IC/91/379

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

LIOUVILLE GRAVITY ON BORDERED SURFACES

Zbigniew Jaskólski

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

IC/91/379

International Atomic Energy Agency

and

United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

LIOUVILLE GRAVITY ON BORDERED SURFACES

Zbigniew Jaskólski *

International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The functional quantization of the Liouville gravity on bordered surfaces in the conformal gauge is developed. It was shown that the geometrical interpretation of the Polyakov path integral as a sum over bordered surfaces uniquely determines the boundary conditions for the fields involved. The gravitational scaling dimensions of boundary and bulk operators and the critical exponents are derived. In particular, the boundary Hausdorff dimension is calculated.

MIRAMARE – TRIESTE

November 1991

* Permanent address: Institute of Theoretical Physics, Wroclaw University, ul. Cybulskiego 36, Wroclaw, Poland.

1 Introduction

The recent progress in the continuum treatment of 2D gravity was initiated by Polyakov's proposal of the light-cone quantization [1]. This method based on SL(2R) current algebra provides an exact solution of the theory on the sphere [2]. Soon after the success of the light-cone approach the results of KPZ have been rederived and generalized for arbitrary genus within the functional integral framework in the conformal gauge [3, 4]. The fundamental assumption of the latter approach is that the Jacobian arising due to change from the field dependent functional measure to the translation invariant one is of the same form as the original Liouville action. This conjecture, crucial for applications of conformal field theory methods, was further justified by the explicit calculation of the Jacobian by the heat-kernel technique [6]. The combined results of Refs [3, 4] and [5, 6] provide an elegant and efficient approach to 2D gravity. Recently this method has been successfully used in some exact calculations in the Liouville gravity [7, 8] yielding a considerable agreement with the matrix model results.

The developments mentioned above concern the Liouville gravity on closed surfaces. There are however some interesting potential applications of a corresponding theory on bordered surfaces so it is desirable to extend the techniques of [3–6] to that case. The basic issue that arises is the choice of a relevant boundary condition for the Liouville field. This choice however depends on the context in which the sum over surfaces appears.

In the Liouville gravity regarded as a noncritical string theory there are two different applications of the Polyakov path integral over bordered surfaces. First, it can be used in the path integral representation of the closed (noncritical) string off-shell amplitudes. In the special case of surface with only one boundary component and several operator insertions one gets a path integral representation of a wavefunction of the theory. This representation of states in Liouville gravity has been recently analysed in [9] where the nonhomogeneous Neumann boundary condition $n^a\partial_a\varphi - k_g + \rho e^{\frac{\varphi}{2}} = 0$ involving the boundary geodesic curvature k_g has been postulated. In general the boundary condition in this case depends on an off-shell extension of the theory and therefore is gauge dependent.

The second application concerns the path integral expression of the open (noncritical) string on-shell amplitudes. Among few recent papers on this subject one can find two different choices: the homogeneous $n^a\partial_a\varphi = 0$ and the nonhomogeneous $n^a\partial_a\varphi \sim e^{\frac{\varphi}{2}}$ Neumann boundary conditions. The first one, conjectured in [7], has been implemented in the recent attempt [10] to identify the boundary operators of 2d gravity with some redundant matrix model operators. The second choice has been used in the operator quantization of the Liouville theory on the strip [11].

The aim of the present paper is to extend the path integral conformal gauge approach of [3–6] to the Liouville gravity on bordered surfaces interpreted as a theory of open noncritical string. In Sect.2 we start with the derivation of boundary conditions based on the geometrical interpretation of the Polyakov path integral as a sum over bordered surfaces. Since in the continuum treatment such an interpretation goes via the Faddeev-Popov procedure we adopt the requirement of consistency of this method as a "first principle" from which the boundary conditions can be derived. This is in logical agreement with the original derivation of the Liouville gravity where the action (expect the matter field conformal anomaly contribution) as well as the functional measure are direct consequences of the Faddeev-Popov method [12]. It is shown that the consistency requirement imposes some strong conditions on a submanifold of the space of all Riemannian metrics which can serve as an integral domain in the Polyakov path integral. All admissible integration domains form a family parameterized by the space of all normal directions along the world sheet boundary. In particular each member of this family consists of metrics with vanishing boundary geodesic curvature. This completely determines the homogeneous Neumann boundary condition for the Liouville field confirming the conjecture of [7] and the choice made in [9].

In the rest of the paper some consequences of the boundary conditions found in Sect.2 are analysed. In Sect.3, following the methods of [5, 6] the transition to the translation invariant measure is performed. It yields the expected renormalization of the central charge. Moreover with the cosmological terms renormalized to zero one gets the free field action. This justifies the calculations of the anomalous dimensions of the bulk and boundary operators based on the free field OPEs [9]. Finally, in Sect.4, using the scaling arguments of [3, 4] the critical exponents are calculated. Due to the special properties of the boundary conditions derived in Sect.2 the calculations of the bulk scaling exponents are not altered by any boundary effects and proceed exactly as in the case of closed surfaces. As an example of the critical exponent specific to the bordered random surface the boundary Hausdorff dimension is derived.

