ for some $x \in A$ and all $0 < i < k$ if and only if $f_A(k) = k$;

(4) $T_A(nm) = T_A(n)(T^n)_{A_1}(m)$, where $A \subset X$ is a minimal set of T, $A_1 \subset A$ is a minimal set of T^n.

Lemma 2.4 *Let $f \in C(I,I), I = [0,1]$. Suppose period of every periodic orbit of f is not an odd number except 1, A is a minimal set of f with $\#A \geq 2$, then there exists a fixed point $x_0 \in (infA, supA)$ such that $f^2(A_1) = A_1, f^2(A_2) = A_2$, where $A_1 = \{x \in X : x < x_0\}, A_2 = \{x \in X : x > x_0\}$.*

Lemma 2.4 is a direct consequence of results of $[X]$ and $[LMPY]$. See Corollary 4.2 $[Y1]$ for details.

3 D-function of uniquely ergodic system

To generalize Sharkovskii's Theorem about the coexistence of periodic orbits of interval mapping to uniquely ergodic subsystems the first difficulty we meet is to describe these subsystems with some useful isomorphic invariant. We find that the D-function of uniquely ergodic system which we will define below is a such invariant. To give the definition at first we show

Theorem 3.1 *Let X be a compact metric space, $T \in C(X,X)$ and (X,T,μ) be uniquely ergodic. Then for every $k \in$ N there exists an integer n dividing k such that $\forall \alpha \in E(X,T^k), E(X,T^k) = \{\alpha, T\alpha, \cdots, T^{n-1}\alpha\}$ and $T^i\alpha \neq T^j\alpha$ if $1 \leq i < j \leq n-1$.*

Proof: By Lemma 2.1 $E(X,T^k) \neq \emptyset$. Let $\alpha \in E(X,T^k)$ then $T^k\alpha = \alpha$. Now we suppose $\beta \in E(X,T^k)$ and want to show $\beta = T^{i_0}\alpha$ for some $0 \leq i_0 \leq k-1$. Because

$$\mu = \frac{1}{k}(\gamma + T\gamma + \cdots + T^{k-1}\gamma), \forall \gamma \in E(X,T^k),$$

we have

$$\mu(\cup_{i=0}^{k-1} G(T^i\alpha)) = \mu(\cup_{i=0}^{k-1} G(T^i\beta)) = 1$$

($T^i\gamma \in E(X, T^k), \forall\gamma \in E(X, T^k)$). So there are $0 \leq j_1, j_2 \leq k-1$ such that $G(T^{j_1}\alpha) \cap G(T^{j_2}\beta) \neq \emptyset$. It implies that $\beta = T^{i_0}\alpha, 0 \leq i_0 \leq k-1$.

Let $n = min\{0 < i \in \mathrm{N} : T^i\alpha = \alpha\}$. It is obvious that $n|k$ and $E(x, T^k) = \{\alpha, T\alpha, \cdots, T^{n-1}\alpha\}$.

Definition 3.2 *A D-function of uniquely ergodic system (X, T, μ), T_μ, is n if $supp(\mu)$ is a periodic orbit of T with period n; is n' if $supp(\mu)$ is not periodic orbit of T but the number of ergodic components of T^k is (n, k) for all $k \in \mathrm{N}$; is a function from N to N such that $T_\mu(k)$ is the number of ergodic components of T^k for all $k \in \mathrm{N}$ and T_μ is not a bounded function.*

Remark 3.3 *By Lemma 2.1 The number of ergodic components for T^k is equal to the integer n obtained in Theorem 3.1. So we see that T_μ is well defined. For convenience we also write $T_\mu(k) = (n, k), \forall k \in \mathrm{N}$ if $T_\mu = n'$.*

Remark 3.4 *A function $s : \mathrm{N} \longrightarrow \mathrm{N}$ belongs to $\mathcal{Z}$ if and only if there exists a function t from the set of all prime numbers to the set $\mathrm{N} \cup \{0, \infty\}$, such that for every $k \in \mathrm{N}$ if $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ is the decomposition of k into prime factors then*

$$s(k) = \prod_1^n p_i^{min(\alpha_i, t(p_i))}$$

or equivalently $s \in \mathcal{Z}$ if and only if (1) $s(mn) = s(m)s(n)$, if $(m, n) = 1$; (2) for prime number p either $s(p^l) = p^l$ for all $l \in \mathrm{N}$ or there exists $l_0 \in \mathrm{N} \cup \{0\}$ such that $s(p^l) = (p^l, p^{l_0})$ for all $l \in \mathrm{N}$. In the future we will use any equivalent statement of $\mathcal{Z}$ by convenience.

Theorem 3.5 *$T_\mu \in \mathcal{Y}$.*

Proof: The proof of this theorem is similar with the proof of Theorem 3.8 of [Y1].

