IC/92/17

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

SO(2,ℂ) INVARIANT RING STRUCTURE OF BRST COHOMOLOGY AND SINGULAR VECTORS IN 2D GRAVITY WITH C<1 MATTER

N. Chair

V.K. Dobrev

and

H. Kanno

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

MIRAMARE-TRIESTE

IC/92/17

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

SO(2,C) INVARIANT RING STRUCTURE OF BRST COHOMOLOGY AND SINGULAR VECTORS IN 2D GRAVITY WITH C < 1 MATTER

N. Chair, V.K. Dobrev * and H. Kanno
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We consider BRST quantized 2D gravity coupled to conformal matter with arbitrary central charge $c^M = c(p,q) < 1$ in the conformal gauge. We apply a Lian-Zuckerman $SO(2,\mathbb{C})$ ((p,q) - dependent) rotation to Witten's $c^M = 1$ chiral ground ring. We show that the ring structure generated by the (relative BRST cohomology) discrete states in the (matter $\otimes$ Liouville $\otimes$ ghosts) Fock module may be obtained by this rotation. We give also explicit formulae for the discrete states. For some of them we use new formulae for $c < 1$ Fock modules singular vectors which we present in terms of Schur polynomials generalizing the $c = 1$ expressions of Goldstone, while the rest of the discrete states we obtain by finding the proper $SO(2,\mathbb{C})$ rotation. Our formulae give the extra physical states (arising from the relative BRST cohomology) on the boundaries of the $p \times q$ rectangles of the conformal lattice and thus all such states in $(1,q)$ or $(p,1)$ models.

MIRAMARE – TRIESTE

January 1992

* Permanent address: Bulgarian Academy of Sciences, Institute of Nuclear Research and Nuclear Energy, 72 Boul. Trakia, 1784 Sofia, Bulgaria.

Introduction.

Recently the continuum approach to 2D gravity [1-4] has received a lot of attention. In particular, there was progress in the computation of the BRST cohomological extra discrete states of 2D gravity coupled to $c^M \leq 1$ conformal matter in the conformal gauge [5-14]. (For the calculation of correlators and fusion rules see [15].) More recently Witten [12] made the observation that for $c^M = 1$ the discrete states of ghost number (-1), parametrized by two negative integers (r,s), give rise to a polynomial ring of ghost number zero operators. This ring is generated by two elements which correspond to the two cohomology states with (r,s) = (-1,-2), (-2,-1). The states of ghost number zero give rise to spin one currents which act as derivatives of this ring and obey the W_∞ - algebra of area preserving diffeomorphisms.

Not much is known about this ring for $c^M < 1$. (Some conjectures are made in [16] in the case of minimal conformal matter.) This is the first question we address in this letter. We use the $SO(2,\mathbb{C})$ rotation trick [6,17,10] to change $c^M = 1$. The operators of BRST cohomologies for $c^M < 1$ are obtained by applying the $SO(2,\mathbb{C})$ to those for $c^M = 1$. We observe that the (chiral) ring structure, defined by the short distance behaviour of operator product, is preserved by this rotation. This includes also the spin one currents which again fall in $SU(2)$ multiplets and generate a W_∞ - algebra. This is done in Section 1.

The second question we address is the construction of explicit representatives of the relative BRST cohomology for $c^M < 1$. In the case $c^M = 1$ prominent role was played [7,8,13,11,14] by the singular vectors of Fock spaces [18]. We try to generalize these formulae, in particular, for the ghost number zero states. We generalize the Goldstone formula for Fock singular vectors for the cases $(r,s) = (1,s),(2,s),(r,1),(r,2),(3,3)$ and arbitrary central charge, and we give a conjecture for the general case. The singular vectors are given as linear combinations of Schur polynomials $S_{k_1,\ldots,k_n}$, ($n = r$ or $n = s$), of all (in general) ordered partitions $k_1 \geq \ldots \geq k_n \geq 0$, $k_1 + \ldots + k_n = rs$, in contrast to the $c = 1$ case in which $k_1 = \cdots = k_r = s$, or $r \longleftrightarrow s$. Our formulae for the discrete states have direct application for the **(1,q) models**, (or for the $(p,1)$ models). In the general case our formulae are valid for (primary) fields on the boundaries of the Kac table. All this is presented in Section 2., while compact formulae for the action of Virasoro generators on the Schur polynomials (needed for the derivation of singular vectors) are derived in the Appendix.

1. SO(2,C) invariance of the ring structure of the discrete states

1.1. We can consider the BRST cohomology of the Virasoro algebra $H^*(Vir,\mathcal{M})$ for any Virasoro module $\mathcal{M}$ with the energy momentum tensor $T(z)$ of the central charge $c = 26$. The BRST operator

$$d = \int \frac{dz}{2\pi i} : (T(z) + \frac{1}{2}T^G(z))c(z) : \tag{1.1}$$

acts on $\mathcal{M} \otimes \mathcal{F}^G$, where $\mathcal{F}^G$ is a fermion Fock module of (b,c) ghosts with the energy momentum tensor $T^G(z)$. We will be concerned with the case in which $\mathcal{M}$ is a tensor product of two Feigin-

Fuchs modules, $\mathcal{M} = \mathcal{F}^M \otimes \mathcal{F}^L$. In the free field realization of 2D quantum gravity coupled to conformal matter, $\mathcal{F}^M$ corresponds to the matter and $\mathcal{F}^L$ represents the Liouville field. We will distinguish between the quantities for matter and the Liouville fields by using the superscript M and L, respectively.