2 Boundary conditions

Let $M_{h,b}$ denotes the compact oriented surface with b–smooth boundary components and h–handles. The contribution to the partition function of all surfaces of the topological type $M_{h,b}$ is proportional to

$$Z_{h,b} = \int \mathcal{D}g\, \mathcal{D}x\, (Vol\mathcal{D}_{h,b})^{-1} \exp\left(-S_M[g,x] - S_G[g]\right) \ , \tag{1}$$

where the matter action is

$$S_M[g,x] = \frac{1}{8\pi}\int d^2z\sqrt{g}g^{ab}\partial_a x^\mu \partial_b x^\mu \ ,$$

and $S_G[g]$ denotes the bare cosmological constant term

$$S_G[g] = \mu_0 \int d^2z\sqrt{g} \ .$$

The standard way to calculate (=to define) the functional integral (1) is to apply the Faddeev-Popov procedure with respect to the group $\mathcal{D}_{h,b}$ of diffeomorphisms of $M_{h,b}$. In the conformal gauge it yields the determinant of the F-P operator $P_g^\dagger P_g$ defined by the conformal Lie derivative

$$(P_g v)_{ab} = \nabla_a v_b + \nabla_b v_a - g_{ab}\nabla_c v^c \ ,$$

and its formal adjoint

$$(P_g^\dagger h)_a = -2\nabla^b h_{ab} \ .$$

These operators act respectively, on the space of vector fields on $M_{h,b}$ (identified as the Lie algebra of the gauge group $\mathcal{D}_{h,b}$) and on the space $T_g\mathcal{M}_{h,b}$ tangent at g to the space $\mathcal{M}_{h,b}$ of all metrics on $M_{h,b}$. In the case of bordered surfaces the operators $P_g^\dagger P_g$, $P_g P_g^\dagger$ are not well defined unless some appropriate boundary conditions are introduced.

The analysis based on the requirement that $P_g^\dagger P_g$, $P_g P_g^\dagger$ are elliptic operators (and therefore it makes sense to calculate their determinants) leads to the following most general structure of admissible boundary conditions [12],

$$\begin{array}{ll} \mathcal{B}_1 v = 0 & \mathcal{B}_2 h = 0 \\ \mathcal{B}_2(P_g v) = 0 & \mathcal{B}_1(P_g^\dagger h) = 0 \ , \end{array}$$

where $\mathcal{B}_1$, $\mathcal{B}_2$ are ultralocal relations making the integral

$$\int_{\partial M_{h,b}} ds\, n^a h_{ab} v^b$$

vanish. The interpretation of vector fields on $M_{h,b}$ as vectors tangent to the group $\mathcal{D}_{h,b}$ at the identity diffeomorphism implies the boundary condition

$$n_a v^a = 0 \ , \tag{2}$$

which has the following unique extension to an admissible system,

$$\begin{aligned} t^a n^b h_{ab} &= 0 \ , & (3)\\ n^a (P_g^\dagger h)_a &= 0 \ , & (4)\\ t^a n^b (P_g v)_{ab} &= 0 \ . & (5) \end{aligned}$$

It follows that if one insists to integrate in (1) over a space of metrics on a fixed manifold $M_{h,b}$ than the boundary conditions (2-5) are uniquely determined. From the geometrical point of view the additional boundary conditions (3,4) mean that one has to integrate in (1) not over the whole space of metrics on $M_{h,b}$ but rather over an integral submanifold of the distribution on $\mathcal{M}_{h,b}$ defined by (3,4).

We will briefly discuss the integrability conditions for this distribution. For this purpose it is convenient to regard the boundary conditions (3,4) as 1-forms on $\mathcal{M}_{h,b}$,

$$\begin{aligned} \mathcal{A}_g(\delta g) &= n^a t^b \delta g_{ab}\, |_{\partial M_{h,b}} \ , \\ \mathcal{B}_g(\delta g) &= n_a \left(P_g^\dagger\left(\delta g - \tfrac{1}{2}\mathrm{Tr}(\delta g) g\right)\right)^a |_{\partial M_{h,b}} \ . \end{aligned}$$

Calculating the exterior derivative of $\mathcal{A}$ one can check that the distribution $ker\mathcal{A}$ is integrable

$$d\mathcal{A} = -(\mathcal{T} + \mathcal{N}) \wedge \mathcal{A} \ .$$

The 1-forms $\mathcal{T}, \mathcal{N}$ in the formula above are defined by

$$\begin{aligned} \mathcal{T}_g(\delta g) &= t^a t^b \delta g_{ab}\, |_{\partial M_{h,b}} \ , \\ \mathcal{N}_g(\delta g) &= n^a n^b \delta g_{ab}\, |_{\partial M_{h,b}} \ . \end{aligned}$$

An integral submanifold $\mathcal{M}^n_{h,b} \subset \mathcal{M}_{h,b}$ of $ker\mathcal{A}$ consists of all metrics $g \in \mathcal{M}_{h,b}$ with the same normal direction n at the boundary $\partial\mathcal{M}_{h,b}$. Let us fix n and consider the integrability conditions for the distribution on $\mathcal{M}^n_{h,b}$ given by the kernel of $\mathcal{B}$. One gets the following relations,

$$\begin{aligned} d\mathcal{B} &= -\frac{1}{2}\mathcal{T} \wedge \mathcal{B} + \frac{1}{4}(\mathcal{T} - \mathcal{N}) \wedge \mathcal{C} \ , \\ d\mathcal{C} &= \frac{1}{2}\mathcal{N} \wedge \mathcal{C} \ , \end{aligned}$$

where the 1-form $\mathcal{C}$ on $\mathcal{M}^n_{h,b}$ is defined by

$$\mathcal{C}_g(\delta g) = n^a \nabla_a (g^{ab}\delta g_{ab}) \mid_{\partial M_{h,b}} \quad .$$

It follows that in order to get an integrable system the boundary conditions $\mathcal{A}$, $\mathcal{B}$ should be supplemented by the new boundary condition $\mathcal{C}$. Recall that the integrability is actually required by the interpretation of the path integral (1) as a sum over surfaces.

