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ABSTRACT

Continuing our work in refs.[1, 2], we study the Classical and Quantum Liouville theory on the Riemann sphere with $n > 3$ punctures. We get the quantum exchange algebra relations between the chiral components in the Liouville theory from our assumption on the principle of quantization.

1. Introduction

The Liouville theory has attracted much attention for a long time. The early interests in it rests mainly on the uniformization theory of two dimensional Riemann surfaces and a lot of mathematicians such as Klein, Poincare, Koebe [3] etc. have done a lot of excellent works in this domain. The recent interests in it revived mainly with the Polyakov string [4] and two dimensional quantum gravity[3-5] where the Liouville action plays the role of Weyl anomaly. It is inevitable for us to study the classical and quantum Liouville theory in order to study the noncritical string theory.

Despite a lot of efforts made by many people[5], there still exist a lot of mysterious aspects in both the classical and quantum Liouville theory. In [3], we have mainly studied the classical Liouville theory on the Riemann sphere with $n > 3$ punctures and interesting results have been obtained there. In ref.[2], we have studied the classical and quantum Liouville theory on the Riemann sphere with $n > 3$ punctures near an arbitrary but fixed puncture. We have found there exists an extra symmetry in our theory to which corresponds a family of solutions of both the classical and quantum Yang-Baxter equation dominating the classical and quantum exchange algebra of the Liouville theory.

Generally, we may assume that the Poisson-Lie symmetry at the classical level has its quantum counterpart in the symmetry with respect to the quantum group. That is to say that if we find that the classical exchange algebra of a set of chiral components is dominated by the classical r-matrice which is a solution of the classical Yang-Baxter equation, then we can suppose that the corresponding quantum exchange algebra is dominated by the quantum R-matrices which is the solution of the quantum Yang-Baxter equation. This is in fact the postulation of quantum integrability.

In our theory, we will meet some monodromy parameters except for the fields depending on the space time. We wish to get a quantum theory including that of the monodromy parameters and that of the space-time dependent fields. To do this, we will adopt a postulation of the quantum exchange properties of the monodromy parameters as well as the space-time dependent fields. Starting from this postulation, we can get the quantum exchange algebra relations of the chiral components from which we can see whetl er our postulation is reasonable or not since the quantum Liouville theory has the $SL(2,R)$ quantum group symmetry. This postulation on the principle of quantization, we find, can be related with the usual canonical on in an indirect way.

This paper is organized as follows: In section 2, we introduce concisely, for the convenience of notations, some background knowledge about the uniformization theory of the Riemann sphere with $n > 3$ punctures and some main results of [1, 2]. In section 3, we present some recent results for the classical Liouville theory on the Riemann sphere with $n > 3$ punctures which is necessary for the quantum theory. In section 4, we present the assumption on the principle of quantization for the monodromy parameters and then give the quantum exchange algebra relations for the chiral components. In section 5, we will give some concluding remarks. Finally, we present two appendices in

one of which we try to understand our assumption on the principle of quantization for the monodromy parameters from the point of view of canonical quantization.

2. Introduction to the Liouville theory on the Riemann sphere with $n > 3$ punctures.

Before going to our recent new results, we first introduce some background knowledge and some main results of [1,2] for conveniences.

The Riemann sphere X with n punctures whose coordinates are $z_1, \cdots, z_{n-3}, 0, 1, and\ \infty$ without loss of generality can be realized as the quotient space H/Γ where H is the upper half plane and $\Gamma \subset PSL(2,\ R)$ is the Fuchsian group. That is to say, there exists a covering $J:\ H \to X$ with $J(\gamma\omega) = J(\omega)$ to arbitrary $\gamma \in \Gamma,\ \omega \in H$.

The uniformization problem of the Riemann sphere is connected with the second order linear differential equation

$$\frac{d^2\eta}{dz^2} + \frac{1}{2}Q_X(z)\eta = 0 \tag{1}$$

by that J^{-1} can be represented as the quotient of the two linearly independent solutions of (1).