Theorem 3.6 *Let (X_i, T_i, μ_i) be uniquely ergodic systems, $i = 1, 2$. If T_1 is spectrally isomorphic to T_2, then $T_{1\mu_1} = T_{2\mu_2}$.*

Proof: By the properties of $\mathcal{Y}$ It is enough for us to show that if $T_{2\mu_2}(n) = n$ for some $n \in \mathrm{N}$, then $T_{1\mu_1}(n) = n$. Because T_1 is spectrally isomorphic to T_2, there is an isomorphic from $L^2(\mu_2)$ to $L^2(\mu_1)$ satisfying $U_{T_1} \cdot W = W \cdot U_{T_2}$. Suppose for some $n \in \mathrm{N}, T_{2\mu_2}(n) = n$. Let $\alpha \in E(X_2, T_2^n), B_i = G(T_2^i\alpha), 0 \leq i \leq n-1$ then $B_i \in \mathcal{B}(X_2)$ and

$$T_2(B_i) = B_{i+1}, T_2^{-1}(B_{i+1}) = B_i, B_i \cap B_j = \emptyset \text{ if } 1 \leq i < j \leq n-1.$$

By the condition satisfied by W we get $U_{T_1}^n W = W U_{T_2}^n$. Hence

$$W(1_{B_i}) = W(U_{T_2}^n 1_{B_i}) = (U_{T_1}^n W)(1_{B_i}) = W(1_{B_i})T_1^n.$$

Therefore

$$W(1_{B_i}) = C_j^i \text{ a.e. for every } \gamma_j \in E(X_1, T_1^n), 1 \leq j \leq T_{1\mu_1}(n) = k.$$

From

$$< W(1_{B_i}), W(1_{B_i}) >=< 1_{B_i}, 1_{B_i} >= \frac{1}{n},$$

$$< W(1_{B_i}), W(1_{B_j}) >=< 1_{B_i}, 1_{B_j} >= 0, 1 \leq i < j \leq n-1,$$

$\mu_1 = \frac{1}{k}(\gamma_1 + \ldots + \gamma_k)$, we have

$$\Sigma_{j=1}^k \mid C_j^i \mid^2 = \frac{k}{n}, 0 \leq i \leq n-1$$

$$\Sigma_{j=1}^k C_j^{i_1} \overline{C}_j^{i_2} = 0, 0 \leq i_1 < i_2 \leq n-1$$

If $k < n$, it is contradict to the fact that if we denote

$$\alpha_i = \sqrt{k/n}(C_1^i, C_2^i, \cdots, C_k^i),$$

then $\mid \alpha_i \mid = 1, \alpha_i\overline{\alpha}_j = 0, 0 \leq i < j \leq n-1$. Because $k \leq n$ we get $k = n$.

Remark 3.7 *Because $\{e^{2\pi ij/T_\mu(n)} : n \in \mathrm{N}, 0 \leq j \leq T_\mu(n)-1\}$ is only a subset of spectrum of uniquely ergodic system (X, T, μ) so the inverse of Theorem 3.6 is not true in the general case.*

Let $\mathcal{Y}$ be the set defined in the introduction. We naturally ask the following question : Given $s \in \mathcal{Y}$ do there exist a compact metric space X, $T \in C(X, X)$ such that (X, T, μ) is uniquely ergodic and $T_\mu = s$? The answer is positive. We have the following theorems that we will prove in section 5:

Theorem 3.8 *For every $s \in Z \backslash E$, there exists strictly ergodic system (A_s, σ, μ_s) which is a subsystem of (Σ_M, σ) such that $\sigma_{\mu_s} = s$.*

Theorem 3.9 *For every $n \in N$, there exists strictly ergodic system (A_n, σ, μ_n) which is a subsystem of (Σ_M, σ) such that $\sigma_{\mu_n} = n'$.*

To get the existence of uniquely ergodic system with a given D-function of interval mapping we need

Theorem 3.10 *Let X, Y be compact metric spaces, $T \in C(X, X)$ and $S \in C(Y, Y)$. Measurable transformation $\varphi : X \longrightarrow Y$ satisfying $S\varphi = \varphi T$ and $\varphi(B) \in \mathcal{B}(Y)$ for every $B \in \mathcal{B}(X)$. φ is one-to-one except a countable set, on which φ is countable-to-one. If (Y, S) is uniquely ergodic system and $supp(\gamma)$ is not a periodic orbit, then (X, T) is also a uniquely ergodic system and $T_\mu = S_\gamma$, where $\{\mu\} = M(X, T), \{\gamma\} = M(Y, S)$.*

Proof: It is easy to see that $\gamma(B) = \mu(\varphi^{-1}(B))$, for every $B \in \mathcal{B}(Y)$ and $\mu \in M(X, T)$. Hence for every $B \in \mathcal{B}(X), \mu \in M(X, T)$ we have $\mu(B) = \mu(\varphi^{-1}(\varphi(B))) = \gamma(\varphi(B))$. This implies that $\#M(X, T) = 1$. Suppose $M(X, T) = \{\mu\}$, then by the properties of D–function we only need to prove

(1)if $S_\gamma(n) = n$, then $T_\mu(n) = n$;

(2)if $T_\mu(n) = n$, then $S_\gamma(n) = n$.

we only show (1), (2) is similar.