The similar analysis can be performed for the boundary condition (5) imposed on vector fields on $M_{h,b}$. A corresponding integral submanifold is the subgroup $\mathcal{D}^n_{h,b} \subset \mathcal{D}_{h,b}$ consisting of all diffeomorphisms preserving a fixed normal direction n.

The next requirement following from the consistency of the Faddeev-Popov procedure is that an integral submanifold of the system $\mathcal{B}$, $\mathcal{C}$ on $\mathcal{M}^n_{h,b}$ which can be used as an integration domain in (1) should be stable under the action of $\mathcal{D}^n_{h,b}$. It yields the following conditions for vector fields,

$$\begin{aligned} \mathcal{B}(\nabla_a v_b + \nabla_b v_a) &= 0 \ . \\ \mathcal{C}(\nabla_a v_b + \nabla_b v_a) &= 0 \ . \end{aligned}$$

Since $\mathcal{B}(\nabla_a v_b + \nabla_b v_a) = n_a(P^\dagger_g P_g v)^a$, for any eigenfunction of the operator $P^\dagger_g P_g$ the first relation is automatically satisfied (note that $P^\dagger_g P_g$ with the boundary conditions (2,5) is an elliptic operator and all its eigenfunctions are smooth). The second relation however implies a new boundary condition

$$t^a \partial_a (k_g t_a v^a) = 0 \quad ,$$

where k_g denotes the boundary geodesic curvature. This is compatible with the boundary conditions (2,5) if and only if $k_g \equiv 0$ along $\partial M_{h,b}$. This last condition uniquely determines the admissible integral submanifold $\mathcal{M}^{n*}_{h,b}$ of the system $\mathcal{B}$, $\mathcal{C}$ on $\mathcal{M}^n_{h,b}$.

The explicit description of $\mathcal{M}^{n*}_{h,b}$ can be given by means of doubling construction [13]. Let $M^D_{h,b}$ denotes the double of $M_{h,b}$ and i – an involution of $M^D_{h,b}$ with $\partial M_{h,b}$ as the set of all fixed points and with the invariant direction along $\partial M_{h,b}$ coinciding with the normal direction n. Then the submanifold $\mathcal{M}^{n*}_{h,b}$ consists of all metrics $g \in \mathcal{M}_{h,b}$ admitting a C^1 extension to an i-symmetric metric g^D on $M^D_{h,b}$ ($g^D \mid_{M_{h,b}} = g$, $i^* g^D = g^D$).

For any n the submanifold $\mathcal{M}^{n*}_{h,b}$ has the structure of a principal fibre bundle over the reduced Teichmüller space $T^R_{h,b}$ of $M_{h,b}$ with the semidirect product $\mathcal{W}^n_{h,b} \odot \mathcal{D}^{no}_{h,b}$ as a structure group [13, 14]. ($\mathcal{W}^n_{h,b}$ denotes the abelian group of real valued functions on $M_{h,b}$ satisfying the Neumann boundary condition $n^a\partial_a\varphi = 0$, and $\mathcal{D}^{no}_{h,b}$ is the connected component of identity in $\mathcal{D}^n_{h,b}$.) This is not surprising since in the infinite-dimensional geometry the ellipticity of a differential operator arising in the map tangent to the group action is a basic ingredient of the proof of the existence of such a structure [15]. In the case under consideration this is just the ellipticity of the Faddeev-Popov operator – a property assumed at the beginning of the present discussion. What the analysis given above actually shows is that the supply of submanifolds of $\mathcal{M}_{h,b}$ caring the required structure is strongly limited. It is precisely the family of submanifolds $\mathcal{M}^{n*}_{h,b}$ parametrized by the set of all normal directions along $\partial M_{h,b}$.

The principal bundle structure of $\mathcal{M}^{n*}_{h,b}$ has some important consequences. First of all, as it has been desired, the Faddeev-Popov technique is perfectly in order to provide a meaning to the formal expression (1). Secondly the boundary condition for the Liouville field

$$n^a \partial_a \varphi = 0$$

is uniquely determined and does not depend on a section

$$T^R_{h,b} \ni t \longrightarrow \hat{g}_t \in \mathcal{M}^{n*}_{h,b} \tag{6}$$

used for the construction of a conformal gauge slice $g = e^{\varphi}\hat{g}_t$.

The question arises to what extent the partition function (1) depends on the choice of an admissible integration domain $\mathcal{M}^{n*}_{h,b}$. Let n and n' denote two different normal directions. There always exists a diffeomorphism f of $M_{h,b}$ such that $f^*n = n'$. Since the functional measures and the action in (1) are explicitly $\mathcal{D}_{h,b}$-invariant the change of variables $(g, x) \longrightarrow (f^*g, f^*x)$ results only in the change of integration domains. It follows that the path integral (1) is in fact independent of n. As we will see in the next section this result can be confirmed by explicit calculations based on the heat kernel technique.

Concluding this section let us stress that the above discussion concerns only the Liouville gravity, i.e. the quantum Liouville theory arising from the Polyakov path integral over bordered surfaces. It is fairly possible to construct a consistent quantum field theory determined by the Liouville action and based on different boundary conditions. In such a case, however, the interpretation of the theory in terms of a sum over random surfaces is unclear.