Let us recall that the Feigin-Fuchs module $\mathcal{F}$ is a boson Fock module of 2D scalar field $\phi(z)$. $\mathcal{F}$ is characterized by the two parameters: the background charge ξ and the momentum p of the vacuum. The energy momentum tensor

$$T(z) = -\frac{1}{2} : \partial\phi(z)\partial\phi(z) : + i\xi\partial^2\phi((z) \tag{1.2}$$

defines an action of Virasoro algebra. The central charge c and the conformal weight h of the vacuum are given by

$$c = 1 - 12\xi^2 , \qquad h = \frac{1}{2}p(p - 2\xi) = \frac{1}{2}(\eta + \xi)(\eta - \xi) . \tag{1.3}$$

Here we have introduced the "shifted" momentum $\eta = p - \xi$, which will be used in the following instead of the "unshifted" momentum p. We will denote the Feigin-Fuchs module with the parameters ξ and η, $\mathcal{F}(\xi, \eta)$.

The BRST cohomology of the Virasoro algebra for $\mathcal{M} = \mathcal{F}^M(\xi^M, \eta^M) \otimes \mathcal{F}^L(\xi^L, \eta^L)$ was studied in [5,6,11,14]. Let us fix the matter central charge $c^M = 1 - 12(\xi^M)^2$. The condition $c^M + c^L = 26$ gives

$$\xi_+\xi_- = -1 , \tag{1.4}$$

where we have defined the light-cone combination

$$\begin{aligned} \xi_\pm &= \frac{1}{\sqrt{2}}(\xi^M \pm i\xi^L) , \\ \eta^\pm &= \frac{1}{\sqrt{2}}(\eta^M \pm i\eta^L) . \end{aligned} \tag{1.5}$$

ξ^M and ξ^L are fixed up to sign. In terms of two (real) parameters r and s defined by

$$r = -\xi_-\eta^+ , \qquad s = -\xi_+\eta^- . \tag{1.6}$$

one can parametrize the momenta in the following way:

$$\begin{aligned} \eta^M(r,s) &= \frac{1}{2}\left[(r+s)\xi^M + (r-s)i\xi^L\right] , \\ i\eta^L(r,s) &= \frac{1}{2}\left[(r-s)\xi^M + (r+s)i\xi^L\right] . \end{aligned} \tag{1.7}$$

Then the result of [11] (for $c^M = 1$ see [6]) is summarized as follows:

Theorem. The relative cohomology $H^*_{rel}(Vir, \mathcal{M})$ is non-trivial only in the following cases.

(i) If $rs = 0$, then $H^*_{rel} = \delta_{n,0}\,\mathbb{C}$.

(ii) If both r and s are positive integers, then $H^*_{rel} = \delta_{n,0}\,\mathbb{C} \oplus \delta_{n,1}\,\mathbb{C}$.

(iii) If both r and s are negative integers, then $H^*_{rel} = \delta_{n,0}\,\mathbb{C} \oplus \delta_{n,-1}\,\mathbb{C}$.

In all cases the non-trivial cohomology states appear at level rs.

The gravitationally dressed primary states of [3,4] are included in the case (i). The cases (ii) and (iii) give the "extra" physical states. (In fact there is a doubling of the physical states since finally one has to take the absolute cohomology into account (cf. [11]).) Especially, when $c^M = 1$ ($\xi^M = 0$), the physical states with ghost number (-1) are basic ingredients for the construction of the chiral ground ring by Witten [12]. At the (discrete) momenta $\eta^M = \pm(r-s)/\sqrt{2}$, $\eta^L = \mp i(r+s)/\sqrt{2}$ with r and s being negative integers, there is an operator $\mathcal{O}_{r,s}$ corresponding to the BRST cohomology class of ghost number (-1). $\mathcal{O}_{r,s}$ has conformal spin zero and ghost number zero, due to a shift of quantum numbers corresponding to the ghost $c(z)$ in the transition from states to operators. By this crucial property the short distance behavior of the product $\mathcal{O}_{r,s}(z)\mathcal{O}_{r',s'}(w)$ defines a ring structure of operators $\mathcal{O}_{r,s}$. Witten has proved that this ring is the polynomial algebra generated by $x = \mathcal{O}_{-2,-1}$ and $y = \mathcal{O}_{-1,-2}$ and that we can identify $\mathcal{O}_{r,s}$ with $x^{-r-1}\,y^{-s-1}$. Hence the ring structure is given by

$$\mathcal{O}_{r,s}\mathcal{O}_{r',s'} = \mathcal{O}_{r+r'+1,s+s'+1} . \tag{1.8}$$

(Note that the indeces are negative integers.) Furthermore, at the same momenta as $\mathcal{O}_{r,s}$ we have a current $W_{r,s}(z)$ with ghost number zero. $W_{r,s}(z)$ is also an extra BRST cohomology class (see the case (iii) above.). $W_{r,s}(z)$ acts on the ground ring as a derivation. The action is defined again in terms of the operator product expansion (OPE):

$$W_{r,s}(z)\mathcal{O}_{r',s'}(w) \sim \frac{\mathcal{O}_{r+r'+1,s+s'+1}(w)}{z-w} + \cdots , \tag{1.9}$$

where dots represent BRST exact terms. From the viewpoint of $SU(2)$ symmetry it is natural to include the tachyonic currents $W_{0,s}$ and $W_{r,0}$. The relation (1.9) is also valid for $r = 0$ or $s = 0$. Finally the symmetry algebra of the ground ring is identified with W_∞ - algebra of area preserving diffeomorphisms [19]:

$$W_{r,s}(z)W_{r',s'}(w) \sim \frac{W_{r+r'+1,s+s'+1}(w)}{z-w} . \tag{1.10}$$

1.2. The existence of extra BRST cohomologies at discrete values of momenta is independent of the value of background charges for Feigin-Fuchs modules, as long as the constraint (1.4) is satisfied. Therefore, one can obtain the operators $\mathcal{O}_{r,s}$ and the currents $W_{r,s}$ for any value of c^M in the same manner as described above. To investigate the OPE relations of the types (1.8) $\sim$ (1.10), we will make use of the $SO(2,\mathbb{C})$ rotation of the form

$$R(m,n) = \frac{1}{2\sqrt{mn}}\begin{pmatrix} m+n & i(m-n) \\ -i(m-n) & m+n \end{pmatrix} . \tag{1.11}$$