The Poisson bracket relations between these two solutions of equation (1) is dominated by

$$\{\eta_i(z), \eta_j(z')\} = S_{ij}^{kl}\eta_k(z)\eta_l(z') \tag{2}$$

where $i, j, k, l = 1, 2$ and

$$S_{ij}^{kl} = -\frac{1}{16}[r_+\theta(|z| - |z'|) + r_-\theta(|z'| - |z|)]_{ij}^{kl} \tag{3}$$

and $\theta(z)$ is the step function. The 4×4 matrices $r_\pm$, called the classical r matrices, are the solutions of the classical Yang-Baxter equation:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0,$$

and can be expressed by the generators of the Lie algebra $sl(2,R)$:

$$\begin{aligned} r_\pm &= \pm H \otimes H \pm 4E_\pm \otimes E_\mp \\ [H, E_\pm] &= \pm 2E_\pm, \quad [E_+, E_-] = H \end{aligned} \tag{4}$$

The projection monodromy group here is just the Fuchsian uniformization group. The elements of this group are parabolic. We can choose in this group a standard system of parabolic generators $M_1, \cdots, M_n$ satisfying the single relation $M_1 \cdots M_n = 1$, $TrM_\lambda = 2$ and M_λ has one fixed point z_λ, the matrices satisfying these conditions can be represented as

$$M_\lambda = \begin{pmatrix} 1 + \alpha_\lambda z_\lambda & -\alpha_\lambda z_\lambda^2 \\ \alpha_\lambda & 1 - \alpha_\lambda z_\lambda \end{pmatrix} \tag{5}$$

and

$$M_n = \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix} \tag{6}$$

where we take $z_n = \infty$. All z_λ and α_λ here are real varial 'es.

When we circle around the λ-th puncture, η_1, η_2 transform according to

$$\begin{pmatrix} \eta_1^\lambda \\ \eta_2^\lambda \end{pmatrix} = \begin{pmatrix} M_\lambda \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \tag{7}$$

we can assume

$$\{\eta_i^\lambda(P), \otimes\eta_j^\lambda(P')\} = S_{ij}^{kl}\eta_k^\lambda(P)\eta_l^\lambda(P').$$

and (8)

$$\{\eta_i^{\lambda\rho}(P), \otimes\eta_j^{\lambda\rho}(P')\} = S_{ij}^{kl}\eta_k^{\lambda\rho}(P)\eta_l^{\lambda\rho}(P').$$

where S_{ij}^{kl} is as shown in (3).

From (8), the following Poisson brackets can be uniquely determined

$$\begin{aligned}
&\{\eta_1, \alpha_\lambda\} = -\tfrac{1}{4}\alpha_\lambda z_\lambda \eta_2, \quad &&\{\eta_1, z_\lambda\} = -\tfrac{1}{8} z_\lambda^2 \eta_2, \\
&\{\eta_2, \alpha_\lambda\} = 0, &&\{\eta_2, z_\lambda\} = -\tfrac{1}{8}\eta_1 \\
&\{\alpha_\lambda, \alpha_\rho\} = 0, &&\{z_\lambda, z_\rho\} = \tfrac{1}{8}(z_\rho^2 - z_\lambda^2) \\
&\{\alpha_\lambda, z_\rho\} = \tfrac{1}{4}\alpha_\lambda z_\lambda
\end{aligned} \tag{9}$$

where $\lambda, \rho = 1, 2, \cdots, k$. $k \geq 2$ is the number of punctures. Surrounded by the level circles. Thus the elements of the monodromy matrices are dynamical variables in our case.

3. Further results on the classical Liouville theory.

In refs. [1, 2], we have presented some results on the classical theory, especially, we have studied the classical and the quantum Liouville theory near one arbitrary but fixed puncture. There we have got a family of solutions of both the classical and quantum Yang-Baxter equations. In this section, we will give some further properties of the classical theory and establish some formalism which is convenient for the quantization which will be presented in the next section.