Suppose $S_\gamma(n) = n$ for some $n \in N$. Then for $\alpha \in E(Y, S^n), E(Y, S^n) = \{\alpha, S(\alpha), \cdots, S^{n-1}(\alpha)\}$ by Theorem 3.1. Let $G(i) = G(S^i(\alpha)), 0 \leq i \leq n-1$, then $G(i) \in \mathcal{B}(Y)$ and $G(i) \bigcap G(j) = \emptyset$ if $i \neq j$. Furthermore $S(G(i)) = G(i+1), S^{-1}(G(i)) = G(i-1)$. Let $X_i = \varphi^{-1}(G(i))$, then $X_i \in \mathcal{B}(X)$ and $T^{-1}(X_i) = X_{i-1}$. Hence $\mu(X_i) = \frac{1}{n}, 0 \leq i \leq n-1$.

Let $\beta \in E(X, T^n)$. Then $E(X, T^n) = \{\beta, T(\beta), \cdots, T^{k-1}(\beta)\}$ and $\beta = T^k(\beta)$. That is to say for every $B \in \mathcal{B}(X), \beta(B) = \beta(T^{-k}(B))$. Hence $\beta(X_i) = \beta(T^{-k}(X_i))$ for every $0 \leq i \leq n-1$. Because $T^{-n}(X_i) = X_i$, we have $\beta(X_i) = 0$, or $1, 0 \leq i \leq n-1$. Let $\beta(X_{i_0}) = 1$, then $1 = \beta(X_{i_0}) = \beta(T^{-k}(X_{i_0})) = (X_{i_0 - k (mod\ n)})$. By the fact that $\beta(X) = 1, \beta(X_i \bigcap X_j) = 0$ if $i \neq j$, we get $k = n$.

4 The proof of Theorem 4.5

In this section we will prove the coexistence of uniquely ergodic subsystems of interval mapping. At first we show

Lemma 4.1 *If for some $n \in N, T_{supp(\mu)}(n) = n$, then $T_\mu(n) = n$. Furthermore $T_{supp(\mu)}(n) | T_\mu(n)$.*

Proof: Obviously minimal components for T^k are also ergodic components for T^k. So if $T_{supp(\mu)}(n) = n$, then $T_\mu(n) = n$. So it is enough for us to show $T_{supp(\mu)}(n) | T_\mu(n)$ for every $n \in N$. (*)

Let p be a prime number. If $T_{supp(\mu)}(p^l) = p^l$ for every $l \in N$, then (*) is true. Hence we may assume that there exists $l_0 \in N \bigcup \{0\}$ such that $T_{supp(\mu)}(p^l) = (p^l, p^{l_0})$ for every $l \in N$. Then

$$T_\mu(p^l) = T_{supp(\mu)}(p^l) = p^l \text{ if } l \leq l_0; T_{supp(\mu)}(p^l) = p^{l_0} | T_\mu(p^l) \text{ if } l \geq l_0 + 1.$$

This implies that for every $n = p_1^{r_1} \cdots p_m^{r_m} \in N$,

$$T_{supp(\mu)}(n) = \prod_{i=1}^{m} T_{supp(\mu)}(p_i^{r_i}) | \prod_{i=1}^{m} T_\mu(p_i^{r_i}) = T_\mu(n),$$

where $p_1 < \cdots < p_m$ are prime numbers.

Lemma 4.2 *For uniquely ergodic system $(X, T, \mu), n, m \in N$, if $T_{supp(\mu)}(n) = n$ we have $T_\mu(nm) = T_\mu(n)(T^n)_\gamma(m)$, where $\{\gamma\} = M(A_1, T^n), A_1 \subset A$ is a minimal set of T^n.*

Proof: Because $T_{supp(\mu)}(n) = n$ we know that $T_\mu(n) = n$. Let $E(X, T^n) = \{\alpha, T\alpha, \cdots, T^{n-1}\alpha\}, E(X, T^{nm}) = \{\beta, T\beta, \cdots, T^{k-1}\beta\}$, where $k = T_\mu(nm)$ and $m_0 = min\{i : T^{ni}\beta = \beta, 0 < i \leq m\}$.Obviously $(1/m_0)\{\beta + T^n\beta + \cdots + T^{n(m_0-1)}\beta\} \in E(X, T^n)$. Let $C_0 = \{\beta, T^n\beta, \cdots, T^{n(m_0-1)}\beta\}$. Then $E(X, T^{nm}) = \bigcup_{i=0}^{n-1} T^i C_0$. It implies that $T_\mu(nm) = k = nm_0 = T_\mu(n)m_0$. By definition of $m_0, m_0 = (T^n)_\gamma(m), \gamma \in E(A_1, T^n), A_1 \subset A$ is a minimal set of T.