Any pair of background charges (ξ^M, ξ^L) satisfying the condition (1.4) can be obtained by making an appropriate $SO(2, \mathbb{C})$ rotation from the "reference" charges $(0, \pm i\sqrt{2})$ for $c^M = 1$ case. For example, the background charges corresponding to (p, q) model are

$$\vec{\xi} = \begin{pmatrix} \xi^M \\ \xi^L \end{pmatrix} = R(q,p) \begin{pmatrix} 0 \\ \pm i\sqrt{2} \end{pmatrix} = \begin{pmatrix} \frac{\pm(p-q)}{\sqrt{2pq}} \\ \frac{\pm i(p+q)}{\sqrt{2pq}} \end{pmatrix} . \tag{1.12}$$

In the following we will use a vector notation like $\vec{\xi} = (\xi^M, \xi^L)^t$ for a pair of quantities for matter and the Liouville systems. The discrete momenta where we have extra physical states are related to the background charges by (1.7) with *integers* r and s. It is quite remarkable that this relation is decomposed into the product of rotation and scaling: $\vec{\eta}(r,s) = \sqrt{rs}\, R(r,s)\, \vec{\xi}$, which is valid for any value of $\vec{\xi}$. The discrete momenta $\vec{\eta}(r,s)$ are obtained from those for $c^M = 1$ case (the "reference" momenta) by the *same* rotation as (1.12):

$$\vec{\eta}(r,s) = \sqrt{rs} R(r,s) R(q,p) \begin{pmatrix} 0 \\ \pm i\sqrt{2} \end{pmatrix} = R(q,p) \begin{pmatrix} \pm\frac{1}{\sqrt{2}}(s-r) \\ \pm\frac{i}{\sqrt{2}}(r+s) \end{pmatrix} . \tag{1.13}$$

($\vec{p} = \vec{\eta} + \vec{\xi}$ is also rotated by the same $R(q,p)$.)

The operators $\mathcal{O}_{r,s}$ and the currents $W_{r,s}$ arising from BRST cohomology for $\mathcal{M} = \mathcal{F}^M \otimes \mathcal{F}^L$ have dependence on the characterestic parameters $\vec{\xi}$ and $\vec{\eta}$ of $\mathcal{M}$,

$$\mathcal{O}_{r,s} = \mathcal{O}_{r,s}(\vec{\xi}, \vec{\eta}), \qquad W_{r,s} = W_{r,s}(\vec{\xi}, \vec{\eta}) . \tag{1.14}$$

This dependence is universal in the sense that the way of constructing BRST physical states is independent of $\vec{\xi}$ with $\vec{\xi}$ and $\vec{\eta}$ treated as free parameters. Let us define an action of $R(m,n)$ by

$$R(m,n) \cdot \mathcal{O}_{r,s}(\vec{\xi}, \vec{\eta}) = \mathcal{O}_{r,s}(R(m,n)\vec{\xi}, R(m,n)\vec{\eta}) , \tag{1.15}$$

and the same for $W_{r,s}$. Then the universality stated above implies

$$\mathcal{O}^{(p,q)}_{r,s} = R(p,q) \cdot \mathcal{O}^{(1,1)}_{r,s} , \qquad W^{(p,q)}_{r,s} = R(p,q) \cdot W^{(1,1)}_{r,s} , \tag{1.16}$$

where the superscript (p,q) specifies the model. We will show some examples later on. In the calculation of OPE, two point function for free scalar fields $\phi^I(z)$ $(I = M, L)$:

$$\langle \phi^I(z)\, \phi^J(w) \rangle = -\delta^{IJ} \ln(z-w) \tag{1.17}$$

is used for contractions. Since two point function (1.17) is invariant under $SO(2, \mathbb{C})$ rotation,we can conclude that the coefficient of OPE are independent of (p,q), which means, for example,

$$\begin{aligned} W^{(p,q)}_{r,s}(z)\mathcal{O}^{(p,q)}_{r',s'}(w) &= R(p,q) \cdot W^{(1,1)}_{r,s}(z) R(p,q) \cdot \mathcal{O}^{(1,1)}_{r',s'}(w) = \\ &= R(p,q) \cdot \left(W^{(1,1)}_{r,s}(z)\mathcal{O}^{(1,1)}_{r',s'}(w)\right) \sim \\ &\sim \frac{1}{z-w} R(p,q) \cdot \mathcal{O}^{(1,1)}_{r+r'+1,s+s'+1}(w) = \\ &= \frac{1}{z-w} \mathcal{O}^{(p,q)}_{r+r'+1,s+s'+1}(w) . \end{aligned} \tag{1.18}$$

Thus, $SO(2, \mathbb{C})$ rotation preserves the OPE structure of BRST cohomology classes of types (1.8) ~ (1.10). The whole algebraic structure appearing in the $c^M = 1$ case remains true for any (p,q). For example, the algebra of $\mathcal{O}^{(p,q)}_{r,s}$'s is identified with the polynomial ring generated by $x^{(p,q)} = \mathcal{O}^{(p,q)}_{-2,-1}$ and $y^{(p,q)} = \mathcal{O}^{(p,q)}_{-1,-2}$. The currents $W^{(p,q)}_{r,s}$ acts on this polynomial ring as derivations like $W^{(p,q)}_{0,-1} = \partial/\partial x^{(p,q)}$ and $W^{(p,q)}_{-1,0} = \partial/\partial y^{(p,q)}$.