Let us reparametrize the monodromy parameters α_λ, z_λ as

$$\alpha_\lambda^{\frac{1}{2}} = Q_\lambda, \qquad z_\lambda \alpha_\lambda^{\frac{1}{2}} = P_\lambda.$$

Then the Poisson bracket relation (9) can be rewritten as

$$\begin{aligned}
&\{Q_\lambda, P_\rho\} = \tfrac{1}{8}P_\lambda Q_\rho, \quad &&\{Q_\lambda, Q_\rho\} = 0 \\
&\{\eta_1, Q_\lambda\} = -\tfrac{1}{8}P_\lambda \eta_2, &&\{P_\lambda, P_\rho\} = 0 \\
&\{\eta_1, P_\lambda\} = 0, &&\{\eta_2, Q_\lambda\} = 0 \\
&\{\eta_2, P_\lambda\} = \tfrac{1}{8}Q_\lambda \eta_1
\end{aligned} \tag{10}$$

The monodromy matrix (5) can be reparametrized as

$$M_\lambda = \begin{pmatrix} 1 + Q_\lambda P_\lambda & -P_\lambda^2 \\ Q_\lambda^2 & 1 - Q_\lambda P_\lambda \end{pmatrix} \tag{11}$$

Comparing with the notation of ref. [2] in the neighbourhood of the λ-th puncture, we have

$$P_\lambda = e^{\frac{1}{2}p_\lambda}, \qquad Q_\lambda = e^{\frac{1}{4}q_\lambda}$$

Generally, if we consider the Poisson bracket relation between the functions $F = F(Q_i.P_i)$ and $G = G(Q_i.P_i)$, we can get from equation (10) and the property of the Poisson bracket,

$$\begin{aligned}
\{F, G\} &= \tfrac{\partial F}{\partial Q_i}\tfrac{\partial G}{\partial Q_j}\{Q_i, Q_j\} + \tfrac{\partial F}{\partial Q_i}\tfrac{\partial G}{\partial P_j}\{Q_i, P_j\} + \tfrac{\partial F}{\partial P_i}\tfrac{\partial G}{\partial Q_j}\{P_i, Q_j\} + \tfrac{\partial F}{\partial P_i}\tfrac{\partial G}{\partial P_j}\{P_i, P_j\} \\
&= \tfrac{1}{8}[\nabla_Q F \cdot \nabla_P G - \nabla_P F \cdot \nabla_Q G]
\end{aligned} \tag{12}$$

where

$$\nabla_Q = \sum_i P_i \frac{\partial}{\partial Q_i}, \qquad \nabla_Q = \sum_i Q_i \frac{\partial}{\partial P_i}.$$

Equation (12) may also be regarded as the defining relation for the Poisson bracket in the parameter space spanned by P_i, Q_i.

From equation (9) and the defining relation (12), we can easily get the differential equations satisfied by η_1 and η_2 in this parameter space:

$$\begin{aligned}
&\textstyle\sum_{i=1}^{n-3} Q_i \frac{\partial \eta_1}{\partial P_i} = \eta_2, \quad &&\textstyle\sum_{i=1}^{n-3} P_i \frac{\partial \eta_2}{\partial Q_i} = \eta_1, \\
&\textstyle\sum_{i=1}^{n-3} P_i \frac{\partial \eta_1}{\partial Q_i} = 0, &&\textstyle\sum_{i=1}^{n-3} Q_i \frac{\partial \eta_2}{\partial P_i} = 0
\end{aligned} \tag{13}$$

The general solution to this equation can be found to be

$$\eta_1 = \sum_{i=1}^{n-3} f_i(z) P_i, \qquad \eta_2 = \sum_{i=1}^{n-3} f_i(z) Q_i \tag{14}$$

where $f_i(z)$ depends only on the space-time coordinates and must satisfy the uniformization equation (1) (see appendix I for detail).