Lemma 4.3 *Let X be a compact metric space, $A \subset X$ be a minimal set of $f \in C(X, X), A = A_1 \bigcup A_2, A_1 \bigcap A_2 = \emptyset, A_i$ be minimal set of $f^2, i = 1, 2$. If (A_i, f^2) is strictly ergodic for some $i \in \{1, 2\}$, then (A, f) is also strictly ergodic.*

Proof: Without lossing of generality we suppose that (A_1, f^2) is strictly ergodic. At first we show that (A_2, f^2) is also strictly ergodic. If it is not the case, then there exist $\mu_1, \mu_2 \in E(A_2, f^2)$ such that $\mu_1(C) = 1, \mu_2(C) = 0$, where $C = G(\mu_1) \subset A_2, f^{-2}(C) = C$. Because μ_1 is regular, there exists a closed set $C_1 \subset C$ such that $\mu_1(C_1) \geq \frac{1}{2}[W]$. For $B \in \mathcal{B}(A_1)$, let $\tilde{\mu}_i(B) = \mu_i(f^{-1}(B))$ then $\tilde{\mu}_i \in M(A_1, f^2), i = 1, 2$. As $f(C_1) \in \mathcal{B}(A_1)$, therefore

$$\tilde{\mu}_1(f(C_1)) = \mu_1(f^{-1}f(C_1)) \geq \mu_1(C_1) \geq \frac{1}{2},$$

$$\tilde{\mu}_2(f(C_1)) = \mu_2(f^{-1}f(c_1)) \leq \mu_2(C) = 0$$

where $f^{-1}f(C_1) \subset f^{-1}f(C) = f^{-1}f^{-1}(C) = f^{-2}(C) = C$. It is a contradiction to the strict ergodicity of (A_1, f^2). Hence (A_2, f^2) is also strictly ergodic. Now let $\alpha_1, \alpha_2 \in M(A, f)$, then $\alpha_i^j(B) = 2\alpha_i(A_j \bigcap B) \in M(A_j, f^2), 1 \leq i, j \leq 2$. Therefore for $B \in \mathcal{B}(A)$

$$\alpha(B) = \alpha_1(B \bigcap A_1) + \alpha_1(B \bigcap A_2) = \frac{1}{2}\alpha_1^1(B) + \frac{1}{2}\alpha_1^2(B) = \frac{1}{2}\alpha_2^1(B) + \frac{1}{2}\alpha_2^2 = \alpha_2(B).$$

This implies that (A, f) is strictly ergodic.

Lemma 4.4 *Let $f \in C(I, I)$ and any period of periodic orbit of f be a power of 2. Then for any uniquely ergodic subsystem (X, f, μ) we have $f_\mu \in Y_\infty^2$ or $f_\mu = 2^n$ for some $n \in \mathbb{N} \cup \{0\}$.*

Proof: Not lossing of generality we suppose that $supp(\mu)$ is not a periodic orbit of f. By $[Mi]$ (X, f, μ) is isomorphic to generalized adding machine (X_Q, D), where $X_Q = \{0, 1\}^{\mathbb{N}}$. So $f_\mu \in Y_\infty^2$ by Theorem 5.2, 5.4.

Theorem 4.5 *(a) If $f \in C(I, I), I = [0, 1]$, then $DF(I, f) = \mathcal{Y}(n), n \in \mathbb{N} \cup \{\infty\} \cup 2^\infty$,*

(b) If $n \in \mathbb{N} \cup \{\infty\} \cup 2^\infty$, then there exists $f \in C(I, I)$, such that $DF(I, f) = \mathcal{Y}(n)$.

Proof: We will prove Theorem 4.5 (a) in the following steps:

(1) If f has a uniquely ergodic subsystem (X, f, μ) with $f_\mu \in Y_i$ for some $0 \leq i < \infty$, then f must have periodic orbit with period $2^i q(q \geq 3$ is odd).