1.3. We show a few examples. The generators $x^{(p,q)}$ and $y^{(p,q)}$ are operators corresponding to the BRST cohomology class with ghost number -1 at level 2:

$$\left[b_{-2} - \frac{\xi_-}{\sqrt{2}}(\phi^M_{-1} + i\phi^L_{-1})b_{-1}\right]|\vec{\eta}(-2,-1)\rangle , \tag{1.19a}$$

and

$$\left[b_{-2} - \frac{\xi_+}{\sqrt{2}}(\phi^M_{-1} - i\phi^L_{-1})b_{-1}\right]|\vec{\eta}(-1,-2)\rangle , \tag{1.19b}$$

respectively. Noting a shift of momenta in going to operators, we obtain

$$\begin{aligned} x^{(p,q)} &= \left[cb - \frac{\xi_-}{\sqrt{2}} i(\partial\phi^M + i\partial\phi^L)\right] \exp -i\frac{\xi_+}{\sqrt{2}}(\phi^M - i\phi^L) , \\ y^{(p,q)} &= \left[cb - \frac{\xi_+}{\sqrt{2}} i(\partial\phi^M - i\partial\phi^L)\right] \exp -i\frac{\xi_-}{\sqrt{2}}(\phi^M + i\phi^L) . \end{aligned} \tag{1.20}$$

The dependence on (p,q) only appears in the parameters $\xi_\pm$ on which $SO(2, \mathbb{C})$ acts as scale transformation. Our next example is $SO(2, \mathbb{C})$ rotated $SU(2)$ current algebra. In terms of a combination

$$\Phi^{(p,q)} = \frac{\xi_+}{\sqrt{2}}(\partial\phi^M - i\partial\phi^L) - \frac{\xi_-}{\sqrt{2}}(\partial\phi^M + i\partial\phi^L) \tag{1.21}$$

the generators of $SU(2)$ current algebra is given by

$$\begin{aligned} T_+ &= W^{(p,q)}_{0,-2} = \exp i\Phi^{(p,q)} , \\ T_3 &= W^{(p,q)}_{-1,-1} = i\partial\Phi^{(p,q)} , \\ T_- &= W^{(p,q)}_{-2,0} = \exp -i\Phi^{(p,q)} . \end{aligned} \tag{1.22}$$

The $SU(2)$ currents acts on the polynomial ring in $x^{(p,q)}$ and $y^{(p,q)}$ as derivations:

$$\begin{aligned} T_+ &= x^{(p,q)} \frac{\partial}{y^{(p,q)}} , \\ T_3 &= \frac{1}{2}\left(x^{(p,q)}\frac{\partial}{\partial y^{(p,q)}} - y^{(p,q)}\frac{\partial}{\partial x^{(p,q)}}\right) , \\ T_- &= y^{(p,q)} \frac{\partial}{\partial x^{(p,q)}} . \end{aligned} \tag{1.23}$$

The currents $W^{(p,q)}_{r,s}$ $(r+s=-n)$ constitutes an $SU(2)$ multiplet of spin $\frac{n}{2}$. The following is an example of spin $\frac{3}{2}$ multiplet:

$$
\begin{aligned}
W^{(p,q)}_{-3,0} &= \exp\frac{i}{\sqrt{2}}\Big(\xi_-(\phi^M + i\phi^L) - 2(\xi_+(\phi^M - i\phi^L)\Big)\,,\\
W^{(p,q)}_{-2,-1} &= \frac{1}{2}(T_3^2 - \partial T_3)\exp -\frac{i}{\sqrt{2}}\xi_+(\phi^M - i\phi^L)\,,\\
W^{(p,q)}_{-1,-2} &= \frac{1}{2}(T_3^2 + \partial T_3)\exp \frac{-i}{\sqrt{2}}\xi_-(\phi^M + i\phi^L)\,,\\
W^{(p,q)}_{0,-3} &= \exp\frac{i}{\sqrt{2}}\Big(\xi_+(\phi^M - i\phi^L) - 2(\xi_-(\phi^M + i\phi^L)\Big)\,.
\end{aligned}
\tag{1.24}
$$

With these examples at lower levels it is easy to check the OPE structure of type (1.8) $\sim$ (1.10) explicitly. The conditions necessary for this calculation are the constraint (1.4) and the fundamental two point functions (1.17).

2. Discrete states in (p,q) models and Fock modules singular vectors

2.1. In this Section we give explicit representatives of the discrete states of $H^{*}_{rel}(Vir, \mathcal{M})$ for arbitrary (p,q) models. We start with the ghost number zero case. For $c^M = 1$ it is well known [7,8,13,11,14] that explicit formulae for the physical states are provided by the singular vectors of the matter Fock spaces expressed in terms of the so called Schur polynomials which are recalled in the Appendix.

Let ξ^M be given as in (1.12) with $p,q \in \mathbb{N}$. Then the Fock module $\mathcal{F}^M(\xi^M, \eta^M)$ is reducible [20] iff η^M is given by (1.7) with $r,s \in \mathbb{Z}, rs > 0$.

Let us consider first the case $r,s \in \mathbb{N}$. For $c^M = 1$ the singular vector of level rs is given by [18] (see also [11,14] whose notation we partly follow):

$$
v^{c=1}_{r,s} = S_{\underbrace{s,\ldots,s}_{r}}\left(\frac{\sqrt{2}}{n}\eta^M_{-n}\right)|\eta^M\rangle\,,
\tag{2.1a}
$$

or, equivalently, by

$$
v^{c=1}_{r,s} = S_{\underbrace{r,\ldots,r}_{s}}\left(-\frac{\sqrt{2}}{n}\eta^M_{-n}\right)|\eta^M\rangle\,,
\tag{2.1b}
$$

the two expressions differing by sign. We recall that a singular vector of a Virasoro highest weight module (here Fock modules) is a weight vector v different from the highest weight vector and obeying $L_n v = 0$, for $n \geq 1$, where L_n are the Virasoro generators. Let us note that the Schur polynomials involved are exactly of length r, (resp. s), i.e., each term involves the product of exactly r, (resp. s), elementary Schur polynomials.