If we consider two functions $F = F(P_i, Q_i, f_i, f_i')$ and $G = G(P_i, Q_i, f_i, f_i')$ in which $f_i = f_i(z)$, $f_i' = f_i(z')$, their Poisson bracket can be found to be

$$\{F,\ G\} = \{F,\ G\}_{(Q_i,\ P_j)} \cdot \{Q_i,\ P_j\} + \{F,\ G\}_{(f_i,\ f_j')} \cdot \{f_i,\ f_j'\} \tag{15}$$

where

$$\{F,\ G\}_{(Q_i,\ P_j)} = \frac{\partial F}{\partial Q_i} \cdot \frac{\partial G}{\partial P_j} - \frac{\partial F}{\partial P_j} \cdot \frac{\partial G}{\partial Q_i}$$

$$\{F,\ G\}_{(f_i,\ f_j')} = \frac{\partial F}{\partial f_i} \cdot \frac{\partial G}{\partial f_j'} - \frac{\partial F}{\partial f_j'} \cdot \frac{\partial G}{\partial f_i}$$

At this stage, we have derived the formalism of η_i in terms of the monodromy parameters and the space-time dependent functions f_i. On the other hand, we already know the Poisson bracket relations between η_i, so we are ready to get the Poisson bracket relations between f_i and f_i' as

$$\{f_i, f_j'\} = \frac{1}{16}\epsilon(|z| - |z'|)(f_i f_j' - 2 f_j f_i') \tag{16}$$

4. The quantum Liouville theory in the general case.

In ref. [2], we have exploited the quantum Liouville theory near an arbitrary but fixed puncture. We can get the corresponding quantum theory by using the canonical quantization. When we consider the effects of all the punctures, one may notice that neither the parameters P_i, Q_i in (10), nor the space-time dependent fields f, f' in (16) can be considered as free fields.

However, in the canonical quantization, when we try to get the quantum theory corresponding to the classical theory, we have made an important assumption- the principle of quantization. This assumption corresponding to the canonical Poisson bracket relation between the canonical variables is the bridge between the classical and quantum theory.

What we face now is: we do not have the canonical variables at hand. So the first thing we should do is to make a principle of quantization corresponding to our Poisson bracket relations. We assume that all the functions such as η_1, η_2, fz and all the monodromy parameters such as Q_i, P_i will be regarded as operators in the quantum case. The expression of η_1, η_2 in terms of $f(z)$, Q_i, P_i is the same as its classical case in equation (14). ¿From equations (10) and (16), we can find that

$$\begin{aligned}\{Q_\lambda + Q_\rho,\ P_\lambda + P_\rho\} &= \tfrac{1}{8}(Q_\lambda + Q_\rho)(P_\lambda + P_\rho)\\ \{f_i + f_j,\ f_i' + f_j'\} &= -\tfrac{1}{16}\epsilon(|z| - |z'|)(f_i + f_j)(f_i' + f_j').\end{aligned} \tag{17}$$

We there call $f_i + f_j$ as the semi-free fields.

Furthermore, we make the following principle of quantization:

$$\begin{aligned}P_k Q_l &= Q_l P_k + (1 - A) P_l Q_k\\ f_l f_k' &= (2 - B) f_k' f_l - 2(1 - B) f_l' f_k\end{aligned} \tag{18}$$

where A and B are some coefficients to be determined (See appendix II for details).

From (18), we can deduce the quantum commutation relations corresponding to (16), i.e.

$$\begin{aligned}[Q_\lambda + Q_\rho,\ P_\lambda + P_\rho] &= (A - 1)(P_\lambda + P_\rho)(Q_\lambda + Q_\rho)\\ [f_i + f_j,\ f_i' + f_j'] &= (B - 1)(f_i' + f_j')(f_i + f_j).\end{aligned} \tag{19}$$

Furthermore, we can get from (14) and (18) these exchange relations

$$\begin{aligned}\eta_1\eta_1' = B\eta_1'\eta_1, &\quad \eta_1\eta_2' = (2 - B)\eta_2'\eta_1 + (B - 2A + AB)\eta_1'\eta_2,\\ \eta_2\eta_2' = B\eta_2'\eta_2, &\quad \eta_2\eta_1' = (2 - B)A(2 - A)\eta_1'\eta_2 - (4 - 3B - 2A + AB)\eta_2'\eta_1.\end{aligned} \tag{20}$$

They can be rewritten compactly as

$$\eta_i\eta_j' = R_{ij}^{mn}\eta_m\eta_n' \tag{21}$$

where

$$R = \begin{pmatrix} B & 0 & 0 & 0\\ 0 & 2 - B & B - 2A + AB & 0\\ 0 & -4 + 3B + 2A - AB & (2 - B)A(2 - A) & 0\\ 0 & 0 & 0 & B\end{pmatrix} \tag{22}$$