If it is not the case, then the biggest period of periodic orbit of f according Sharkovskii' s order is $2^{i+j} q(q \geq 3$ is odd and $j \geq 1)$ or any period of periodic orbit of f is a power of 2. By Lemma 4.4 the latter case is impossible. Repeatly using Lemma 2.4 we get $f_A(2^{i+1}) = 2^{i+1}$ for every minimal set A of f with $\#A \geq 2^{i+1}$. By Lemma 4.1, $f_\mu(2^{i+1}) = f_A(2^{i+1}) = 2^{i+1}$. This implies that $f_\mu \notin Y_i$. Hence f must have a periodic orbit with period $2^i q(q \geq 3$ is odd).

(2) If the biggest period of periodic orbit of f according to Sharkovskii's order is $q(q \geq 3$, odd), then for every $s \in \mathcal{Z} \backslash E$, there exists uniquely ergodic system (X, f, μ) such that $f_\mu = s$.

Using results of [Bl] and the structure of odd periodic orbit discussed in [St] we know that for every strictly ergodic subsystem $(A_s, \sigma, \mu_s) \subset (\Sigma_M, \sigma)$ satisfying $\sigma_{\mu_s} = s$ (Theorem 5.5) there exist a compact subset $X_s \subset I, h \in C(X_s, A_s)$ such that $f(X_s) \subset X_s, hf = \sigma h$ and h is one-to-one except at a countable subset at which h is two-to-one. By Theorem 3.10 step (2) follows.

(3) If the biggist period of periodic orbits of f according to Sharkovskii's order is $2^i q(q \geq 3,$ odd), then for every $s \in Y_i$ there exists a uniquely ergodic subsystem (X, f, μ) such that $f_\mu = s$.

Let $g = f^{2^i}$, then g has a periodic orbit of period q. By step (2) there exists uniquely ergodic subsystem $(\tilde{X}, g, \tilde{\mu})$ such that $g_{\tilde{\mu}} = s'$, where $s = s' \cdot s_i$ and $s' \in Y_0, s_i : \mathbb{N} \to \mathbb{N}$ satisfies $s_i(2^j m) = (2^j, 2^i), j \geq 0, m$ odd. Let $X = \bigcup_{j=0}^{2^i} f^j(supp(\tilde{\mu}))$. By Lemma 4.3 (X, f) is strictly ergodic. Let $\{\mu\} = E(X, f)$. If p is odd, then

$$f_\mu(p) f_\mu(2^i) = f_\mu(p 2^i) = f_\mu(2^i) g_{\tilde{\mu}}(p).$$

This implies $f_\mu(p) = g_{\tilde{\mu}}(p)$. At the same time

$$f_\mu(2^{i+1}) = f_\mu(2^i) g_{\tilde{\mu}}(2) = f_\mu(2^i) = 2^i$$

(by Lemma 4.2 and Lemma 2.4). Hence $f_\mu = s$.

(4) If periods of periodic orbits are not all power of 2, then for every $s \in Y_\infty^1 \cup Y_\infty^2$, there exists uniquely ergodic subsystem (X, f, μ) such that $f_\mu = s$.

Let $2^i q(q \geq 3,$ odd) be the period of some periodic orbit of f and $g = f^{2^i}$. Then for every $s \in Y_\infty^1 \cup Y_\infty^2$, there exists uniquely ergodic subsystem $(\tilde{X}, g, \tilde{\mu})$ such that $g_{\tilde{\mu}} = s$. Let $X = \bigcup_{j=0}^{2^i} f^j(supp(\tilde{\mu}))$, then (X, f) is the uniquely ergodic subsystem that we need.

(5) If f has a uniquely ergodic subsystem (X, f, μ) such that $f_\mu \in Y_\infty^2$, then for every $n \in \mathbb{N} \cup \{0\}, f$ has a periodic orbit of period 2^n.

If it is not the case, then there exists n_0 such that any period of periodic orbit of f is an element of set $\{2^n : 0 \leq i \leq n_0\}$ (Sharkovskii's order). Let $g = f^{2^{n_0}}$, then $\overline{P(f)} = \overline{P(g)} = \overline{F(g)} = F(g) = P(f)$[CH]. This implies $supp(\mu) \subset \overline{P(f)} = P(f)$. Hence $f_\mu(2^{n_0+1}) = 2^{n_0}$. It is contradict to $f_\mu \in Y^2_\infty$.

(6) If f has a uniquely ergodic subsystem $(\tilde{X}, f, \tilde{\mu})$ with $f_{\tilde{\mu}} \in Y^1_\infty$, then there exists a uniquely ergodic subsystem (X, f, μ) such that $f_\mu \in Y^2_\infty$.

At first we claim that periods orbits of f are not all power of 2. If it is not the case by Lemma 4.4 we know $f_{\tilde{\mu}} \in Y^2_\infty$ or $f_{\tilde{\mu}} = 2^n$ for some $n \in \mathrm{N} \cup \{0\}$. So our claim is true. By step (4), (6) holds.

