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ABSTRACT

Some critical properties of an equilibrium superradiant model are discussed, taking into account the quantum fluctuations of the field variables. The critical region is calculated using the Ginsburg criterion, underlining the role of the atomic concentration as a control parameter of the phase transition.

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The critical properties of low dimensional models present a special interest for the general theory of phase transitions, due to their nonconformal behaviour regarding the concept of universality [1, 2].

Thus, if the dimension d of the model is diminished under the critical one $d_c=4$, the results of the mean field theory (MFT) are restricted and the effect of fluctuations has to be considered.

Although the gaussian theory of fluctuations leads to pathological behaviour of the models, this approach is able to give information about the critical region: the interval of external parameters where the fluctuations could not be more neglected. The Ginsburg criterion or similar analogous statements can be applied to obtain it [1, 2, 3].

In this paper we intend to estimate the temperature critical interval, for a superradiant model belonging to a large class of models much used in quantum optics [4, 5]. During the last 20 years, their critical properties were extensively studied [6, 7, 8, 9, 10], especially in the mean field approximation.

Using the Method of Functional Integration [10, 11, 12], one can go beyond this approach and include both the effect of the statistical fluctuations of the field variables [12, 13], and that of the quantum fluctuations [14] which are of thermal origin for an equilibrium model.

The aim of this paper is to discuss critical aspects of a superradiant model within the gaussian approximation and including the quantum fluctuations of field variables.

Superradiance is a collective effect of spontaneous emission exhibited by a system of N two-level atoms prepared in a fully excited state.

For a small system, whose characteristic length is less than the wavelength of the spontaneous emitted radiation, a mean field model (that is a 0-dimensional one, in which the propagation effects are neglected) is appropriate.

The hamiltonian of an atomic system interacting with a single resonant mode of the electromagnetic field in the dipole approximation, is expressed in the second quantization formalism, as:

$$H = \omega a^{+}a + \frac{\varepsilon}{2}\sum_j (c^{+}_{j2}c_{j2} - c^{+}_{j1}c_{j1}) - \frac{i\lambda}{\sqrt{2\omega N}}\sum_j (a - a^{+})(c^{+}_{j2}c_{j1} + c^{+}_{j1}c_{j2}) \quad (1)$$

where:

- $a^{+}(a)$ are the bosonic operators of the resonant mode of energy ω (in units of $\hbar=1$)
- $c^{+}_{j1,2}(c_{j1,2})$ are the fermionic operators associated with the j-th atom ($j=1 \div N$) having one of the energies $\pm\varepsilon/2$
- $-\lambda = D_{12}\omega(N/\varepsilon_0 V)^{1/2}$ is the interaction coupling, and
  - D_{12} is the electric-dipole matrix element
  - ε_0 is the permittivity
  - V is the effective volume of the cavity which contains the emitters [15].

The statistical properties of the model are obtained from the partition function, written as a functional integral over the complex bosonic and the Grassmann fermionic variables [11]:

$$Z/Z_0 = \frac{\int Da^{+}DaDc^{+}Dc\ \exp[S]}{\int Da^{+}DaDc^{+}Dc\ \exp[S_0]} \quad (2)$$

The functional S which appears under the integral is the quasiclassical "action" defined by:

$$S = \int_0^{\beta} d\tau \left[a^{+}(\tau)\partial_\tau a(\tau) + \sum_j \sum_\alpha c^{+}_{j\alpha}(\tau)\partial_\tau c_{j\alpha}(\tau) - H(\tau) \right] \quad (3)$$

where: τ is the Matsubara imaginary time, $\beta = 1/(k_B T)$, with T -the temperature and k_B -the Boltzmann constant. In eq. (3) $\alpha = 1,2$ and $S_0 = S(\lambda = 0)$.