3 Functional measures

The application of the Faddeev-Popov procedure to the path integral (1) with the integration domain $\mathcal{M}^{n*}_{h,b}$ proceeds along the standard way [12, 13]. In the conformal gauge (6) it yields the following expression,

$$\begin{aligned} Z_{h,b} \;=\; & \int\limits_{[T^R_{h,b}]} [dt] \int\limits_{\mathcal{W}^n_{h,b}} \mathcal{D}^{e^{\varphi}\hat{g}_t}\varphi \; \exp\left(-\tfrac{26-d}{48\pi} S_L[\hat{g}_t, \varphi]\right) \times \\ & \times \int \mathcal{D}^{\hat{g}_t} b\, \mathcal{D}^{\hat{g}_t} c \int \mathcal{D}^{\hat{g}_t} x \; \exp\left(-S_{Gh}[\hat{g}_t, b, c] - S_M[\hat{g}_t, x]\right) \ , \end{aligned} \tag{7}$$

where $[dt]$ stands for the φ-independent measure on the moduli space and $[T^R_{h,b}]$ is a fundamental domain of the modular group $\mathcal{D}^n_{h,b}/\mathcal{D}^{no}_{h,b}$. The Liouville action $S_L[g, \varphi]$ in (7) takes the following form,

$$S_L[g,\varphi] = \int\limits_{M_{h,b}} d^2z\sqrt{g}\left(\tfrac{1}{2}g^{ab}\partial_a\varphi\partial_b\varphi + R_g\varphi + \mu e^{\varphi}\right) + \lambda \int\limits_{\partial M_{h,b}} ds\, e^{\frac{1}{2}\varphi} \ .$$

Note that as a consequence of our choice of integration domain, the only boundary term in the formula above is the cosmological one. Finally, the functional measure $\mathcal{D}^{e^{\varphi}\hat{g}_t}$ for the Liouville field is formally interpreted as the volume form related to the weak Riemannian structure on $\mathcal{W}^n_{h,b}$ given by

$$\bar{W}^g_{\varphi}(\delta\varphi, \delta\varphi') = \int\limits_{M_{h,b}} \sqrt{g}\, d^2z\, e^{\varphi}\, \delta\varphi\, \delta\varphi' . \tag{8}$$

Following [5, 6] we will calculate the Jacobian emerging from the transformation of the measure $\mathcal{D}^{e^{\varphi}\hat{g}_t}\varphi$ to the translation invariant measure $\mathcal{D}^{\hat{g}_t}\varphi$ related to the field independent Riemannian metric

$$W^g_\varphi(\delta\varphi, \delta\varphi') = \int_{M_{h,b}} \sqrt{g} d^2z\, \delta\varphi\, \delta\varphi' . \tag{9}$$

The comparison of (8) and (9) leads to the formal relation

$$\mathcal{D}^{e^{\varphi}\hat{g}_t}\varphi = \sqrt{\det L}\, \mathcal{D}^{\hat{g}_t}\varphi \ ,$$

where L is given by the integral kernel

$$L(z-z') = e^{\varphi(z)}\delta^{(2)}(z-z') \ .$$

The short distance divergence $\delta^{(2)}(0)$ appearing in the formal formula for variation,

$$\begin{aligned} \delta \log\det L &= \delta \operatorname{Tr} \log L = \\ &= \int d^2z\, \delta^{(2)}(0)\, \delta\varphi(z) = \\ &= \int \sqrt{g} d^2z\, \delta^{(2)}_g(0)\, \delta\varphi(z) \ , \end{aligned}$$

can be regularized by the short time heat kernel of some elliptic operator natural for the problem under consideration. In our case this is the covariant Laplacean $\triangle_g$ acting on the space of scalar functions on $M_{h,b}$ satisfying the Neumann boundary condition. One gets the following formula for the variation of regularized determinant,

$$\delta \log \det{}_R L = \lim_{\varepsilon\to 0} \int_{M_{h,b}} \sqrt{g} d^2z\, e^{-\varepsilon\triangle_g}(z,z)\, \delta\varphi(z) \ . \tag{10}$$

The formula above depends on the choice of metric g however. One way to avoid this ambiguity has been proposed in [6]. The reasoning given there involves the family $e^{s\varphi}\hat{g}_t$, $s \in [0,1]$ of interpolating metrics and is based on the observation that for a small variation δs the ambiguity in the choice of metric is of order $(\delta s)^2$ and can be neglected. Integrating out an infinitesimal expression one gets the result equivalent to the choice $g = e^{\varphi}\hat{g}_t$ in the formula (10). Another justification of this choice can by obtained by a slightly modification of the symmetry arguments used in [3, 4]. The original measure $\mathcal{D}^{e^{\varphi}\hat{g}_t}\varphi$ is invariant under the Weyl transformations

$$g \longrightarrow e^{\sigma}g \ , \quad \varphi \longrightarrow \varphi - \sigma \ .$$

One can show that the choice $g = e^{\varphi}\hat{g}_t$ is the only one which yields the Weyl-invariant regularization.

The short time expansion of the heat kernel for the scalar Laplacean with the Neumann boundary condition is given by

$$\begin{aligned} \operatorname{Tr}(f\exp(-\varepsilon\triangle_g)) &= \frac{1}{4\pi\varepsilon}\int_{M_{h,b}} \sqrt{g}d^2z - \frac{1}{8\sqrt{\pi\varepsilon}}\int_{\partial M_{h,b}} ds\, f + \\ &- \frac{1}{8\pi}\int_{\partial M_{h,b}} ds\, n^a\partial_a f + \\ &+ \frac{1}{12\pi}\left(\frac{1}{2}\int_{M_{h,b}} \sqrt{g}d^2z\, R_g f + \int_{\partial M_{h,b}} ds\, k_g f\right) + O(\sqrt{\varepsilon}) \ . \end{aligned} \tag{11}$$