Combining (1)-(6) and using Sharkovskii's theorem we conclude that $DF(I, f) = \mathcal{Y}(n), n \in \mathrm{N} \cup \{\infty\} \cup 2^\infty$.

Proof of Theorem 4.5 (b): It is a well known fact that for every $n \in \mathrm{N}$, there exists $f \in C(I, I)$ such that $\{k : k$ is a period of periodic orbit of f $\} = \{n\} \cup \{k : n \rhd k, k \in \mathrm{N}\}$ or $\{2^l : l = 0, 1, \cdots, n-1\}$ [see ALM]. So by above argument, we remain to show that there exists $f \in C(I, I)$ such that $DF(I, f) = \mathcal{Y}(\infty)$ or $\mathcal{Y}(2^\infty)$. Any chaotic function in the sense of Li-York with zero topological entropy is the function with $DF(I, f) = \mathcal{Y}(\infty)$ [Sm]. Any function which satisfies $\overline{P}(f) = P(f)$ and $\mathcal{Y}(2^\infty) \subset DF(I, f)$ is the example that $DF(I, f) = \mathcal{Y}(2^\infty)$.

Proof is completed.

5 Uniquely ergodic system with a given D-function

For the purpose to prove Theorem 4.5 we give the examples of uniquely ergodic systems with given D-functions in subshift of finite type Σ_M. Let

$$M = (m_{ij}) = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{p \times p}$$

$$\Sigma_M = \{x = (x_1, x_2, \cdots) : m_{x_j x_{j+1}} = 1, 0 \leq x_j \leq p-1, \text{ for all } j \in \mathrm{N}\},$$

$\sigma : \Sigma_M \longrightarrow \Sigma_M$ is the shift defined by $(\sigma(x))_j = x_{j+1}$ for $x \in \Sigma_M, j \in N$.

Recall that Σ_M with the product topology is a compact metrizable space. Let $P = \{0, 1, \cdots, p-1\}$. For $B \in P^r$ we call B a block over P of length r and write $l(B) = r$. If $A = (a_1 \cdots a_n)$, $B = (b_1 \cdots b_m)$ are two blocks, then we denote $AB = (a_1 \cdots a_n b_1 \cdots b_m)$, $A^\infty = AAA \cdots = (a_1 \cdots a_n a_1 \cdots a_n \cdots)$.

Definition 5.1 *Let $Q = \{q_i\}_1^\infty, q_i \geq 2, 1 \leq i < \infty, X_{q_i} = \{0, 1, \cdots, q_i - 1\}$ and $X_Q = \prod_{i=1}^\infty X_{q_i}$. A transformation D from X_Q into X_Q is defined as follows: $D(q_1 - 1, q_2 - 1, \cdots, q_n - 1, x_{n+1}, \cdots) = (0, 0, \cdots, 0, x_{n+1} + 1, \cdots)$ if $x_{n+1} < q_{n+1} - 1, n \in \mathrm{N} \cup \{0\}$. Specially $D(q_1 - 1, q_2 - 1, \cdots) = (0, 0, \cdots)$. The couple (X_Q, D) is called a generalized adding machine, shortly, GAM. It is a well known fact that (X_Q, D) is strictly ergodic.*

Theorem 5.2 *Let $(X_Q, D), Q = \{q_i\}_1^\infty$ be a GAM. If $q_i = p_1^{\alpha_i(1)} p_2^{\alpha_i(2)} \cdots p_{r_i}^{\alpha_i(r_i)}$ is the decomposition q_i into prime factors then for every prime number $p = p_j, D_{X_Q}(p^l) = p^l$ for all $l \in \mathrm{N}$ if $\Sigma_{i=1}^\infty \alpha_i(j) = \infty; D_{X_Q}(p^l) = (p^l, p^{l_0})$ if $\Sigma_{i=1}^\infty \alpha_i(j) = l_0$ or more simply for every $n \in \mathrm{N}, D_{X_Q} = \lim_j (n, q_1 \cdots q_j)$, where $p_1 < p_2 < \cdots$ are all prime numbers $\alpha_i(j) \geq 0, \alpha_i(r_i) > 0$.*

Proof: Notice that for each j D permutes cyclically all $\prod_{i=1}^j$ cylinders of length j. That is to say $D_{X_Q}(q_1 \cdots q_j) = q_1 \cdots q_j$ for every $j \in \mathrm{N}$. So if $n | q_1 \cdots q_j$ for some $n, j \in \mathrm{N}$ then $D_{X_Q}(n) = (n, D_{X_Q}(q_1 \cdots q_j)) = n$.