We would like to generalize these formulae for $c < 1$. For $r = 1$ or $s = 1$ one may check that the singular vectors are given by

$$
v_{1,s} = S_s\left(\frac{\sqrt{2}}{n}\xi_+\eta^M_{-n}\right)|\eta^M\rangle\,,\quad r = 1\,,
\tag{2.2a}
$$

$$
v_{r,1} = S_r\left(\frac{\sqrt{2}}{n}\xi_-\eta^M_{-n}\right)|\eta^M\rangle\,,\quad s = 1\,,
\tag{2.2b}
$$

which reduce to (2.1) for $c = 1, p = q = 1, \xi_\pm = \pm 1$. The derivation of (2.2) and the formulae, necessary for the derivation of the singular vectors given below, are given in the Appendix.

The cases $r, s > 1$ are much more interesting. In particular, for $r = 2$ or $s = 2$ we have:

$$
v_{2,s} = \sum_{k=0}^{s}\beta^s_k(\xi_+)\,S_{2s-k,k}\left(\frac{\sqrt{2}}{n}\xi_+\eta^M_{-n}\right)|\eta^M\rangle\,,\quad r = 2\,,
\tag{2.3a}
$$

$$
v_{r,2} = \sum_{k=0}^{r}\beta^r_k(\xi_-)\,S_{2r-k,k}\left(\frac{\sqrt{2}}{n}\xi_-\eta^M_{-n}\right)|\eta^M\rangle\,;\quad s = 2\,,
\tag{2.3b}
$$

$$
\beta^n_k(\xi_\pm) = \frac{(-1)^k\Gamma(2(n+u_\pm))}{\Gamma(2n-k+u_\pm+1)\Gamma(k+u_\pm)} = (-1)^k\binom{2(n+u_\pm)-1}{k+u_\pm-1}\,,
\tag{2.3c}
$$

where $u_\pm = \gamma_\pm(\eta^M - \xi^M) - 1 = \xi^2_\pm - n$, (cf. the Appendix), and only the expression in terms of Γ - functions can be used if $u_\pm$ is not integer.

Our first observation is that for $c < 1$ and $r = 2$, (resp. $s = 2$), all Schur polynomials of degree $2n$ and of length ≤ 2 are involved, since the term with $k = 0$ involves the elementary Schur polynomial $S_{2n} = S_{2n,0}$. Of course, there are partial cases when not all terms are present in (2.3a,b). Thus for $p = 1$, resp. $q = 1$, we have $u_+ = q - s$ for (2.3a), resp. $u_- = p - r$ for (2.3b), and $\beta^n_k(\xi_\pm) = 0$ for $k < 1 - u_\pm$. Finally, for $c = 1, p = q = 1$ we have $\beta^n_k(\pm 1) = 0$ for $k < n$ and (2.3a,b) collapse (up to signs) to (2.1a,b). Note that $\beta^n_n \neq 0$ in all cases.

Let us give also the example of a singular vector in the case $r = s = 3$:

$$
v_{3,3} = \sum_{9-j-s\geq j\geq s\geq 0}\beta^9_{j,s}(\xi_+)\,S_{9-j-s,j,s}\left(\frac{\sqrt{2}}{n}\xi_+\eta^M_{-n}\right)|\eta^M\rangle\,,
\tag{2.4a}
$$

$$
\beta^9_{j,s}(\xi_+) = \frac{(-1)^{j+s}\beta_{j,s}}{\Gamma(10-j-s+u)\Gamma(j+u)\Gamma(s+u-1)u}\,,
\tag{2.4b}
$$

$$
\begin{aligned}
&\beta_{0,0} = \beta_{1,0} = (u-3)(u+8)\,,\ \beta_{1,1} = \beta_{2,1} = (u-1)(u+6)\,,\\
&\beta_{4,0} = \beta_{4,1} = -(u+2)(u+3)\,,\ \beta_{2,2} = \beta_{3,2} = \beta_{3,3} = u(u+5)\,,\\
&\beta_{2,0} = -18\,,\ \beta_{3,1} = -6\,,\ \beta_{3,0} = -(u^2+5u+12)\,.
\end{aligned}
\tag{2.4c}
$$

where $u = u_+ = \gamma_+(\eta^M(3,3) - \xi^M) - 1 = 2\xi^2_+ - 3 = 2q/p - 3 > -3$, (cf. the Appendix). Note that $\beta^9_{j,s}$ have no poles (even for $u = 0$). For $c^M = 1, p = q = 1 = -u$, all coefficients are zero except $\beta^9_{3,3}$ and (2.4) goes into (2.1). For $u = -2, 0, 1, 3$ there are also some vanishing coeffiecients (always $\beta^9_{0,0}$). Note that $\beta^9_{3,3} \neq 0$ in all cases. The same expression is valid if we replace $\xi_+ \longrightarrow \xi_-$, $u = u_+ \longrightarrow u_- = 2\xi^2_- - 3 = 2p/q - 3$.

Thus we are lead to the conjecture that the general expression for the singular vectors should be:

$$v_{r,s} = \sum_{\substack{k_1 \geq \dots \geq k_r \geq 0 \\ k_1+\dots+k_r = rs}} \frac{(-1)^{k_1} \beta_{k_1,\dots,k_r}(\xi_+)}{\Gamma(k_1+u_++1)\dots\Gamma(k_r+u_+-r+2)} S_{k_1,\dots,k_r}\left(\frac{\sqrt{2}}{n}\xi_+\eta^M_{-n}\right)|\eta^M\rangle \quad (2.5a)$$

or, up to sign,

$$v_{r,s} = \sum_{\substack{k_1 \geq \dots \geq k_s \geq 0 \\ k_1+\dots+k_s = rs}} \frac{(-1)^{k_1} \beta_{k_1,\dots,k_s}(\xi_-)}{\Gamma(k_1+u_-+1)\dots\Gamma(k_s+u_--s+2)} S_{k_1,\dots,k_s}\left(\frac{\sqrt{2}}{n}\xi_-\eta^M_{-n}\right)|\eta^M\rangle \quad (2.5b)$$