Note that with our choice of the integration domain $n^a\partial_a\delta\varphi = 0$ and $k_g = 0$ for any $g \in \mathcal{M}^{n*}_{h,b}$. Therefore inserting the expansion (11) in the formula (10) and integrating one gets (up to φ–independent terms)

$$\begin{aligned} \log\sqrt{\det{}_R L} &= \lim_{\varepsilon\to 0}\left(\frac{1}{8\pi\varepsilon}\int_{M_{h,b}} \sqrt{\hat{g}}d^2z\, e^{\varphi} + \frac{1}{4\sqrt{\pi\varepsilon}}\int_{\partial M_{h,b}} d\hat{s}\, e^{\frac{1}{2}\varphi}\right) + \\ &+ \frac{1}{48\pi}\int_{M_{h,b}} \sqrt{\hat{g}}d^2z\, \left(\tfrac{1}{2}\hat{g}^{ab}\partial_a\varphi\partial_b\varphi + \hat{R}\varphi\right) \ . \end{aligned}$$

The final expression for the partition function reads

$$\begin{aligned} Z_{h,b} &= \int_{[T^R_{h,b}]} [dt] \int_{W^n_{h,b}} \mathcal{D}^{\hat{g}_t}\varphi \int \mathcal{D}^{\hat{g}_t}b\, \mathcal{D}^{\hat{g}_t}c \int \mathcal{D}^{\hat{g}_t}x \times \\ &\times \exp\left(-S'_L[\hat{g}_t,\varphi] - S_{Gh}[\hat{g}_t,b,c] - S_M[\hat{g}_t,x]\right) \ , \end{aligned} \tag{12}$$

where the effective Liouville action $S'_L[g,x]$ is given by

$$S'_L[g,\varphi] = \frac{1}{8\pi}\int_{M_{h,b}} d^2z\sqrt{g}\left(g^{ab}\partial_a\varphi\partial_b\varphi + QR_g\varphi\right) + \frac{\mu}{8\pi\gamma^2}\int_{M_{h,b}} e^{\gamma\varphi} + \frac{\lambda}{4\sqrt{\pi}\gamma}\int_{\partial M_{h,b}} ds\, e^{\frac{1}{2}\gamma\varphi} \tag{13}$$

with $Q = \sqrt{\frac{25-d}{3}}$ and $\gamma = \frac{2}{Q}$. This form results from the rescaling $\varphi \longrightarrow \sqrt{\frac{12}{25-d}}\varphi$ performed in (12) in order to achieve the standard normalization of the kinetic term.

In the case of closed surfaces the crucial observation was that setting the renormalized cosmological constant equal to zero one gets a simple conformal field theory [4]. This allows to employ the known machinery of the free field OPE [16]. In particular the Liouville exponential interaction term can be treated as a marginal deformation of the free action.

As follows from the formula (13) in the bordered surface case essentially the same interpretation is possible. In fact with the bulk and the boundary cosmological constants renormalized to zero one obtains a free conformal field theory with the background charge $Q = \sqrt{\frac{25-d}{3}}$. Moreover, due to the special properties of the boundary conditions derived in Sect.2, one can apply, via the doubling construction, all methods of conformal field theory on closed surfaces. In particular the combined results of [17] and [16] provide an efficient method of calculating the correlation functions of boundary operators.

4 Critical exponents

Another consequence of the boundary conditions derived in Sect.2 is that the anomalous dimensions and the critical exponents can be calculated by a straighforward generalization of the scaling method developed in [4]. In this section we will mainly consider the boundary Hausdorff dimension which provides a nontrivial example of a critical exponent specific to the bordered random surface. Although the definition of this notion is rather tricky in the range $d < 1$ we believe that it still contains a good deal of information about the Liouville gravity. At least the derivation of this exponent can serve as a good illustration of possible boundary effects in the theory.

The interest to the Hausdorff dimension comes from its standard interpretation as a measure of interaction of a system. Let us recall that it could be roughly defined as the dimension h of the subset of the target space swept out by a random trajectory. If the dimension of the target space $d > 2h$ then two generic random walks do not intersect and the system behaves like a free one. For $d < 2h$, however, the random walks necessarily intersect and the interaction becomes crucial for properties of the system. Following this interpretation in the noncritical open string the boundary Hausdorff dimension measures the cubic string interaction while the bulk Hausdorff dimension can be related to the quartic one.

The first attempt to calculate the Hausdorff dimension of the closed Polyakov surface within the continuum approach has been made in [19]. More recently this issue has been discussed in [20] by means of the conformal field theory techniques developed in [4]. For a more comprehensive discussion of the Hausdorff dimension and its interpretation we refer to [20].

For completeness we will start with a brief exposition of some results of [9] concerning the bulk and the boundary anomalous dimensions. Since the anomalous dimensions concern local properties of the theory it is enough to work in a local chart of $M_{h,b}$, which can be chosen as the upper half plane with the flat metric. (Note that locally it is always possible to deform the metric $\hat{g}_t$ to the flat metric keeping the boundary geodesic.) In this case the (improved) stress energy tensor derived from the action (13) takes the following familiar form,

$$T_\varphi(z) = -\frac{1}{2}(\partial_z\varphi)^2 + \frac{Q}{2}\partial_z^2\varphi.$$

Using the free field propagator satisfying the Neumann boundary condition along the real axis

$$\langle\varphi(z)\varphi(w)\rangle = -\log|z-w|^2 - \log|\bar{z}-w|^2 \ , \qquad (14)$$

and the free field normal ordering one can calculate the bulk and the boundary OPE's. For the operator $:e^{\alpha\varphi(z)}:$ one gets

$$T_\varphi(z)\, e^{\alpha\varphi(w)} = \frac{1}{(z-w)^2}\left(\frac{\alpha^2}{2} - \frac{\alpha Q}{2}\right) e^{\alpha\varphi(z)} + \frac{1}{z-w}\partial_z e^{\alpha\varphi}, \qquad (15)$$

$$T_\varphi(s)\, e^{\alpha\varphi(s')} = \frac{1}{(s-s')^2}\left(-2\alpha^2 - \alpha Q\right) e^{\alpha\varphi(s)} + \frac{2}{s-s'}\partial_s e^{\alpha\varphi}. \qquad (16)$$