Let p be a prime number such that $p^{l_0} | q_1 \cdots q_j$ for j large enough and $p^{l_0+1} \not| q_1 \cdots q_j$ for all $j \in \mathrm{N}$. Then $(p, q_1 \cdots q_j / p^{l_0}) = 1$. So for j large enough there are k_j such that $q_1 \cdots q_j / p^{l_0} = 1 + k_j p$. This implies $q_1 \cdots q_j = p^{l_0} + k_j p^{l_0+1}$. Because $D^{q_1 \cdots q_j}(0, 0, \cdots) = (\overbrace{0, 0, \cdots, 0}^{j} 10 \cdots)$ we conclude that $0 \in \omega(D^{p^{l_0}}(0), D^{p^{l_0+1}})$, where $0 = (0, 0, \cdots)$. By Lemma 2.3 $D_{X_Q}(p^{l_0+1}) < p^{l_0+1}$, consequently $D_{X_Q}(p^{l_0+1}) = p^{l_0}$.

Combining obtained above, we get the theorem.

Lemma 5.3 *(X_Q, D) is topologically conjugate to (X'_Q, D) if and only if they have same D-functions.*

Proof: See [HR, p.417] and use Theorem 5.2.

Lemma 5.4 *Let $M(X_Q, D) = \{\mu\}$ then $D_{X_Q} = D_\mu$.*

Proof: Let p be a prime number. If $D_{X_Q}(p^l) = p^l, \forall l \in N$ then $D_\mu(p^l) = D_{X_Q}(p^l) = p^l$. So without loss of generality we suppose there is l_0 such that $D_{X_Q}(p^l) = (p^l, p^{l_0}), \forall l \in N$. By Lemma 4.2 $D_\mu(p^l) = D_{X_Q}(p^l) = p^l, \forall j \leq l_0$. So it enough to $D_\mu(p^{l_0+1}) = p^{l_0}$. Let $A = \omega(0, D^{p^{l_0+1}})$, where $0 = (0, 0, \cdots)$. By [E] $D^{p^{l_0+1}}$ is a rotation on the compact metrizable group and $(A, D^{p^{l_0+1}})$ is minimal. So it is strictly ergodic [W, p.162]. It implies that $D_\mu(p^{l_0+1}) = p^{l_0}$.

Now we are ready to prove Theorem 3.8

Proof of Theorem 3.8 : Let $s \in \mathcal{Z}\backslash E$, and p_{i_0} be a prime number such that $s(p_{i_0}) = p_{i_0} > p$. Let $s_1 : N \to N$ satisfying $s_1(p^l) = s(p^l)$, if $p \neq p_{i_0}$ or $s(p_{i_0}^l) = p_{i_0}^l, \forall l \in N$; $s_1(p_{i_0}^l) = (p_{i_0}^l, p_{i_0}^{l_0-1})$ if $s(p_{i_0}^l) = (p_{i_0}^l, p_{i_0}^{l_0}). \forall l \in N$ Suppose A is a GAM such that $D_{X_Q} = s_1$. By the theorem of Jewett and Krieger [W, p.161] there exists $A_1 \subset \Sigma_2$ such that (A_1, σ) is strictly ergodic and is isomorphic to (X_Q, D). So $\sigma_\mu = s_1$, $\{\mu\} = M(A_1, \sigma)$.

Now let θ be a bijection from $\{0, 1, \cdots, p-1\}$ to $\{0, 1, \cdots, p-1\}^{p_{i_0}}$ such that

$$\theta(0) = 0 \cdots 00 \cdots 0, \theta(1) = 0 \cdots 01 \cdots (p-1).$$

Let $x \in A_1$. The induced transformation $\theta_* : A_1 \to \omega(\theta_*(x), \sigma^{p_{i_0}})$ is defined by $\theta_*(y) = \theta(Y_1)\theta(Y_2)\cdots$ if $y \in A_1$ and $y = Y_1 Y_2 \cdots, l(Y_i) = p_{i_0}, 1 \leq i < \infty$. θ_* is one-to-one and continuous. It is easy to check that $\theta_* \sigma = \sigma^{p_{i_0}} \theta_*$. So by Lemma 4.2 and Theorem 3.6 we immediately get $\sigma_{A_s} = s$, where $A_s = \omega(\theta(x), \sigma)$.

Proof of Theorem 3.9 : For $n \in N$ choose an irrational numbers β such that $0 \leq \beta < \frac{1}{p}$. We define $x \in \Sigma_{\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)}$ such that $x_i = 0$ if $i\beta \in [k(i)(1+\beta), k(i)(1+\beta)+1)$; $x_i = 1$ if $i\beta[k(i)(1+\beta)+1, (k(i)+1)(1+\beta)), k(i) \in N \cup \{0\}$. Then $\omega(x, \sigma)$ is the Sturmian dynamical system defined by Hedlund [He]. By the choosing of β we know $x = C_1 C_2 \cdots$, where $C_i = 0 \cdots 01$ and $l(C_i) = k$ or $k+1, k > p$.