Applying the rules of integration [12] over the fermionic and bosonic variables (those which are considered to be irrelevant for the critical behaviour), one obtains the effective "action" of the model:

$$S_{eff}\{A(\tau)\} = -N/2 \int_0^{\beta} d\tau [\omega^2 A^2(\tau) + (\partial_\tau A)^2] + N\ \mathrm{Tr}_\tau \left[1 - \lambda^2 G_1 A G_2 A \right] \quad (4)$$

In the equation above,

$$A(\tau) = \frac{i}{\sqrt{2\omega N}} [a(\tau) - a^{+}(\tau)] \quad (5)$$

is the relevant variable which will be associated to the order parameter, $G_{1,2}(\tau - \tau')$ are the fermionic Green functions and Tr_τ indicates the integration over all the "quasiclassical trajectories" described by the continuous parameter τ ($\tau \in [0, \beta]$).

If one works in the class of functions independent of the Matsubara time τ, the critical properties of a "static" model are obtained. As a result, we mention here only the equation for the critical temperature of the phase transition, expressed in terms of the concentration of the atomic system $n = N/V$, namely:

$$k_B T_c = \frac{\varepsilon}{2} \left[\ln \frac{1 + n_c/n}{1 - n_c/n} \right]^{-1} \quad (6)$$

where $n_c = (\varepsilon_0 \varepsilon)/(2D_{12}^2)$ has the significance of the critical concentration for which an equilibrium superradiant state can be achieved.

In ref. [13], the maximum value of the critical region was calculated for a "static" model and some conclusions about the effect of fluctuations were drawn.

A "dynamic" model, in which the quantum fluctuations of the field variables are included [14,16], has to be considered for low temperatures (for instance, in the case of a zero temperature cavity, if the condition upon the concentration is satisfied and the ordered superradiant state is obtained).

Some critical results of the dynamic superradiant model are already discussed in ref. [16]. It is important to remember here the 1-dimensional behaviour of this one, in the absence of propagation phenomena.

In order to calculate the critical region, we shall use here the thermodynamic results upon the specific heat and the definition of the size of the critical region [1], as:

$$(\Delta T)_{cr} = T_c \zeta_T \quad (7)$$

where the parameter ζ_T appears in the ratio between the singular term (introduced by fluctuations) in the specific heat C_s, and the jump of the same quantity ΔC (calculated at $T = T_c$), obtained in a mean field approach. So,

$$\frac{C_s}{\Delta C} = \left[\frac{\zeta_T}{|T - T_c|/T_c} \right]^{3/2} \quad (8)$$

for a 1-dimensional model; we intend to identify the parameter ζ_T, from a direct calculation of the specific heat.

We are using the results of ref. [16], where the separate contributions of the static mode (η) and of the dynamic ones ($b(\omega_n)$) were pointed in the equation of the effective action

(the new variables η and $b(\omega_n)$ are the Fourier transforms of the field variables $A(\tau)$). So,

$$S(\eta,\{b(\omega_n)\}) = -N\beta\omega^2\eta^2/2 + 2N\ ln\ ch(\beta\omega_F/2) - N/2\sum_{n\neq 0}[\omega^2+\omega_n^2+2\lambda^2/\beta\ K(\omega_n;\beta)]b(\omega_n)b(-\omega_n)+\ldots \quad (9)$$

where $\omega_n = 2\pi n/\beta$ are the bosonic Matsubara frequencies ($n \in Z$), $K(\omega_n;\beta)$ can be calculated as a sum over the fermionic Matsubara frequencies and $\omega_F = [(\varepsilon/2)^2+\lambda^2\eta^2]^{1/2}$.

The expansion of the effective action (eq.9) up to the second order in the coupling constant λ and around the mean field values of the variables (η and $\{b(\omega_n)\}$) would correspond to a Ginzburg-Landau theory. In this way, both types of gaussian terms in the functional can be integrated (eq.2) and the partition function Z is obtained for both sides of the critical point.

In consequence, the thermodynamic potential will contain several contributions:

- a mean field term which is proportional to the number of atoms N
- nonsingular gaussian contributions from the higher modes ($\omega_n \neq 0$, $n>0$) which become negligible in the thermodynamic limit
- singular contribution from the lower modes ($\omega_n \geq 0$) which proceeds from the neighbourhood of the static mode ($n=0$ and $n\neq 0$, but small ω_n). This last contribution is also negligible in the thermodynamic limit and for $T \neq T_c$, but becomes important in a certain vicinity of the critical temperature, that is inside the critical region, even for large value of N.