If $A = \frac{4-3B}{2-B}$, we will get

$$R_+ = \begin{pmatrix} B & 0 & 0 & 0\\ 0 & 2 - B & -4 + 4B & 0\\ 0 & 0 & \frac{B(4-3B)}{2-B} & 0\\ 0 & 0 & 0 & B\end{pmatrix}. \tag{23}$$

If $A = \frac{B}{2-B}$, we will get

$$R_- = \begin{pmatrix} B & 0 & 0 & 0\\ 0 & 2 - B & 0 & 0\\ 0 & -4 + 4B & \frac{B(4-3B)}{2-B} & 0\\ 0 & 0 & 0 & B\end{pmatrix}. \tag{24}$$

It is easy to show that both $R+$ and R_- satisfy the quantum Yang-Baxter equation. If we let $B = exp(\frac{i\hbar}{16})$, we find

$$R_+|_{\hbar\to 0} = I + \frac{i\hbar}{16}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & -4 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} + o(\hbar^2) = I + \frac{i\hbar}{16}r_+ + o(\hbar^2). \tag{25}$$

If we let $B = exp(-\frac{i\hbar}{16})$, we find

$$R_-|_{\hbar\to 0} = I - \frac{i\hbar}{16}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 4 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} + o(\hbar^2) = I - \frac{i\hbar}{16}r_- + o(\hbar^2). \quad (26)$$

5. Concluding remarks.

At this stage, we have the quantum Liouville theory on the Riemann sphere with $n > 3$ punctures. To make the problems clearer, let us present the logic as follows: we are handling the Liouville theory on the Riemann sphere with $n > 3$ punctures, the chiral components which are two independent solutions of the uniformization equation depend in general on the space-time coordinates and the monodromy parameters, its classical exchange algebra relations is dominated by the classical r-matrix and its quantum exchange algebra relation, in principle, can be assumed to be dominated by the quantum R-matrices. This is in fact the quantum integrability condition. In our case, we know that the usual canonical quantization procedure does not do in the case of more than four distinguished punctures, however, we want to get the quantum theory of all the dynamical variables which include the monodromy parameters. For this reason, we adopt the assumption of the quantum exchange relations of the monodromy parameters and that of the space-dependent fields. Then we try to get the quantum exchange algebra relations between the chiral components according to our assumption, i.e. equation (18). If our assumption can lead to the quantum integrability condition, we can say that this assumption is reasonable and hence we have the quantum exchange relations between all the dynamical variables.

Appendix I

In this appendix, we give the solutions to equation (13).

From (13), we can get another four differential equations:

$$\begin{aligned} &Q_i\frac{\partial\eta_1}{\partial Q_i} = 0, \quad P_i\frac{\partial\eta_2}{\partial P_i} = 0,\\ &P_i\frac{\partial\eta_1}{\partial P_i} = \eta_1, \quad Q_i\frac{\partial\eta_2}{\partial Q_i} = \eta_2 \end{aligned} \quad (27)$$

where $i = 1, 2, \cdots, n-3$. From these equations, together with (13), we can see that η_1 and η_2, generally, are functions of the monodromy parameters P_λ, Q_λ ($\lambda = 1, 2, \cdots, n-3$) and the space-time coordinates. In the monodromy parameter space, η_1 and η_2 can expressed as the Taylor expansion form:

$$\eta_i = \sum_{m_i, n_j \geq 0} a^{(i)}(m_1, \cdots, m_M, n_1, \cdots, n_N)P_1^{m_1}\cdots P_N^{m_M}Q_1^{n_1}\cdots Q_N^{n_N} \quad (28)$$

where $i = 1, 2$. Substituting this expression into the first two equations of (27), we can find that

$$\begin{aligned} \eta_1 &= \textstyle\sum_{m_i\geq 0} a^{(1)}(m_1, \cdots, m_M)P_1^{m_1}\cdots P_N^{m_M}\\ \eta_2 &= \textstyle\sum_{n_j\geq 0} a^{(2)}(n_1, \cdots, n_N)Q_1^{n_1}\cdots Q_N^{n_N}. \end{aligned} \quad (29)$$