Therefore the bulk and the boundary anomalous dimensions are given by

$$\Delta_v(e^{\alpha\varphi}) = -\frac{\alpha^2}{2} - \frac{\alpha Q}{2} \ ,$$
$$\Delta_b(e^{\alpha\varphi}) = -2\alpha^2 - \alpha Q \ .$$

The requirement that the bulk and the boundary integrals of the operator $:e^{\alpha\varphi}:$ be reparametrization invariant leads to the conditions

$$\Delta_v(e^{\alpha_v\varphi}) = \Delta_b(e^{\alpha_b\varphi}) = 1 \ .$$

Solving for α_v, α_b one obtains

$$\alpha_v = 2\alpha_b = -\frac{1}{2\sqrt{3}}\left(\sqrt{25-d} \mp \sqrt{1-d}\right). \qquad (17)$$

Following the same way one can determine the bulk,

$$V[\Psi(z),\varphi(z)] = :e^{\beta_v\varphi(z)}\Psi(z): \ ,$$
$$\beta_v = -\frac{1}{2\sqrt{3}}\left(\sqrt{25-d} \mp \sqrt{1-d+24\Delta_v(\Psi)}\right), \qquad (18)$$

and the boundary,

$$V[\Psi(s),\varphi(s)] = :e^{\beta_b\varphi(s)}\Psi(s): \ ,$$
$$\beta_b = -\frac{1}{4\sqrt{3}}\left(\sqrt{25-d} \mp \sqrt{1-d+24\Delta_b(\Psi)}\right) \ , \qquad (19)$$

gravitational dressing of the spinless primary field Ψ of a conformal field theory coupled to the Liouville gravity.

The choice of the minus sign in the solutions (17,18,19) is determined by the semiclassical analysis [18] which leads to the inequalities [9]:

$$\alpha_v, \beta_v \le \frac{Q}{2} \ , \qquad (20)$$
$$\alpha_b, \beta_b \le \frac{Q}{4} \ . \qquad (21)$$

The simplest example of the bulk critical exponent is the open string susceptibility. Using the scaling arguments presented in [4] it can be easily derived from the effective action (13),

$$\Gamma_{open}(h,b) = \left(1 - h - \frac{b}{2}\right)\frac{d - 25 - \sqrt{(25-d)(1-d)}}{12}.$$

In order to determine the boundary Hausdorff (or fractal) dimension we will use a continuum version [20] of the microcanonical discrete approach proposed in [21]. The basic idea of this method is to use some scaling property of random path (or surface) as a definition of the Hausdorff dimension. An important advantage of this approach is that one can use some scaling arguments which are hardly available when one starts with the standard definition by coverings.

Let us fix a boundary component $\Sigma \in \partial M_{h,b}$ and denote by l its intrinsic length. According to [21, 20] the boundary Hausdorff dimension h_b can be defined by the scaling limit (for large enough l) of the "mean square size" of the embedded boundary $x^\mu(\Sigma)$,

$$\langle x_\Sigma^2\rangle_l \sim l^{\frac{2}{h_b}},$$

where x_Σ^2 is formally defined as

$$x_\Sigma^2 = \frac{\int_\Sigma d\hat{s}_t\, x^\mu(s)x_\mu(s)}{\int_\Sigma d\hat{s}_t}.$$

The fixed length expectation value $\langle x_\Sigma^2\rangle_l$ of the boundary operator x_Σ^2 is given by:

$$\langle x_\Sigma^2\rangle_l = l^{-1}\frac{\int\mathcal{D}\mu\int\mathcal{D}\varphi\ \delta(\int_\Sigma d\hat{s}_t\, e^{\alpha_b\varphi} - l)\ \int_\Sigma d\hat{s}_t\, V[x^2(s),\varphi(s)]\ e^{-S_L'[\hat{g}_t,\varphi]}}{\int\mathcal{D}\mu\int\mathcal{D}\varphi\ \delta(\int_\Sigma d\hat{s}_t\, e^{\alpha_b\varphi} - l)\ e^{-S_L'[\hat{g}_t,\varphi]}} \qquad (22)$$

where $\mathcal{D}\mu$ stands for the finite dimensional integration over the moduli as well as for the contribution of the ghost and the matter system, and $V[x^2(s),\varphi(s)]$ denotes the gravitational dressing of the local boundary operator $:x^2:$. In addition we have to assume that the functional integral over x-fields in (22) is restricted to the orthogonal complement of the subspace of zero modes [21, 20].

There are two remarks in order. First of all, let us note that the definition above has been designed for the special case of the matter system consisting of d-scalar free fields. Unfortunately, the scaling properties of expectation values, which we wish to use in calculating the l-dependence of $\langle x_\Sigma^2 \rangle_l$, are available only in the range $d \leq 1$. In this range the geometrical interpretation of the Hausdorff dimension is rather problematic. Moreover for $d < 1$ it is hard to recognize a counterpart of the operator $:x^2:$ between operators of the matter system and the definition (22) does not make sense anymore.