where $u_\pm = \gamma_\pm(\eta^M(r,s) - \xi^M) - 1 = \sqrt{2}\xi_\pm(\eta^M(r,s) - \xi^M) - 1$, (cf. the Appendix), and the coefficients obey a simple recursion relation (from the action of L_1):

$$\beta_{k_1,\dots,k_n} = \sum_{j=2}^{n} \beta_{k_1-1,\dots,k_j+1,\dots,k_n}, \quad (2.5c)$$

and a more complicated one (from L_2) which we omit for the lack of space. Note that in (2.5a), (resp. (2.5b)), $s \leq k_1 \leq rs$, (resp. $r \leq k_1 \leq rs$), $0 \leq k_r \leq s$, (resp. $0 \leq k_s \leq r$), $\beta_{s,\dots,s} \neq 0$, $\beta_{r,\dots,r} \neq 0$. Thus according to our conjecture Schur polynomials of degree rs, and of arbitrary length $\leq r$, (or $\leq s$), are involved, i.e., every term involves the product of at most r, (resp. s), elementary Schur polynomials.

The **Proof** that the states given in (2.2) - (2.5) are representatives of $H^{(0)}_{rel}(Vir,\mathcal{M})$ is analogous to the case $c^M = 1$ [11]. Since these are singular vectors they are BRST closed. Consider (2.5b). It is enough to note that by (2.2b), (2.3b), (2.4b), and our conjecture, the term $S_{r,\dots,r}(\sqrt{2}\xi_-\eta^M_{-n}/n)\ |\eta^M,\eta^L\rangle$ is always present which is the representative for $c^M = 1$ and then we can repeat the proof of [11]. In the last step we are reduced to the term $(\tilde\eta^+_{-r})^s = (\xi_-)^s(\eta^+_{-r})^s$ which is representative of $H^{(0)}_{rel}(Vir,\mathcal{M})$ [11]. Analogously is considered (2.5a).

Let us note that we could have used also the singular vectors of the same degree in the Liouville sector.

Finally, let us remark that representatives for the case $c^M < 1$ may be obtained from (2.1) by the $SO(2,\mathbb{C})$ rotation of [6], however, these would not be singular vectors.

2.2. Next we consider the ghost number zero case with $r,s \in -\mathbb{N}$. The physical importance of these sates comes from the fact that they are associated to the currents $W_{r,s} = W^{(p,q)}_{r,s}$ discussed in Section 1.

First we note that for $c^M = 1, \xi^M = 0$, from (1.6) we have that $\eta^M(r,s) = \eta^M(-s,-r)$. Thus the cohomology representatives are singular vectors $v^{c=1}_{-s,-r}$ as given in (2.1) (cf. also [14]).

For $c^M < 1$ things are rather more complicated with one exception, namely the case

$$r = -kp\,, \quad s = -\ell q\,, \quad k,\ell \in \mathbb{N}. \quad (2.6a)$$

In this case we have (cf. (1.7)):

$$\eta^M(r,s) = \eta^M(p\ell, kq)\,, \quad (2.6b)$$

and the cohomology representatives at level $rs = k\ell pq$ are singular vectors $v_{p\ell,kq}$ from (2.5) (and also by (2.2) - (2.4) whenever applicable). Let us remark that these r,s correspond to corners of the conformal lattice with dimension p,q [21,22,23,20], (see also [24] for an explicit parametrization of the reducible Virasoro highest weight modules).

In the generic situation when (2.6a) is not fulfilled there is no singular vector of degree rs in the matter sector (and also in the Liouville sector). (There is only the so called cosingular vector [20].) Thus we can obtain representatives only by implementing the proper $SO(2,\mathbb{C})$ rotation of [6] as discussed in Section 1. In this case we have the following representatives

$$\psi^{(0)}_{r,s} = \underbrace{S_{-r,\dots,-r}}_{-s}(x)\ |\eta^M,\eta^L\rangle\,, \quad \text{or} \quad (2.7a)$$

$$\psi^{(0)}_{r,s} = \underbrace{S_{-s,\dots,-s}}_{-r}(-x)\ |\eta^M,\eta^L\rangle\,. \quad (2.7b)$$

$$x_n = \frac{i}{n}(\xi^L\eta^M_{-n} - \xi^M\eta^L_{-n})\,. \quad (2.7c)$$

For $c^M = 1, \xi^M = 0, \xi^L = -i\sqrt{2}$, formulae (2.7) go into (2.1).

2.3. Finally we consider the cases of **nonzero ghost number**. In these cases the cohomology representatives are given in terms of elementary Schur polynomials and the formulae may be obtained from [14], Theorem 2.2., by the replacement $\eta^\pm \longrightarrow \tilde\eta^\pm$, for $r,s \in \pm\mathbb{N}$. We mention only the partial cases $|r| = 1$ or $|s| = 1$, since these are simplified in comparison with [14]. We have:

$$\psi^{(1)}_{r,1} = \sum_{j=1}^{r} S_{r-j}\left(-\tilde\eta^+_{-n}/n\right) c_{-j}\ |\eta^M,\eta^L\rangle\,, \quad r \in \mathbb{N}\,, \quad (2.8a)$$

$$\psi^{(1)}_{1,s} = \sum_{j=1}^{s} S_{s-j}\left(-\tilde\eta^-_{-n}/n\right) c_{-j}\ |\eta^M,\eta^L\rangle\,, \quad s \in \mathbb{N}\,; \quad (2.8b)$$

$$\psi^{(-1)}_{r,-1} = \sum_{j=1}^{-r} S_{-r-j}\left(-\tilde\eta^+_{-n}/n\right) b_{-j}\ |\eta^M,\eta^L\rangle\,, \quad r \in -\mathbb{N}\,, \quad (2.9a)$$

$$\psi^{(-1)}_{-1,s} = \sum_{j=1}^{-s} S_{-s-j}\left(-\tilde\eta^-_{-n}/n\right) b_{-j}\ |\eta^M,\eta^L\rangle\,, \quad s \in -\mathbb{N}\,. \quad (2.9b)$$

Formulae (2.9a) for $r = -2$ and (2.9b) for $s = -2$ were given in (1.19) as the states associated to the operators (1.20).