Then substituting equation (29) into the last two equations of (27), we can further get

$$\begin{aligned} &\textstyle\sum_{m_i\geq 0} a^{(1)}(m_1, \cdots, m_M)(m_1 + \cdots + m_M)P_1^{m_1}\cdots P_M^{m_M}\\ &= \textstyle\sum_{m_i\geq 0} a^{(1)}(m_1, \cdots, m_M)P_1^{m_1}\cdots P_N^{m_M}\\ &\textstyle\sum_{n_i\geq 0} a^{(1)}(n_1, \cdots, n_N)(n_1 + \cdots + n_N)P_1^{N_1}\cdots P_N^{n_N}\\ &= \textstyle\sum_{N_i\geq 0} a^{(1)}(N_1, \cdots, n_N)P_1^{n_1}\cdots P_N^{n_N} \end{aligned} \quad (30)$$

So we get

$$m_1 + \cdots + m_M = 1, \quad n_1 + \cdots + n_N = 1 \quad (31)$$

Hence the general solution of equation (13) is

$$\eta_1 = \sum_i a_i^{(1)}P_i, \qquad \eta_2 = \sum_j a_j^{(2)}Q_j. \quad (32)$$

where $a_i^{(1)}$ and $a_i^{(2)}$ depend only on the space-time coordinates. Substituting equation (32) back into (13), we, get

$$a_i^{(1)} = a_i^{(2)} = f_i(z).$$

Appendix II

In this appendix, we try to understand our basic assumption, equation (18), from the point of view of canonical quantization.

In order to find out the conjugate variables suitable for canonical quantization from the Poisson relations (10), we try to enlarge the manifold M_1 spanned by P_i and Q_i and the manifold M_2 spanned by f_i and f_i' by introducing their conjugate variables x_i, y_i and u_i, v_i respectively, i.e.

$$\begin{aligned} &\{q_i, x_j\} = \delta_{ij}, \quad \{q_i, q_j\} = 0, \quad \{x_i, x_j\} = 0\\ &\{p_i, y_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0, \quad \{y_i, y_j\} = 0\\ &\{\varphi_i, u_j\} = \delta_{ij}, \quad \{\varphi_i, \varphi_j\} = 0, \quad \{\varphi_i', \varphi_j'\} = 0\\ &\{\varphi_i', v_j\} = \delta_{ij}, \quad \{u_i, u_j\} = 0, \quad \{v_i, v_j\} = 0 \end{aligned} \quad (33)$$

where

$$p_i = 2\ln P_i, \quad q_i = 4\ln Q_i, \quad \varphi_i = 4\ln f_i, \quad \varphi_i' = 4\ln f_i'$$

. Furthermore take

$$x_i = \frac{2}{a}\ln X_i, \quad y_i = \frac{4}{a}\ln Y_i, \quad u_i = \frac{4}{b}\ln U_i, \quad v_i = \frac{4}{b}\ln V_i,$$

Then we can equivalently express (12) as

$$\begin{aligned}
&\{F,\ G\} = \tfrac{8}{a^2} \\
&[\textstyle\sum_i P_i Q_i^{-1} X_i^{-1}\{F,\ X_i\} \sum_j Q_j P_j^{-1} Y_j^{-1}\{G,\ Y_j\} \\
&-\textstyle\sum_j Q_j P_j^{-1} Y_j^{-1}\{F,\ Y_j\} \sum_i P_i Q_i^{-1} X_i^{-1}\{G,\ X_i\}] \\
&+\tfrac{16}{b^2}\epsilon(|z|-|z'|)\textstyle\sum_i\sum_j [U_i^{-1}\{F,\ U_i\} V_j^{-1}\{G,\ V_j\} - V_j^{-1}\{F,\ V_j\} U_i^{-1}\{G,\ U_i\} \\
&+2 f_j f_j'^{-1} V_j^{-1}\{F,\ V_j\} f_i f_i'^{-1} U_i^{-1}\{G,\ U_i\} - 2 f_i f_i'^{-1} U_i^{-1}\{F,\ U_i\} f_j f_j'^{-1} V_j^{-1}\{G,\ V_j\}]
\end{aligned} \tag{34}$$

where F and G are arbitrary functions of P_i, Q_i, f_i and f_i'.