In order to provide a meaning to the expression (22) in the range $d \leq 1$ we will use, after [20], the following formal procedure. We start with a positive integer d and perform calculations using the free field stress energy tensor

$$T_x(z) = -\frac{1}{2}\partial_z x^\mu(z)\partial_z x_\mu(z),$$

and the free field normal ordering. The final formulae are then continued to the region $d \leq 1$ where the scaling exponents involved become real.

The second problem with the formula (22) is that, even with the zero modes projected out, $:x^2(s):$ is not a conformal boundary operator as it can be easy recognized from its OPE,

$$T_x(s)\, x^2(s') = -2\frac{d}{(s-s')^2} + \frac{2}{s-s'}\partial_s x^2. \tag{23}$$

For this reason the formula (19) for gravitational dressing of conformal operators cannot be applied. However, extending the idea of dressing to nonconformal operators, one can assume that after switching on the gravity the operator $:x^2(s):$ acquires a dressing of the following more general form [20],

$$V[x^2(s),\varphi(s)] = :f(\varphi(s))x^2(s): + :g(\varphi(S)):.$$

Note that the second term is necessary for cancellation of the (first) anomalous term in the OPE (23).

Using (16) and (23) one can calculate the OPE for the operator $V[X^2(s),\varphi(s)]$,

$$\begin{aligned}(T_x(s') + T_\varphi(s'))\, V[x^2(s),\varphi(s)] &= \frac{1}{(s'-s)^2}\left[-2f''(\varphi(s)) - Qf'(\varphi(s))\right] x^2(s) - \\ &- \frac{1}{(s'-s)^2}\left[2df(\varphi(s)) - 2g''(\varphi(s)) - Qg'(\varphi(s))\right] + \\ &+ \frac{2}{s'-s}\partial_s V[x^2(s),\varphi(s)]\ .\end{aligned}$$

It follows that the operator $V[x^2(s),\varphi(s)]$ is a conformal boundary operator with $\Delta_b(V[x^2(s),\varphi(s)]) = 1$ provided that the following equations hold,

$$2f'' + Qf' + f = 0\,, \tag{24}$$
$$2g'' + Qg' + g = -2df. \tag{25}$$

The most general solution of the first equation containing only those exponents which satisfy the inequality (21) has the following form,

$$f(\varphi) = \begin{cases} ce^{\alpha_b\varphi} & \text{for } d \neq 1 \\ (c_1 + c_2\varphi)e^{\alpha_b\varphi} & \text{for } d = 1 \end{cases}\ , \tag{26}$$

where c, c_1, c_2 are arbitrary constants. Solving the equation (26) with f given by (27) and using again the bound (21) to reject a part of solutions one gets

$$g(\varphi) = \begin{cases} \left(k - d\sqrt{\frac{12}{1-d}}c\varphi\right) e^{\alpha_b\varphi} & \text{for } d \neq 1 \\ \left(k_1 + k_2\varphi - \frac{1}{2}c_1\varphi^2 - \frac{2}{3}c_2\varphi^3\right) e^{\alpha_b\varphi} & \text{for } d = 1 \end{cases}\ , \tag{27}$$

where k, k_1, k_2, are new arbitrary constants.

Substituting (27) and (28) into (24) we have, in the range $d \neq 1$ the following formula for $V[x^2(s),\varphi(s)]$,

$$V[x^2(s),\varphi(s)] = c:\left(x^2(s) + \frac{k}{c} - d\sqrt{\frac{12}{1-d}}\varphi(s)\right) e^{\alpha_b\varphi(s)}:. \tag{28}$$

Performing the shift $\varphi \longrightarrow \varphi + \frac{1}{\alpha_b}\log l$ in the path integral (22) and using the transformation property of the vertex operator $V[x^2(s),\varphi(s)]$,

$$V[x^2(s),\varphi(s)] \xrightarrow{\varphi\to\varphi+\frac{1}{\alpha_b}\log l} l\left(V[x^2(s),\varphi(s)] + c\tfrac{d}{\alpha_b}\sqrt{\tfrac{12}{1-d}}\, :e^{\alpha_b\varphi(s)}: \log l\right)$$

one gets the explicit l-dependence of $\langle x_\Sigma^2 \rangle$

$$\langle x_\Sigma^2\rangle_l = B + c\frac{d}{\alpha_b}\sqrt{\frac{12}{1-d}}\log l\,, \tag{29}$$

where B is a constant given by some functional integral.

According to the standard interpretation of the logarithmical dependence, the scaling low (29) implies that in the range $d < 1$ the boundary of a random surfaces has the infinite (logarithmically divergent) Hausdorff dimension .

It is of some interest to compare the result above with the bulk Hausdorff dimension. Calculating the gravitational dressing of the bulk operator $:x^2(z):$ and using the fixed area expectation value one obtains

$$\langle x_M^2\rangle_A = B' + c'\frac{d}{\alpha_v}\sqrt{\frac{12}{1-d}}\log A\,. \tag{30}$$

As it was already mentioned the calculations of bulk exponents are not altered by any boundary effects and the derivation of (30) is essentially the same as in the case of closed surfaces [20]. The formula (30) however does not contain the power dependent term found in [20]. This discrepancy has the following origin. The solutions of the bulk counterparts of the equations (25,26) which yield power terms in the scaling low (30) contain exponents of φ not satisfying the inequality (20). According to the semiclassical analysis [18] such exponents do not exist as local operators in the quantum Liouville theory and therefore the solutions containing them are rejected in the present analysis.

Although the coefficients in the scaling lows are not supposed to be universal, it is interesting to observe that the formula (29) can be obtained from (30) by the simple substitution $A \sim l^2$. This is a consequence of the close relation between the bulk (15) and the boundary (16) OPE's determined by the free field propagator (14).