2.4. The states considered above represent the extra physical states for the **(1,q) models** (or equivalently, $(p,1)$ models), since in these cases all (r,s) are on the boundaries of the $(p = 1) \times q$ rectangles of the conformal lattice. (Of course, one has in addition as many states as these taking into account the absolute cohomology.)

This follows from the more general observation that even for arbitrary (p,q) our formulae represent the extra physical states for (r,s) on the corners of the $p \times q$ rectangles of the conformal lattice, i.e, for $r = kp$, $s = \ell q$, $k, \ell \in \mathbb{Z}$. Furthermore, our formulae are applicable also for (r,s) on the boundaries of the of the conformal lattice, i.e, for $r = kp$, $s \neq \ell q$, or $s = \ell q$, $r \neq kp$, $k, \ell \in \mathbb{Z}$. We only have to change $\eta^M \longrightarrow \eta^L$, $u_\pm \longrightarrow \gamma_\pm((\eta^L)^2 - (\xi^L)^2) - 1$ in (2.2) - (2.5).

Our formulae are not directly applicable for r, s in the interiors of rectangles of the conformal lattice, i.e., for $r \neq kp$, $s \neq \ell q$, $k, \ell \in \mathbb{Z}$. The reason is that in these cases the physical states in $L(\xi^M, \eta^M) \otimes \mathcal{F}(\xi^L, \eta^L) \otimes \mathcal{F}^G$, where $L(\xi, \eta)$ is the irreducible highest weight module with highest weight $h = (\eta^2 - \xi^2)/2$, can have arbitrary large ghost number (depending on ξ^L, η^L) [5,11,14]).

Appendix. Action of Virasoro generators on Schur polynomials

Let us recall (cf., e.g., [18]) that the elementary Schur polynomials S_k are defined by the exponential generating function:

$$exp\left(\sum_{k\in\mathbb{N}} t^k x_k\right) = \sum_{k\in\mathbb{Z}} t^k S_k(x) , \quad x = (x_1, x_2, \ldots), \tag{A.1}$$

which implies

$$S_k(x) = 0 , \ k < 0 , \quad S_0(x) = 1 , \tag{A.2a}$$

$$S_k(x) = \sum_{k_1 + 2k_2 + \ldots = k} \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{k_2!} \ldots , \quad k > 0 . \tag{A.2b}$$

For any partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots\}$ one associates a Schur polynomial:

$$S_\lambda(x) = S_{\lambda_1,\lambda_2,\ldots}(x) = \det(S_{\lambda_j+k-j}(x))_{j,k} . \tag{A.3}$$

For a given partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$ with $\lambda_n > 0$ we shall call n the *length* of λ.

The Virasoro generators corresponding to the energy-momentum tensor $T(z)$ are given in terms of the modes η_n as:

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} : \eta_k \eta_{n-k} : -(n+1)\xi\eta_n , \tag{A.4a}$$

where ξ is the background charge and

$$[\eta_k , \eta_n] = k\delta_{k,-n} . \tag{A.4b}$$

The Fock module $\mathcal{F}(\eta, \xi)$ is determined by:

$$\eta_n|\eta\rangle = 0 , \ n > 0 , \quad \eta_0|\eta\rangle = (\eta + \xi)|\eta\rangle , \tag{A.5a}$$

from which follows:

$$L_n|\eta\rangle = 0 , \ n > 0 , \quad L_0|\eta\rangle = h|\eta\rangle , \quad h = \frac{1}{2}(\eta^2 - \xi^2) . \tag{A.5b}$$

From (A.4) follows :

$$[L_n , \eta_k] = -k\eta_{n+k} - n(n+1)\xi\delta_{n,-k} . \tag{A.6}$$

Denote for $k \neq 0$ $x_k = \gamma\eta_{-k}/k$, $\gamma = \gamma_\pm = \sqrt{2}\xi_\pm$, then we have:

$$[L_n , x_k] = (k - n)x_{k-n} + \gamma(\eta - k\xi)\delta_{n,k} , \tag{A.7}$$

where $\eta = \eta_0 - \xi$ anticipating the action of η_0 on $|\eta\rangle$.

For the derivation of the singular vectors we need only the action of L_1 and L_2 (since they generate all L_n , $n \geq 3$). Consider a function $f(x)$ of $x = (x_1, x_2, \ldots)$. Then from (A.7) we obtain at once:

$$[L_1, f(x)] = \sum_{j=1}^{\infty} j x_j \frac{\partial f}{\partial x_{j+1}} + \sqrt{2}\gamma(\eta - \xi)\frac{\partial f}{\partial x_1} , \tag{A.8}$$

and after some manipulation:

$$[L_2, f(x)] = \sum_{j=1}^{\infty} j x_j \frac{\partial f}{\partial x_{j+2}} + \sqrt{2}\gamma(\eta - 2\xi)\frac{\partial f}{\partial x_2} + \gamma^2\frac{\partial^2 f}{\partial x_1^2} + 2\gamma^2\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} . \tag{A.9}$$

The last term in (A.9) indicates that L_2 is not a derivative operators on functions of x. We need the following properties of elementary Schur polynomials [11] :

$$\frac{\partial S_k(x)}{\partial x_n} = S_{n-k} , \quad \sum_{m=j+1}^{k} (m - j)x_{m-j}S_{k-m} = (k - j)S_{k-j} . \tag{A.10}$$