Let φ be the bijection from $\{0, 1, \cdots, p-1\}^k$ onto $\{0, 1, \cdots, p-1\}^k$ satisfying

$$\varphi(\overbrace{0 \cdots 01}^{i} 0 \cdots 0) = \begin{cases} 0 \cdots 01 \cdots (p-1)0 \cdots 0, & \text{if } i > p-1 \\ (p-i) \cdots (p-1)0 \cdots 01 \cdots (p-i-1), & \text{if } i \leq p-1 \end{cases},$$

$$\varphi(00 \cdots 0) = (0 \cdots 01 \cdots (p-2)).$$

Denote $\overline{x} = \varphi_*(x)$, where φ_* is the induced transformation. Then φ_* is a topological conjugation between $(\omega(x, \sigma), \sigma)$ and $(\omega(\overline{x}, \sigma), \sigma)$. Hence $\omega(\overline{x}, \sigma), \sigma)$

is a strictly ergodic system and $\sigma_{\mu_{\overline{x}}} = 1'$ by the fact that $(\omega(x, \sigma), \sigma)$ is a strictly ergodic system and $\sigma_{\mu_x} = 1'$, where $\{\mu_y\} = M(\omega(y, \sigma), \sigma), y = x$ or $\overline{x}$.

Now let $\psi = \psi_n$ be a mapping from $\{0, 1, \cdots, p-1\}$ into $\{0, 1, \cdots, p-1\}^n$ satisfying

$$\psi(0)\psi(1) \cdots \psi(p-1) = \overbrace{0 \cdots \cdots 01 \cdots (p-1)}^{np}.$$

We have following commutative graph:

$$\begin{array}{ccc} \omega(\overline{x}, \sigma) & \xrightarrow{\sigma} & \omega(\overline{x}, \sigma) \\ \downarrow \psi_* & & \downarrow \psi_* \\ \omega(x', \sigma^n) & \xrightarrow{\sigma^n} & \omega(x', \sigma^n) \end{array}$$

where $x' = \psi_*(\overline{x})$.

We prove that ψ_* is one-to-one. Let $z \neq y \in \omega(\overline{x}, \sigma)$, then there exists $i_0 \in N$ such that $z_i = y_i, 1 \leq i \leq i_0, z_{i_0+1} \neq y_{i_0+1}$. By the construction of $\psi_*(\overline{x})$ there are two posibilities :

$(1) z_{i_0+1} = 1, y_{i_0+1} = 0$;

$(2) z_{i_0+1} = 0, y_{i_0+1} = 1$.

In the first case $z_{i_0+i} = i, 1 \leq i \leq p-1$. If $y_{i_0+i} = 0, 1 \leq i \leq p-1$, then $\psi_*(z) \neq \psi_*(y)$, if $y_{i_0+i_1} = 1$ for some $2 \leq i_1 \leq p-1$, then

$$(\psi_*(z))_{n(i_0+p-1)} = p-1 \neq \begin{cases} 0 \\ k-1-i_1 \end{cases} = (\psi_*(y))_{n(i_0+p-1)}.$$

That is to say $\psi_*(z) \neq \psi_*(y)$. Case (2) is similar with (1). So we conclude that ψ_* is one-to-one, onto and satisfies $\psi_* \sigma = \sigma^n \psi_*$, so a is topological conjugation between $(\omega(\overline{x}, \sigma), \sigma)$ and $(\omega(x', \sigma^n), \sigma^n)$.

Hence $(\sigma^n)_{\mu'} = \sigma_{\mu_{\overline{x}}} = 1'$, where $\{\mu^{prime}\} = M(\omega(x', \sigma^n), \sigma^n)$. Let $A_n = \omega(x', \sigma)$, then (A_n, σ) is a strictly ergodic system. So we have

$$\sigma_{\mu_n}(nm) = (\sigma^n)_{\mu'}(m)\sigma_{\mu_n}(n) = \sigma_{\mu_n}(n) = n$$

for every $m \in N$ by the fact $\sigma_{A_n}(n) = n$ and Lemma 4.2, 3.6, where $\{\mu_n\} = M(A_n, \sigma)$. It implies that $\sigma_{\mu_n}(m) = (m, n)$ for all $m \in N$. In other words $\sigma_{\mu_n} = n'$.

To sum up we get

Theorem 5.5 *Let Σ_M be one-sided subshift of finite type with $k \times k$ matrix $M = (m_{ij})$ satisfying that $m_{01} = m_{12} = \cdots = m_{(k-1)0} = m_{00} = 1$ and σ the left shift. Then $DF(\Sigma_M, \sigma) \supset (\mathcal{Y}\backslash E)$.*

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