As both formulae (29) and (30) exhibit the same logarithmic divergence one can say that the random surface and its boundary have the same (infinite) Hausdorff dimension. Note that both scaling lows are independent on the global topology of a random surface. In particular the boundary Hausdorff dimension does not depend on the presence of other boundary components. It is tempting to conclude that the cubic, quartic as well as open-closed string interactions equally contribute to the full interaction picture of the open string. In view of the failure of the standard geometric interpretation of the Hausdorff dimension in the range $d < 1$, the meaning of such a statement is however highly problematic.

The remarks above concerning the comparison of the bulk and the boundary Hausdorff dimensions apply in the case $d = 1$ as well. As follows from (27) ,(28) the dressing is given by

$$V[x^2(s), \varphi(s)] = \\ = :\left[x^2(s)(c_1 + c_2\varphi(s)) + k_1 + k_2\varphi(s) - \frac{d}{2}c_1\varphi^2(s) - \frac{2}{3}c_2\varphi^3(s)\right] e^{\alpha_b\varphi(s)}: .$$

The change under the shift

$$V[x^2(s), \varphi(s)] \xrightarrow{\varphi \to \varphi + \frac{1}{\alpha_b}\log l} l\, V[x^2(s), \varphi(s)] + \\ + \frac{l}{\alpha_b} :\left[c_2 x^2(s) + k_2 - c_1\varphi(s) - 2c_2\varphi^2(s)\right] e^{\alpha_b\varphi(s)}: \log l + \\ + \frac{l}{\alpha_b^2} :\left[-\frac{1}{2}c_1 - 2c_2\varphi(s)\right] e^{\alpha_b\varphi(s)}: (\log l)^2 + \\ - \frac{2l}{3\alpha_b^3}c_2 :e^{\alpha_b\varphi(s)}: (\log l)^3$$

leads to the following scaling low,

$$\langle x_\Sigma^2 \rangle_l = B_0 + B_1 \log l + B_2 (\log l)^2 + \frac{4\sqrt{2}}{3} c_2 (\log l)^3 \quad , \tag{31}$$

where the constants B_1, B_2, B_3 are given by functional integrals.

In the bulk, according to [20] we have:

$$\langle x_M^2 \rangle_A = B_0' + B_1' \log A + B_2' (\log A)^2 + \frac{1}{3\sqrt{2}} c_2' (\log A)^3 \quad .$$

The coefficients in the formula (31) depends on four independent parameters c_1, c_2, k_1, k_2 which cannot be calculated using simple scaling arguments. Let us observe that whatever the values of these parameters are the scaling low (31) always contains a term with higher power of $\log l$. It follows that for $d = 1$ the large l behaviour of the mean square size of the boundary is qualitatively different from the one valid in the range $d < 1$. This can be interpreted as a sign of a phase transition at the critical point $d = 1$.

ACKNOWLEDGMENTS

The author is grateful to Professor J. Eells and Professor A. Verjovsky for the warm hospitality at the Mathematics Group of the International Centre for Theoretical Physics, where this work was carried out. He would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for support.

References

[1] A.Polyakov, Mod.Phys.Lett. A2 (1987) 893

[2] V.Knizhnik, A.Polyakov and A.Zomolodchikov, Mod.Phys.Lett. A3 (1988) 819

[3] F.David, Mod.Phys.Lett. A3 (1988) 1651

[4] J.Distler and K.Kawai, Nucl.Phys. B321 (1989) 509

[5] N.Mavromatos and J.Miramontes, Mod.Phys.Lett. A4 (1989) 1849

[6] E.D'Hoker and P.S.Kurzepa, Mod.Phys.Lett. A5 (1990) 1411

[7] A.Gupta, S.P.Trivedi and M.B.Wise, Nucl.Phys. B340 (1990) 85

[8] M.Bershadsky and I.R.Klebanov, Phys.Rev.Lett. 65 (1990) 3088
P.D.Francesco and D.Kutasov, Nucl.Phys. B342 (1990) 589 ; Phys.Lett. B261 (1991) 385
M.Goulian and M.Li, Phys.Rev.Lett. 66 (1991) 2051
Y.Kitazawa, HUTP-91/A013 (1991) preprint
Vl.I.Dodsenko, PAR-LPTHE 91-18 (1991) preprint
K.Aoki and E.D'Hoker, UCLA/91/TEP/32 (1991) preprint

[9] E.Martinec, G.Moore and N.Seiberg, RU-14-91 (1991) preprint

[10] G.Moore, N.Seiberg and M.Stauder, RU-91-11 (1991) preprint

[11] J.-L.Gervais and A.Neveu, Nucl.Phys. B238 (1984) 125; 396
E.Cremmer, J.-L.Gervais, LPTENS 90/32 (1990) preprint

[12] O.Alvarez, Nucl.Phys. B216 (1983) 125

[13] Z.Jaskólski, Commun.Math.Phys. 128 (1990) 285

[14] C.Earle and A.Schatz, J.Diff.Geom. 4 (1970) 169

[15] A.Fisher and A.Tromba, Math.Ann. 267 (1984) 514

[16] Vl.I.Dodsenko and V.A.Fateev, Nucl.Phys B240 (1984) 312, B251 (1985) 691

[17] J.L.Cardy, Nucl.Phys. B240 (1984) 514

[18] N.Seiberg, RU-90-29 (1990) preprint

[19] J.Jurkiewicz and A.Krzywicki, Phys.Lett. 148B (1984) 148

[20] J.Distler, Z.Hlousek and H.Kawai, Int.J.Mod.Phys. A5 (1990) 1093

[21] D.J.Gross, Phys.Lett. 138B (1984) 185