At this stage, it is appropriate to study the quantum theory via canonical quantization. Firstly, we can get from (33), according to the ordinary procedure of canonical quantization, these canonical commutation relations:

$$\begin{aligned}
&[q_i,\ x_j] = i\hbar\delta_{ij}, \qquad [p_i,\ y_j] = i\hbar\delta_{ij} \\
&[\varphi_i,\ u_j] = i\hbar\delta_{ij}, \qquad [\varphi_i,\ v_j] = i\hbar\delta_{ij}.
\end{aligned} \tag{35}$$

as well as

$$\begin{aligned}
&[Q_i,\ X_j] = (-1+A) X_j Q_i \delta_{ij}, \qquad [P_i,\ Y_j] = (-1+A) Y_j P_i \delta_{ij} \\
&[f_i,\ U_j] = (-1+B) U_j f_i \delta_{ij}, \qquad [f_i',\ V_j] = (-1+B) V_j f_i' \delta_{ij}
\end{aligned} \tag{36}$$

where A and B are some constants depending on $\hbar$.

Suppose that F, G are some composed operators of Q_i and P_i, we define their commutation relation as

$$\begin{aligned}
&[F,\ G] = \\
&c\big(\textstyle\sum_i\sum_j P_i Q_i^{-1} X_i^{-1}[F,\ X_i] Q_j P_j^{-1} Y_j^{-1}\{G\ \ Y_j\} \\
&+\textstyle\sum_i\sum_j P_i Q_i^{-1} X_i^{-1}\{F,\ X_i\} Q_j P_j^{-1} Y_j^{-1}[G,\ Y_j] \\
&-\textstyle\sum_i\sum_j P_i Q_i^{-1} X_i^{-1}[G,\ X_i] Q_j P_j^{-1} Y_j^{-1}\{F,\ Y_j\} \\
&-\textstyle\sum_i\sum_j P_i Q_i^{-1} X_i^{-1}\{G,\ X_i\} Q_j P_j^{-1} Y_j^{-1}\{F,\ Y_j\}\big)
\end{aligned} \tag{37}$$

where $c = \frac{4}{a}$. On the right hand side of (37) we arrange the operators such that P is on the left of Q. With the same procedure we define the Lie bracket of the operators composed of f_i and f_i':

$$\begin{aligned}
&[F,\ G] = \\
&c'\textstyle\sum_i\sum_j\big(-V_j^{-1}\{G,\ V_j\} U_i^{-1}[F,\ U_i] - V_j^{-1}[G,\ V_j] U_i^{-1}\{F,\ U_i\} \\
&+V_j^{-1}[F,\ V_j] U_i^{-1}\{G,\ U_i\} + V_j^{-1}[F,\ V_j] U_i^{-1}\{G,\ U_i\} \\
&+?f_i' f_i^{-1} U_i^{-1}\{F,\ U_i\} f_j f_j'^{-1} V_j^{-1}[G,\ V_j] + 2 f_i' f_i^{-1} U_i^{-1}[F,\ U_i] f_j f_j'^{-1} V_j^{-1}\{G,\ V_j\} \\
&-2 f_i' f_i^{-1} U_i^{-1}\{G,\ U_i\} f_j f_j'^{-1} V_j^{-1}[F,\ V_j] - 2 f_i' f_i^{-1} U_i^{-1}[G,\ U_i] f_j f_j'^{-1} V_j^{-1}\{F,\ V_j\}
\end{aligned} \tag{38}$$

where $c' = \frac{8}{b}$. On the right hand side of (41), we have put f' on the left hand side of f after calculation. The idea for defining (37) and (38) is that the quantum effects of the operators P, Q and f are able to be observed in each subspace acted by its own conjugate operators.

From (37) and (38), it is easy for us to get equation (18), which is just our basic assumption.

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