Using these properties we obtain from (A.8), (A.9) the formulae necessary for the derivation of the singular vectors:

$$[L_1, S_k(x)] = (k - 1 + \gamma(\eta - \xi))S_{k-1}(x) , \tag{A.11}$$

$$[L_2, S_k(x)] = (k - 1 + \gamma(\eta - \xi))S_{k-2}(x) + \gamma^2 S_{k-1}\frac{\partial}{\partial x_1} . \tag{A.12}$$

As a direct application of (A.11) and (A.12) let us derive (2.2a). From (1.7) we have for $r = 1$:

$$\gamma(\eta^M - \xi^M) = \gamma_+(\eta^M(1,s) - \xi^M) = 1 - s , \tag{A.13a}$$

and from (A.11) and (A.12) we have :

$$[L_n, S_k] = (k-s)S_{k-n} \Rightarrow L_n S_s(x)|\eta^M\rangle = 0, \quad n=1,2, \tag{A.13}$$

the last equality meaning that $S_s(x)|\eta^M\rangle$ is a singular vector.

For the action of L_n on arbitrary Schur polynomials it is convenient to use the following formulae which are derived easily from (A.11) and (A.12):

$$[L_1, S_{k_1,\dots,k_n}(x)] = \sum_{j=1}^{n}(k_j - j + \gamma(\eta-\xi))\, S_{k_1,\dots,k_j-1,\dots,k_n}(x), \tag{A.14}$$

$$\begin{aligned}[L_2, S_{k_1,\dots,k_n}(x)] = &\sum_{j=1}^{n}(k_j - j + \gamma(\eta-\xi))\, S_{k_1,\dots,k_j-2,\dots,k_n}(x) + \\ &+ (\gamma^2-1)\sum_{1\le i<j\le n} S_{k_1,\dots,k_i-1,\dots,k_j-1,\dots,k_n}(x),\end{aligned} \tag{A.15}$$

where we have omitted the terms with derivatives which would act if we apply L_2 to $S_{k_1,\dots,k_n}(x)\, f(x)$. Note that application of (A.14),(A.15) involves Schur polynomials of unordered partitions which are defined again by (A.3) and we have to use:

$$S_{k_1,\dots,k-1,k,\dots,k_n} = 0, \quad S_{k_1,\dots,k-2,k,\dots,k_n} = -S_{k_1,\dots,k-1,k-1,\dots,k_n}. \tag{A.16}$$

Acknowledgments

The authors would like to thank E. Gava, K.S. Narain and M.H. Sarmadi for discussions. They would like also to thank Professor Abdus Salam for hospitality at the ICTP. V.K.D. was partially supported by the Bulgarian National Foundation for Science, Grant Φ – 11.

References

[1] A.M. Polyakov, Phys. Lett. **103B** (1981) 207; Mod. Phys. Lett. **A2** (1987) 893.

[2] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819.

[3] J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509.

[4] F. David, Mod. Phys. Lett. **A3** (1988) 1651.

[5] B.H. Lian and G.J. Zuckerman, Phys. Lett. **254B** (1991) 417.

[6] B.H. Lian and G.J. Zuckerman, Phys. Lett. **266B** (1991) 21.

[7] A.M. Polyakov, Mod. Phys. Lett. **A6** (1991) 635.

[8] S. Mukherji, S. Mukhi and A. Sen, Phys. Lett. **266B** (1991) 337.

[9] C. Imbimbo, S. Mahapatra and S. Mukhi, preprint INFN Genova & Tata Institute, GEF-TH 8/91 & TIFR/TH/91-27 (May 1991).

[10] K. Itoh, preprint CTP-TAMU-42/91 (June 1991).

[11] P. Bouwknegt, J. McCarthy and K. Pilch, preprints CERN-TH.6162/91 (July 1991) & CERN-TH.6196/91 (August 1991).

[12] E. Witten, preprint IASSNS-HEP-91/51 (August 1991).

[13] I.R. Klebanov and A.M. Polyakov, Mod. Phys. Lett. **A6** (1991) 3273.

[14] P. Bouwknegt, J. McCarthy and K. Pilch, preprint CERN-TH.6279/91 (October 1991).

[15] M. Goulian and M. Li, Phys. Rev. Lett. **66** (1991) 2051; P. Di Francesco and D. Kutasov, Phys. Lett. **261B** (1991) 385; Y. Kitazawa, Phys. Lett. **265B** (1991) 262; N. Sakai and Y. Tanii, Prog. Theor. Phys. **86** (1991) 547; L. Alvarez-Gaumé, J.L.F. Barbón and C. Gómez, Nucl. Phys. **B368** (1992) 57; L. Alvarez-Gaumé and C. Gómez, preprint CERN-TH.6175/91 (July 1991); Vl.S. Dotsenko, preprint Paris VI PAR-LPTHE 91-18 (October 1991); M. Li, preprint UCSBTH-91-47 (October 1991).

[16] D. Kutasov, E. Martinec and N. Seiberg, preprint PUPT-1293 (November 1991).

[17] Y. Cai and G. Siopsis, Mod. Phys. Lett. **A6** (1991) 1261.

[18] J. Goldstone, unpublished; G. Segal, Comm. Math. Phys. **80** (1981) 301.

[19] I. Bakas, Phys. Lett. **228B** (1989) 57; C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. **236B** (1990) 173.

[20] B.L. Feigin and D.B. Fuchs, Representations of the Virasoro algebra, Seminar on supermanifolds No. 5, Ed. D. Leites (1985).

[21] V.G. Kac, Lecture Notes in Physics, Vol. 94 (1979) pp.441–445.

[22] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. **241** (9184) 333.

[23] B.L. Feigin and D.B. Fuchs, Funkts. Anal. Prilozh. **17** (3) 91-92 (1983).

[24] V.K. Dobrev, in: Proc. XIII Int. Conf. Diff.-Geom. Meth. Theor. Phys., (Shumen, 1984), Eds. H.D. Doebner and T.D.Palev (World Sci., Singapore, 1986) p. 348.