In ref. [16], the equations for the specific heat were also obtained and the same singular contributions (up to a numerical factor) are shown, namely:

$$C_s = |T-T_c|^{-3/2}\lambda/(16\sqrt{2}k_B)\ ch^{-1}z \quad (10)$$

where $z = \beta_c\varepsilon/4$.

The critical exponent of the specific heat $\alpha=3/2$ is connected with the Josephson rule for a $d=1$ model, namely:

$$d\nu = 1-\alpha \quad (11)$$

where $\nu=1/2$ is the critical exponent of the correlation length.

The mean field terms in the specific heat show a jump, already calculated by many authors. We shall use the result obtained in ref. [13]:

$$\Delta C = N\varepsilon^3\{32k_B^2T_c^3ch^4z\ [thz-z/ch^2z]\}^{-1} \quad (12)$$

In comparison with the eq.12, the contribution expressed in eq.10 shows a nonthermodynamical, but singular behaviour when $T\to T_c$.

Now we can apply the equation (8), using also eqs. (10) and (12), obtaining, up to a unit numerical factor:

$$\zeta_T = \frac{\lambda^{2/3}}{N^{2/3}\varepsilon^2}k_BT_c\ ch^2z\ [thz-z/ch^2z]^{2/3} \quad (13)$$

with z already defined.

The experimental conditions in which a superradiant state are optimally obtained [15], that is in a low-temperature resonant cavity ($T_{cav}\cong 0K$) and only for values of the atomic concentration exceeding a critical one, suggest that a better parameter to control the critical properties of the model is the ratio $x=n_c/n$ (already pointed in eq.6).

So, we shall express the critical region $(\Delta T)_{cr}$ as a function of x, as:

$$(\Delta T)_{cr} = \frac{\varepsilon}{k_B}(xN^2)^{-1/3}[ln\frac{1+x}{1-x}]^{-2}(1-x^2)^{-1}[x-\frac{1-x^2}{2}ln\frac{1+x}{1-x}]^{2/3} \quad (14)$$

where $x \in [0,1]$.

If the reduced critical temperature k_BT_c/ε depends only of the control parameter x (continuous curve on Fig. 1), the reduced critical interval $k_B(\Delta T)_{cr}/\varepsilon$ is also a function of the number of atoms N (discontinuous curves on Fig. 1). For larger N the critical region is more and more decreased.

Even if the number of atoms is not sufficiently large, the critical region is much smaller than the critical temperature, for $x\to 0$. Instead, for $x\to 1$, that is in the neighbourhood of the critical concentration (for which the critical temperature itself tends to zero), the value of the critical region remains very large.

Approximate equations for the critical interval in both limits are obtained ($N_c = Vn_c$):

$$(\Delta T)_{cr} \sim (N_cN)^{-1/3} \quad , \quad for x\to 0 \quad (15)$$

and

$$(\Delta T)_{cr} \sim N^{1/3}/(N-N_c) \quad , \quad for x\to 1 \quad (16)$$

A preliminary conclusion based on a briefly analysis of these results, compared with those of ref. [13], is that the

critical properties of the superradiant 1-dimensional model (the dynamic approach) are less sensitive to the effect of fluctuations than the static model, especially in the limit of large atomic concentrations. The fluctuations continue to be important in the neighbourhood of the critical concentration.

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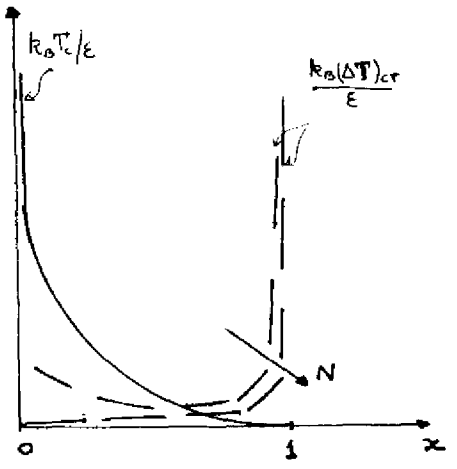

Fig. 1. The critical temperature (continuous curve) and the critical interval (discontinuous curves) represented as a function of the parameter $x=n_c/n$.